On the Generalization for Some Power-Exponential-Trigonometric Inequalities

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Abstract: In this paper, we introduce and prove several generalized algebraic-trigonometric inequalities by considering negative exponents in the inequalities.

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1. Introduction

In recent years, an increasing amount of attention has been paid to the study of power-exponential inequalities [1–10]. A review of some problems and historical landmarks are given in [2,11]. In particular, in order to contextualize, we recall that the basic problem of comparing $a^b$ and $b^a$ for all positive real numbers $a$ and $b$ was presented in [12–14]. Increasing in algebraic difficulty, the comparison of $a^a + b^b$ and $a^b + b^a$ was studied independently by Laub–Ilani and Zeikii–Cirtoaje–Berndt, see [15–18], respectively. The result is the fact that the inequality

$$a^a + b^b \geq a^b + b^a, \quad a, b \in [0, \infty] \tag{1}$$

holds. An extension of (1) was proposed, analyzed and proved by Matejíčka, Cirtoaje and Coronel-Huancas in [2,17,19] obtaining the inequality

$$a^a + b^b \geq a^b + b^a, \quad a, b \in [0, \infty] \quad r \in [0, e]. \tag{2}$$

More recently, other extensions and generalizations of (1) were introduced, proved and conjectured by Özban in [11], where, in particular, the author proved the following inequalities:

$$(\sin x)^{\sin x} + (\sin y)^{\sin y} > (\sin x)^{\sin y} + (\sin y)^{\sin x}, \quad 0 < x < y < \pi/2, \tag{2.1}$$
$$(\cos x)^{\cos x} + (\cos y)^{\cos y} > (\cos x)^{\cos y} + (\cos y)^{\cos x}, \quad 0 < x < y < \pi/2, \tag{2.2}$$
$$(\cos x)^{\sin x} + (\cos y)^{\sin y} < (\cos x)^{\sin y} + (\cos y)^{\sin x}, \quad 0 < x < y \leq 1, \tag{2.3}$$
$$(\cos x)^x + (\cos y)^y < (\cos x)^y + (\cos y)^x, \quad 0 < x < y \leq \pi/2, \tag{2.4}$$
$$(\sin x)^y + (\sin y)^y > (\sin x)^y + (\sin y)^y, \quad 0 < x < y \leq \pi/2, \tag{2.5}$$
$$x^{\cos x} + y^{\cos y} < x^{\cos y} + y^{\cos x}, \quad 0 < x < y, \quad 1 \leq y \leq \pi/2, \tag{2.6}$$
$$x^{\sin x} + y^{\sin y} > x^{\sin y} + y^{\sin x}, \quad 0 < x < y \leq \pi/2. \tag{2.7}$$

In order to extend or generalize (2) and (3), it seems natural to ask some questions: What happens with the inequality (2) when $r \in \mathbb{R} - [0, e]$? and what happens with the inequalities in (3) if we include a negative power $r$? We note that the powers in question exist, since the basis of powers in (2) and (3)
are positive. Indeed, in this article, we study \( x \in [\gamma, 0] \) and establish reverse inequalities for some cases. Moreover, we study the generalization of the inequalities in (3) with negative power \( r \).

The main results of the paper are the following theorems:

**Theorem 1.** Let the function \( f_A : \mathbb{R} \to \mathbb{R} \) be defined by \( f_A(x) = x^a \) for each \( a > 1 \) and consider the following sets:

\[
\begin{align*}
A_{\text{old}} &= \left\{ (a,b,r) \in \mathbb{R}^3 : a \geq 0, \ b \geq 0, \ r \in [0,e] \right\}, \\
A_{\text{new}}^{d} &= \left\{ (a,b,r) \in \mathbb{R}^3 : a > 1, \ b > 1, \ r < 0, \ f_A(r) > f_A(r) \right\} \\
A_{\text{new}}^{d} &= \bigcup \left\{ (a,b,r) \in \mathbb{R}^3 : a > 1, \ b > 1, \ r < 0, \ f_A(r) > f_A(r), \ a^r < \gamma \right\}, \quad (4)
\end{align*}
\]

where \( \gamma \in ]0,1[ \) is such that \( \gamma \neq b^r \) and \( (\gamma)^{1/r} - \gamma - b^r \neq 0 \). Then, the following inequalities

\[
\begin{align*}
& a^r + b^r \geq a^r + b^r, \quad (a,b,r) \in A_{\text{old}} \cup A_{\text{new}}^{d} \\
& a^r + b^r \leq a^r + b^r, \quad (a,b,r) \in A_{\text{new}}^{d}
\end{align*}
\]

are satisfied.

**Remark 1.** The inclusion of the notation \( \gamma \) is related with the fact that the argumentation of the proof is based on the properties of function \( f(t) = (t)^r - t - \gamma^r + \gamma \) with \( t = a^r \) and \( \gamma = b^r \). In particular, we observe that, if \( 0 < t < \gamma < 1 \), there are two solutions of \( f(t) = 0 \) on the interval \([0,1] \); one solution is clearly \( \gamma \) and the other solution is difficult to get explicitly and is denoted by \( \gamma \).

**Theorem 2.** If \( x, y \in (0, \frac{\pi}{2}) \) and \( r < 0 \), then

\[
\begin{align*}
& (\sin x)^r \sin x + (\sin y)^r \sin y \leq (\sin x)^r \sin x + (\sin y)^r \sin x, \\
& (\cos x)^r \cos x + (\cos y)^r \cos y \leq (\cos x)^r \cos x + (\cos y)^r \cos x, \\
& (\cos x)^r \sin x + (\cos y)^r \sin y \geq (\cos x)^r \sin x + (\cos y)^r \sin x.
\end{align*}
\]

**Theorem 3.** If \( x, y \in (0, \frac{\pi}{2}) \) and \( r < 0 \), then

\[
\begin{align*}
& (\cos x)^r \cos x + (\cos y)^r \cos y \geq (\cos x)^r \cos x + (\cos y)^r \cos x, \\
& (\sin x)^r \sin x + (\sin y)^r \sin y \leq (\sin x)^r \sin x + (\sin y)^r \sin x.
\end{align*}
\]

**Theorem 4.** If \( x, y \in (0, \frac{\pi}{2}) \), min\{\( x, y \)\} \( \in (0,1] \) and \( r < 0 \), then

\[
\begin{align*}
& x^r \cos x + y^r \cos y \geq x^r \cos x + y^r \cos x, \\
& x^r \sin x + y^r \sin y \leq x^r \sin x + y^r \sin x.
\end{align*}
\]

The rest of the paper is dedicated to the proof of Theorems 1–4.
2. Proofs of Main Results

2.1. Proof of Theorem 1

For completeness and self-contained structure of the proof, we recall the notation and a result given in [1]. Indeed, let us consider \( s \in \mathbb{R}^+ \) and we define the functions \( f \) and \( g \) from \( \mathbb{R}^+ \) to \( \mathbb{R} \) by the relations

\[
f(t) = t^b - t - \gamma^b + \gamma, \quad g(t) = \begin{cases} e^{-\ln(t)/(t-1)}, & \text{for } t \not\in \{0,1\}, \\ e^{-1}, & \text{for } t = 1, \\ 0, & \text{for } t = 0. \end{cases}
\]

Then, the following properties are satisfied: \( f(\gamma) = 0 \) and \( f(0) = f(1) = -\gamma^b + \gamma; \) if \( s > 1 \) (resp. \( s < 1 \)), \( f \) is strictly increasing (resp. decreasing) on \( [0,g(s)] \) and strictly decreasing (resp. increasing) on \( [0,\mathbb{R}] \). Indeed, let us consider \( \phi \) given in [1]. Moreover, if we consider the function \( \phi : \mathbb{R}^+ \to \mathbb{R} \) defined in the enunciate of the theorem, we observe that the following assertions are satisfied: \( \phi(0) = \phi(1) = 0; \) if \( s > 1 \) (resp. \( s < 1 \)) \( w \) has a maximum at \( g(s) \) (resp. minimum at \( g(s) \)); \( \phi_a(0) = 0 \); \( \phi_a \) has a minimum at \( m^* = -1/\ln(a) \); \( \phi_b \) has an inflection point at \( m^* = -2/\ln(a) \). Thus, from both subcases, we conclude that the inequality \( a^a + b^b < a^b + b^a \) is satisfied.

Let us consider \( t = a^b, \gamma = b^b, \) and \( s = a/b \) and we observe that

\[
f(t) = (a^b)^{a/b} - a^b - (b^b)^{a/b} + b^b = a^a - a^b + b^a + b^b. \tag{14}
\]

Then, the proofs of (5) and (6) are reduced to analyze the sign of \( f(t) \) for \( t \in [0,\gamma] \). Indeed, without loss of generality and by the symmetric form of the inequalities in (5) and (6), we assume that \( 0 \leq b < a \) (i.e., \( s = a/b > 1 \)) and consider three cases:

(i) Let \( a, b \) such that \( 1 > a > b \geq 0 \). Then, for \( r < 0 \), we note that \( 1 < a^r < b^r \) or equivalently we have that \( 1 < t \gamma < \gamma \). Moreover, observing that \( s > 1 \) and \( g(s) < 1 \), by the strictly increasing behavior of \( f \) on \( [g(s),\infty) \), we deduce that \( f(g(s)) < f(1) < f(t) < f(\gamma) = 0 \). Thus, from (14) and \( f(t) < 0 \), we follow that the inequality \( a^a + b^b < a^b + b^a \) is satisfied.

(ii) Let \( a, b \) such that \( a > 1 > b \geq 0 \). In this case, we have that \( a^r < 1 < b^r \) or equivalently \( t < 1 < \gamma \). We note that \( t > 1 \) implies the strictly decreasing behavior of \( f \) on \( [0,g(s)] \) and the strictly increasing behavior of \( f \) on \( [g(s),\infty) \). Moreover, observing that \( g(s) \in [0,1] \), we deduce that \( f(t) < f(1) = -a^b + \gamma := \xi(\gamma) \) for any \( t < 1 < \gamma \). Now, by the fact that \( \xi \) is decreasing on \( [g(s),\infty) \), we have that \( \xi(\gamma) < \xi(1) = 0 \) for any \( \gamma > 1 \). Thus, \( f(t) < \xi(\gamma) < 0 \) for \( t < 1 < \gamma \) and, from (14), the inequality \( a^a + b^b < a^b + b^a \) is satisfied.

(iii) Let \( a, b \) such that \( a < b > 1 \). Similarly to cases (i) and (ii), we have that \( s > 1 \) and \( 0 < a^r < b^r < 1 \) or equivalently \( 0 < t \gamma < 1 \). Here, we distinguish two subcases: \( \gamma \leq g(s) \) and \( g(s) \leq \gamma < 1 \). First, if \( \gamma \leq g(s) \), we have that \( f \) is strictly decreasing on \( [0,\gamma] \) and consequently \( f(t) \geq f(\gamma) = 0 \) for \( t \in [0,\gamma] \). Second, if \( g(s) < \gamma < 1 \), by the fact that \( f(0) = \xi(\gamma) > 0 = f(\gamma) > f(g(s)) \), we have that there exists \( \tau \in [0,g(s)] \) such that \( f(\tau) = 0 \). Then, \( f(t) \geq f(\tau) = 0 \) for \( t \in [0,\tau] \) and \( f(t) \leq f(\tau) = f(\gamma) = 0 \) for \( t \in [\tau,\gamma] \). Thus, from both subcases, we conclude that the inequality \( a^a + b^b < a^b + b^a \) is satisfied for \( t \in [\tau,\gamma] \) with \( \gamma \in [g(s),1] \). And the inequality \( a^a + b^b < a^b + b^a \) is satisfied for \( t \in [0,\tau] \) with \( \gamma \in [g(s),1] \) or for \( t \in [0,\gamma] \) with \( \gamma \in [0,g(s)] \).

On the other hand, by the definition of \( \gamma, s, g \), and \( \phi_b \), we observe that \( \gamma < g(s) \) (resp. \( \gamma > g(s) \)) is equivalent to \( \phi_b(rb) > \phi_b(r) \) (resp. \( \phi_b(rb) < \phi_b(r) \)). Moreover, the relation \( t > \tau \) (resp. \( t < \tau \)) is equivalent to \( a^b > \tau \) (resp. \( a^b < \tau \)). Thus, the subcases can be characterized in terms of the function \( \phi_b \) and \( a^b > \tau \) or \( a^b < \tau \).
Hence, translating (i), (ii) and (iii) to the corresponding notation in (4) and observing that the set \( A_{old} \) is the set for the inequality in (2), we conclude the proof the theorem.

2.2. Proof of Theorem 2

Since \( \sin t, \cos t > 0 \) for \( t \in (0, \pi/2) \), Theorem 1 immediately implies inequalities (7) and (8). To prove (9), we define

\[
f'(t) = (\cos t)^{r \sin t} + (\cos y)^{r \sin y} - (\cos t)^{r \sin y} - (\cos y)^{r \sin t}
\]

for \( y \) is fixed and arbitrarily selected such that \( y \in (0, \pi/2) \) and \( 0 < t \leq y \). We note that \( f(y) = 0 \), then the result follows if \( f \) is decreasing. Indeed, to see this, we write

\[
f'(t) = r \left( \frac{g(t) \cos t}{\cos t} + \frac{\sin t}{\cos t} h(t) \right),
\]

where

\[
g(t) = (\cos t)^{r \sin t} \ln(\cos t) - (\cos y)^{r \sin t} \ln(\cos y),
\]
\[
h(t) = (\cos t)^{r \sin y} \sin y - (\cos t)^{r \sin t} \sin t.
\]

Now, since \( r < 0 \), it is enough to show that \( g(t), h(t) > 0 \). For \( g \), we have that

\[
g(t) = -\int_{t}^{y} \frac{d}{ds} (\cos s)^{r \sin t} \ln(\cos s)
\]
\[
= \int_{t}^{y} ((\cos s)^{r \sin t-1} \sin s)(1 + r \sin t \ln(\cos s)) ds > 0
\]

and, similarly for \( h \), we deduce that

\[
h(t) = \int_{t}^{y} \frac{d}{ds} (\cos t)^{r \sin y} \sin s
\]
\[
= \int_{t}^{y} ((\cos t)^{r \sin y} \cos s)(1 + r \sin s \ln(\cos t)) ds > 0.
\]

2.3. Proof of Theorem 3

Set \( 0 < t \leq y < \pi/2 \) and \( r < 0 \) arbitrarily. Along the proofs, we will use that \( \sin s, \cos s > 0 \) for \( s \in (0, \pi/2) \).

In order to prove (10), let us consider \( f_1(t) = (\cos t)^{r t} + (\cos y)^{r y} - (\cos t)^{r y} - (\cos y)^{r t} \). Observing that \( f_1(y) = 0 \), it is enough to show that \( f_1 \) is decreasing. Indeed, the decreasing behavior of \( f_1 \) follows immediately since

\[
f_1'(t) = r \left[ g_1(t) + \frac{\sin t}{\cos t} h_1(t) \right],
\]

where

\[
g_1(t) = (\cos t)^{r t} \ln(\cos t) - (\cos y)^{r t} \ln(\cos y) = -\int_{t}^{y} \frac{d}{ds} (\cos s)^{r t} \ln(\cos s)
\]
\[
= \int_{t}^{y} ((\cos s)^{rt-1} \sin s)(1 + rt \ln(\cos s)) ds > 0
\]
and
\[
h_1(t) = y(\cos t)^r y^r - t(\cos t)^r t = \int_t^y \frac{d}{ds}(\cos t)^r s \hspace{1cm}
\]
\[
= \int_t^y (\cos t)^r (1 + rs \ln(\cos t)) \, ds > 0.
\]

We prove (11) by analogous arguments to the proof of (10). Indeed, let us introduce the notation
\[
f_2(t) = (\sin t)^y + (\sin y)^y - (\sin t)^y - (\sin y)^y.
\]
We observe that
\[
f_2'(t) = r \left[ g_2(t) + \cos t \sin t h_2(t) \right] < 0,
\]
since
\[
g_2(t) = (\sin y)^y \ln(\sin y) - (\sin t)^y \ln(\sin t) = \int_t^y \frac{d}{ds}(\sin s)^y \ln(\sin s)
\]
\[
= \int_t^y ((\sin s)^y \cos s)(1 + rt \ln(\sin s)) \, ds > 0
\]
and
\[
h_2(t) = y(\sin t)^r y^r - t(\sin t)^r t = \int_t^y \frac{d}{ds}(\sin t)^r s \hspace{1cm}
\]
\[
= \int_t^y (\sin t)^r (1 + rs \ln(\sin t)) \, ds > 0.
\]

Thus, (11) is a consequence of the decreasing behavior of \( f_2 \) and the fact that \( f_2(y) = 0 \).

2.4. Proof of Theorem 4

We set \( 0 < x \leq y < \pi /2 \) with \( x \leq 1 \) and \( r < 0 \) arbitrarily selected. Then, by the fact that \( \cos x \geq \cos y > 0 \), we deduce the following estimate:
\[
x^r \cos x - x^r \cos y = x^r \cos y \left( x^r (\cos x - \cos y) - 1 \right) \geq y^r \cos y \left( y^r (\cos x - \cos y) - 1 \right) = y^r \cos x - y^r \cos y,
\]
which implies (12). Similarly, using the fact that \( \sin y \geq \sin x > 0 \) implies that
\[
x^r \sin y - x^r \sin x = x^r \sin x \left( x^r (\sin y - \sin x) - 1 \right) \geq y^r \sin x \left( y^r (\sin y - \sin x) - 1 \right) = y^r \sin y - y^r \sin x,
\]
and we get the proof of (13).

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References


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