mathematics

MDPI

Article

Directed Strongly Regular Cayley Graphs over Metacyclic Groups of Order $4n$

Tao Cheng [1,†], Lihua Feng [2,†,*] and Weijun Liu [2,†]

1 School of Mathematics and Statistics, Shandong Normal University, Jinan 250014, China; taocheng@sdnu.edu.cn
2 School of Mathematics and Statistics, Central South University New Campus, Changsha 410083, China; wjliu@csu.edu.cn
* Correspondence: fenglh@csu.edu.cn
† These authors contributed equally to this work.

Received: 15 September 2019; Accepted: 22 October 2019; Published: 24 October 2019

check for updates

Abstract: We construct several new families of directed strongly regular Cayley graphs (DSRCGs) over the metacyclic group $M_{4n} = \langle a, b \mid a^n = b^4 = 1, b^{-1}ab = a^{-1} \rangle$, some of which generalize those earlier constructions. For a prime p and a positive integer $\alpha > 1$, for some cases, we characterize the DSRCGs over M_{4p^α}.

Keywords: directed strongly regular graph; Cayley graph; metacyclic groups

1. Introduction

The directed strongly regular graph [1] is one generalization of the undirected strongly regular graphs (SRG), which is an interesting topic in algebraic graph theory.

A *directed strongly regular graph* (DSRG) with parameters (n, k, μ, λ, t) is a k-regular directed graph on n vertices such that every vertex is on t 2-cycles (which may be considered as undirected edges), and the number of paths of length two from a vertex u to a vertex v is λ if there is an arc from u to v, and it is μ if there is no arc from u to v. There is also another definition of a DSRG regarding the adjacency matrix. For a directed graph X of order n, its *adjacency matrix* is $A = A(X) = (a_{ij})_{n\times n}$, where $a_{ij} = 1$ if v_i and v_j are adjacent, and $a_{ij} = 0$ otherwise. We use $I = I_n$ to denote the $n \times n$ identity matrix; $J = J_n$ the all-ones matrix. Then X is a DSRG with parameters (n, k, μ, λ, t) if and only if $JA = AJ = kJ$ and $A^2 = tI + \lambda A + \mu(J - I - A)$. When $t = k$, the DSRG is just the undirected SRG. When $t = 0$, the DSRG is the doubly regular tournaments [1]. Therefore we assumed that $0 < t < k$ in the rest of the paper.

For the SRGs and DSRGs, they share many analog properties. In particular, their eigenvalues are extremely similar. However, for a DSRG, its adjacency matrix is non-symmetric, this leads to more difficulties and makes it an interesting subject. Observe that the DSRGs have several parameters, there has been many constructions oriented to obtain several infinite families of DSRGs, also, some sporadic examples are known in the literature. Although many scholars have studied the existence and constructions of DSRGs for different parameters (one may refer to [2–5]), there are also plenty of DSRGs whose existence cannot be determined. As such, the complete characterization of DSRGs is far from being solved.

By using character theory of finite groups, He and Zhang [6] generalized the semidirect product method in [2] and obtained a large family of directed strongly regular Cayley graph (DSRCG). Technically, they constructed some DSRCGs over dihedral groups, which partially generalize the earlier results in [5]. These results reveals that representation theory is a powerful tool in this subject.

For more results regarding the interplay between algebraic graph theory and representation theory, one may refer to [7–9] and the references therein.

The purpose of this paper is to construct several new infinite families of DSRGs by making use of the representation theory. Borrowing ideas from [6], we consider the DSRCGs over the metacyclic group $M_{4n} = \langle a, b \mid a^n = b^4 = 1, b^{-1}ab = a^{-1} \rangle$ of order $4n$ [10]. Let $C_n = \langle a \rangle$ be a cyclic multiplicative group of order n. The metacyclic group M_{4n} can be viewed as the semidirect product of $C_n = \langle a \rangle$ of order n and $C_4 = \langle b \rangle$ of order 4. As mentioned in [7], if n is odd, the metacyclic group M_{4n} is the dicyclic group T_{4n}. Therefore it would be interesting to consider this group for various applications.

This paper is organized as follows. At first, we give some sufficient and necessary conditions for the Cayley graph $\mathcal{C}(T_{4n}, X \cup Yb \cup Xb^2 \cup Yb^3)$ with $X = Y$ and $X \subseteq Y$ to be directed strongly regular, and we construct several new classes of DSRCGs over metacyclic groups. Then, for prime p, we characterize the DSRCGs $\mathcal{C}(T_{4p^\alpha}, a^X \cup a^Y b \cup a^X b^2 \cup a^Y b^3)$ when $X = Y$ or $X \subseteq Y$.

2. Preliminaries

In this section, we present the fundamental concepts, we also present several lemmas which will be used later. In the sequel, $F = \mathbb{Q}[\omega]$ is the n-th cyclotomic field over the rationals, where ω is the primitive n-th root of unity.

For a multiset M, we define the multiplicity function $\Delta_M : M \to \mathbb{N}$, where $\Delta_M(x)$ is the number of times the element x appears in M. For two multisets M and N, the sum of M and N is denoted by $M \uplus N$, then $\Delta_{M \uplus N} = \Delta_M + \Delta_N$. For a positive integer n, the scalar multiplication of M by n is defined as $n \uplus M$, then we have $\Delta_{n \uplus M} = n\Delta_M$. The difference of M and N is defined as $M \backslash N$, then we have $\Delta_{M \backslash N}(x) = \max\{\Delta_M(x) - \Delta_N(x), 0\}$ for any $x \in M$. For instance, if $M = \{2, 3, 3, 4, 4\}$ and $N = \{1, 1, 2, 2, 3\}$, then we have $M \uplus N = \{1, 1, 2, 2, 2, 3, 3, 4, 4\}$, $2 \uplus M = \{2, 2, 3, 3, 3, 3, 4, 4, 4, 4\}$, and $M \backslash N = \{3, 4, 4\}$.

For a finite group G with the identity element e (we sometimes use 1 if there is no confusion), and a non-empty subset S of G, we denote by $S^{(-1)}$ the set $\{s^{-1} \mid s \in S\}$. Assume now that $e \notin S$, then the graph $\Gamma = \mathcal{C}(G, S)$ is called the *directed Cayley graph* over G with respect to S, if $V(\Gamma) = G$ and $x \to y$ (means there is an arc from x to y) if and only if $yx^{-1} \in S$ for any $x, y \in G$.

Let G be a group and $\mathbb{C}$ be the complex field. We denote the *group algebra* of G over $\mathbb{C}$ by $\mathbb{C}G$, and we denote the element of $\mathbb{C}G$ by $\overline{X}$ for any multisubset X of G. Thus we can write $\overline{X}$ as

$$\overline{X} = \sum_{x \in X} \Delta_X(x)x.$$

By using the group algebra, we have

Lemma 1 ([1]). *A Cayley graph $\mathcal{C}(G, S)$ of group G with the multiset $S \subseteq G$ is a DSRG with parameters (n, k, μ, λ, t) if and only if $|G| = n$, $|S| = k$ and*

$$\overline{S}^2 = te + \lambda\overline{S} + \mu(\overline{G} - e - \overline{S}).$$

The following relations will be frequently used in the context.

Lemma 2. *For the metacyclic group $M_{4n} = \langle a, b \mid a^n = b^4 = 1, b^{-1}ab = a^{-1} \rangle$, we have*

(i) $a^k b = ba^{-k}$; $a^k b^2 = b^2 a^k$; $a^k b^3 = b^3 a^{-k}$;
(ii) $(a^k b)^{-1} = a^k b^3$; $(a^k b^2)^{-1} = a^{-k} b^2$.

Proof. By relations $a^n = b^4 = 1$ and $b^{-1}ab = a^{-1}$, the results follow immediately. □

2.1. Fourier Transformation on $\mathbb{Z}_n$

The following statement and notations are coincided with [6,11]. Let M, N be the multisubsets of $\mathbb{Z}_n$ and $k \in \mathbb{Z}_n$. We denote $kM = \{km \mid m \in M\}$, $k + M = \{k + m \mid m \in M\}$, and $k - M = \{k - m \mid m \in M\}$. The sum of multisubsets M and N is $M + N = \{m + n \mid m \in M, n \in N\}$, and the multiplicity function of $M + N$ is

$$\Delta_{M+N}(c) = \sum_{m+n=c} \Delta_M(m)\,\Delta_N(n),$$

for any $c \in \mathbb{Z}_n$. And let

$$x^M = \biguplus_{m \in M} \Delta_M(m) \oplus \{x^m\}.$$

Then $C_n = x^{\mathbb{Z}_n}$ and x^M is a multisubset of C_n.

Let $\mathbb{Z}_n^*$ be the multiplicative group of the units in the ring $\mathbb{Z}_n$. Then $\mathbb{Z}_n^*$ has an action on $\mathbb{Z}_n$ by multiplication, and hence $\mathbb{Z}_n$ is the union of some $\mathbb{Z}_n^*$-orbits. Each $\mathbb{Z}_n^*$-orbit consists of all elements of a given order in the additive group $\mathbb{Z}_n$. We denote the $\mathbb{Z}_n^*$-orbit containing all elements of order r by $\mathcal{O}_r$, where r is a positive divisor of n. Thus

$$\mathcal{O}_r = \left\{ z \mid z \in \mathbb{Z}_n, \frac{n}{(n,z)} = r \right\} = \left\{ c\frac{n}{r} \mid 1 \leqslant c \leqslant r, c \in \mathbb{Z}_n^* \right\},$$

and $|\mathcal{O}_r| = \varphi(r)$.

We denote all functions $f : \mathbb{Z}_n \to \mathbb{F}$ mapping from $\mathbb{Z}_n$ to the field $\mathbb{F}$ by $\mathbb{F}^{\mathbb{Z}_n}$. By defining the multiplication point-wise, the $\mathbb{F}$-algebra obtained from $\mathbb{F}^{\mathbb{Z}_n}$ will be denoted by $\left(\mathbb{F}^{\mathbb{Z}_n}, \cdot\right)$. And the $\mathbb{F}$-algebra obtained from $\mathbb{F}^{\mathbb{Z}_n}$ by defining the multiplication as the *convolution* will be denoted by $\left(\mathbb{F}^{\mathbb{Z}_n}, *\right)$, where the convolution is defined by:

$$(f * g)(z) = \sum_{i \in \mathbb{Z}_n} f(i)g(z - i), \quad f, g \in \mathbb{F}^{\mathbb{Z}_n}. \tag{1}$$

The *Fourier transformation*, as an isomorphsim between the $\mathbb{F}$-algebra $\left(\mathbb{F}^{\mathbb{Z}_n}, \cdot\right)$ and $\left(\mathbb{F}^{\mathbb{Z}_n}, *\right)$, is defined as

$$\mathcal{F} : \left(\mathbb{F}^{\mathbb{Z}_n}, *\right) \to \left(\mathbb{F}^{\mathbb{Z}_n}, \cdot\right), \quad (\mathcal{F}f)(z) = \sum_{i \in \mathbb{Z}_n} f(i)\omega^{iz}.$$

For any multisubsets M and N of $\mathbb{Z}_n$, we have

$$\mathcal{F}\Delta_{(-M)} = \overline{\mathcal{F}\Delta_M}, \quad \mathcal{F}\Delta_{M+N} = \mathcal{F}\left(\Delta_M * \Delta_N\right) = \left(\mathcal{F}\Delta_M\right)\left(\mathcal{F}\Delta_N\right). \tag{2}$$

Then, for $r\mathbb{Z}_n = \{0, r, 2r, \ldots, n - r\}$, where r is a positive divisor of n, we have

$$\mathcal{F}\Delta_{rZ_n} = \frac{n}{r}\Delta_{\frac{n}{r}}\mathbb{Z}_n, \quad \mathcal{F}\Delta_{\mathbb{Z}_n} = n\Delta_0, \quad \mathcal{F}\Delta_0 = \Delta_{\mathbb{Z}_n} = 1.$$

The following lemmas will be used in the sequel.

Lemma 3 ([11]). *Let $f : \mathbb{Z}_n \to \mathbb{F}$ be a function and* $\mathrm{Im}(f) \subseteq \mathbb{Q}$. *Then we have* $\mathrm{Im}(\mathcal{F}f) \subseteq \mathbb{Q}$ *if and only if* $f = \sum_{r|n} \alpha_r \Delta_{\mathcal{O}_r}$ *for some* $\alpha_r \in \mathbb{Q}$.

From the Ramanujan's sums, we have

$$\left(\mathcal{F}\Delta_{\mathcal{O}_r}\right)(z) = \mu\left(\frac{r}{(r,z)}\right)\frac{\varphi(r)}{\varphi\left(\frac{r}{(r,z)}\right)} \in \mathbb{Z}. \tag{3}$$

Let v be a positive divisor of n. Then we can define a homomorphism

$$\psi_v : \mathbb{Z}_n \to \mathbb{Z}_v, i \mapsto i \ (\bmod v) + v\mathbb{Z}_n \tag{4}$$

with $Ker\psi_v = v\mathbb{Z}_n$. Then it follows that

Lemma 4 ([6]). *If $F(z)$ is a complex variables function, then we have*

$$F\left(\left(\mathcal{F}\Delta^{(v)}_{\psi_v(H)}\right)(z)\right) = 0, \forall z \in \mathbb{Z}_v \Leftrightarrow F\left(\left(\mathcal{F}\Delta_H\right)(z)\right) = 0, \forall z \in \frac{n}{v}\mathbb{Z}_n.$$

2.2. Some Lemmas

Throughout this section, we always assume that p is a prime and $\alpha \geqslant 1$ is an integer.

Let $\nu_p(z)$ be the maximum power of the prime p that divides z. Note that the set of divisors of p^α is $\{1, p, p^2, \cdots, p^{\alpha-1}, p^\alpha\}$, so all the $\mathbb{Z}^*_{p^\alpha}$-orbits are $\mathcal{O}_1 = \mathcal{O}_{p^0}, \mathcal{O}_p, \mathcal{O}_{p^2}, \cdots, \mathcal{O}_{p^\alpha}$, where

$$\mathcal{O}_{p^z} = \left\{cp^{\alpha-z} \mid 1 \leqslant c \leqslant p^z, (p, c) = 1\right\} = \left\{i \mid i \in \mathbb{Z}_{p^\alpha}, \nu_p(i) = \alpha - z\right\}.$$

We denote $\mathcal{O}_{p^i}$ by O_i for simplicity hereafter. In particular, $\mathcal{O}_1 = \left\{i \mid i \in \mathbb{Z}_{p^\alpha}, \nu_p(i) = \alpha\right\} = \{0\}$.

Note that $p^{\alpha-\beta}\mathbb{Z}_{p^\alpha}$ is a subgroup of $\mathbb{Z}_{p^\alpha}$ and

$$p^{\alpha-\beta}\mathbb{Z}_{p^\alpha} = \bigcup_{i=0}^{\beta} \mathcal{O}_{p^i} = \bigcup_{i=0}^{\beta} O_i,$$

for each $0 \leqslant \beta \leqslant \alpha$.

In this section, we assume that X is a subset of $\mathbb{Z}_{p^\alpha}$ such that $0 \notin X$ and

$$\left(\mathcal{F}\Delta_{X \uplus (-X)}\right)(z) = \left(\mathcal{F}\Delta_X\right)(z) + \overline{\left(\mathcal{F}\Delta_X\right)(z)} \in \{0, -m\}, \tag{5}$$

for any $0 \neq z \in \mathbb{Z}_{p^\alpha}$, where m is a positive integer. We also assume $X \neq -X$. Then we have

Lemma 5 ([6]). *Let X be a subset of $\mathbb{Z}_{p^\alpha}$ such that*

$$\left(\mathcal{F}\Delta_{X \uplus (-X)}\right)(z) = \left(\mathcal{F}\Delta_X\right)(z) + \overline{\left(\mathcal{F}\Delta_X\right)(z)} \in \{0, -m\},$$

where m is a positive integer. Then there exists some integers $1 \leqslant r_1 < r_2 < \cdots < r_s \leqslant \alpha$ and $1 \leqslant r_{s+1} < r_{s+2} < \cdots < r_t \leqslant \alpha$ satisfy

$$X \uplus (-X) = (2 \oplus O_{r_1}) \uplus (2 \oplus O_{r_2}) \uplus \cdots \uplus (2 \oplus O_{r_s}) \uplus \left(O_{r_{s+1}} \cup O_{r_{s+2}} \cup \cdots \cup O_{r_t}\right).$$

Let $\mathcal{I}_1 = \{r_1, r_2, \cdots, r_s\}$ and $\mathcal{I}_2 = \{r_{s+1}, r_{s+2}, \cdots, r_t\}$. Then $\mathcal{I}_2 \neq \emptyset$ as $X \neq -X$. Thus $|\mathcal{I}_1| = s$ and $|\mathcal{I}_2| = t - s \geqslant 1$.

Lemma 6 ([6]). *Let $\mathcal{I}_1$ and $\mathcal{I}_2$ be the sets defined above. Then $\mathcal{I}_1$ and $\mathcal{I}_2$ form a partition of $\{\beta + 1, \beta + 2, \cdots, \alpha\}$ for some integer $0 \leq \beta \leq \alpha - 1$. Hence*

$$X \uplus (-X) = (O_{r_1} \cup O_{r_2} \cup \cdots \cup O_{r_s}) \uplus \left(\mathbb{Z}_{p^\alpha} \backslash p^{\alpha-\beta}\mathbb{Z}_{p^\alpha}\right).$$

Lemma 7 ([6]). *Let X be a subset of $\mathbb{Z}_{p^\alpha} \setminus \{0\}$ satisfying the condition (5), and $\mathcal{I}_1$, $\mathcal{I}_2$ be the sets defined above. If p is an odd prime, then $\mathcal{I}_1 = \emptyset$ and*

$$X \uplus (-X) = \mathbb{Z}_{p^\alpha} \backslash p^{\alpha-\beta}\mathbb{Z}_{p^\alpha}.$$

Lemma 8 ([6]). *Let X be a subset of $\mathbb{Z}_{2^\alpha} \setminus \{0\}$ satisfies the condition (5), and $\mathcal{I}_1, \mathcal{I}_2$ be the sets defined above. If $\mathcal{I}_1 \neq \emptyset$, then $\mathcal{I}_1 = \{\beta+1\}, \mathcal{I}_2 = \{\beta+1, \beta+2, \cdots, \alpha\}$ and*

$$X \uplus (-X) = O_{\beta+1} \uplus (\mathbb{Z}_{2^\alpha} \backslash 2^{\alpha-\beta}\mathbb{Z}_{p^\alpha}).$$

Lemma 9 ([6]). *Let X be a subset of $\mathbb{Z}_{p^\alpha}$ and $0 < \gamma \leqslant \alpha$ be a positive integer. If X satisfies $(\mathcal{F}\Delta_X)(z) = 0$ for all $z \notin p^\gamma \mathbb{Z}_{p^\alpha}$, then $X = T' + p^{\alpha-\gamma}\mathbb{Z}_{p^\alpha}$ for some subset T' of $\{0, 1, \cdots, p^{\alpha-\gamma} - 1\}$.*

3. The DSRCGs over M_{4n}

In this section, we will provide several constructions of DSRCGs over M_{4n}.

Let $\Lambda_1 = \overline{a^X} + \overline{a^{(-X)}} = \overline{a^{X \uplus (-X)}}$ and $\Lambda_2 = \overline{a^Y}\, \overline{a^{(-Y)}} - \overline{a^X}\, \overline{a^{(-X)}}$.

We now give a criterion for the Cayley graph $\mathcal{C}(M_{4n}, a^X \cup a^Y b \cup a^X b^2 \cup a^Y b^3)$ to be directed strongly regular.

Lemma 10. *The Cayley graph $\mathcal{C}(M_{4n}, a^X \cup a^Y b \cup a^X b^2 \cup a^Y b^3)$ is a DSRCG with parameters $(4n, 2(|X| + |Y|), \mu, \lambda, t)$ if and only if*

(i) $t = \mu$;
(ii) $2\overline{a^Y}\Lambda_1 = (\lambda - \mu)\overline{a^Y} + \mu\overline{C_n}$;
(iii) $2(\overline{a^X}\Lambda_1 + \Lambda_2) = \overline{a^X}^2 + \overline{a^Y}\, \overline{a^{(-Y)}} = (\lambda - \mu)\overline{a^X} + \mu\overline{C_n}$.

Proof. By Lemma 2, we have

$$\begin{aligned}
& \left(\overline{a^X} + \overline{a^Y b} + \overline{a^X b^2} + \overline{a^Y b^3}\right)^2 \\
= \; & \overline{a^X}^2 + \overline{a^X}\,\overline{a^Y} b + \overline{a^X}^2 b^2 + \overline{a^X}\,\overline{a^Y} b^3 + \overline{a^Y}\,\overline{a^{(-X)}} b + \overline{a^Y}\,\overline{a^{(-Y)}} b^2 \\
& + \overline{a^Y}\,\overline{a^{(-X)}} b^3 + \overline{a^Y}\,\overline{a^{(-Y)}} + \overline{a^X}^2 b^2 + \overline{a^X}\,\overline{a^Y} b^3 + \overline{a^X}^2 + \overline{a^X}\,\overline{a^Y} b + \\
& \overline{a^Y}\,\overline{a^{(-X)}} b^3 + \overline{a^Y}\,\overline{a^{(-Y)}} + \overline{a^Y}\,\overline{a^{(-X)}} b + \overline{a^Y}\,\overline{a^{(-Y)}} b^2 \\
= \; & 2\left(\overline{a^X}^2 + \overline{a^Y}\,\overline{a^{(-Y)}}\right) + 2\left(\overline{a^X}\,\overline{a^Y} + \overline{a^Y}\,\overline{a^{(-X)}}\right) b \\
& + 2\left(\overline{a^X}^2 + \overline{a^Y}\,\overline{a^{(-Y)}}\right) b^2 + 2\left(\overline{a^X}\,\overline{a^Y} + \overline{a^Y}\,\overline{a^{(-X)}}\right) b^3 \\
= \; & 2(\overline{a^X}\Lambda_1 + \Lambda_2) + 2\overline{a^Y}\Lambda_1 b + 2(\overline{a^X}\Lambda_1 + \Lambda_2) b^2 + 2\overline{a^Y}\Lambda_1 b^3.
\end{aligned}$$

Thus, from Lemma 1, the Cayley graph $\mathcal{C}(M_{4n}, a^X \cup a^Y b \cup a^X b^2 \cup a^Y b^3)$ is a DSRCG with parameters $(4n, 2(|X| + |Y|), \mu, \lambda, t)$ if and only if

$$\begin{aligned}
& \left(\overline{a^X} + \overline{a^Y b} + \overline{a^X b^2} + \overline{a^Y b^3}\right)^2 \\
= \; & te + \lambda\left(\overline{a^X} + \overline{a^Y b} + \overline{a^X b^2} + \overline{a^Y b^3}\right) + \mu\left(\overline{C_n} + \overline{C_n b} + \overline{C_n b^2} + \overline{C_n b^3}\right) \\
& - \mu e - \mu\left(\overline{a^X} + \overline{a^Y b} + \overline{a^X b^2} + \overline{a^Y b^3}\right) \\
= \; & \left((t - \mu)e + (\lambda - \mu)\overline{a^X} + \mu\overline{C_n}\right) + \left((\lambda - \mu)\overline{a^Y} + \mu\overline{C_n}\right) b \\
& + \left((\lambda - \mu)\overline{a^X} + \mu\overline{C_n}\right) b^2 + \left((\lambda - \mu)\overline{a^Y} + \mu\overline{C_n}\right) b^3.
\end{aligned}$$

Comparing the above two equations, we complete the proof. □

Setting $X = Y$ in Lemma 10, we have

Lemma 11. *The Cayley graph $\mathcal{C}(M_{4n}, a^X \cup a^X b \cup a^X b^2 \cup a^X b^3)$ is a DSRCG with parameters $(4n, 4|X|, \mu, \lambda, t)$ if and only if $t = \mu$ and*

$$2\overline{a^X}\Lambda_1 = (\lambda - \mu)\overline{a^X} + \mu\overline{C_n}.$$

We now define

$$\mathbf{r}(z) = (\mathcal{F}\Delta_X)(z) = \sum_{i \in X} \omega^{iz} \text{ and } \mathbf{t}(z) = (\mathcal{F}\Delta_Y)(z) = \sum_{i \in Y} \omega^{iz}.$$

Then $\mathbf{r}(z) + \bar{\mathbf{r}}(z) = \left(\mathcal{F}\Delta_{X \uplus (-X)}\right)(z)$. The following lemma gives a characterization of the Cayley graph $\mathcal{C}(M_{4n}, a^X \cup a^Y b \cup a^X b^2 \cup a^Y b^3)$ to be directed strongly regular by using $\mathbf{r}(z)$ and $\mathbf{t}(z)$.

Lemma 12. *The Cayley graph $\mathcal{C}(M_{4n}, a^X \cup a^Y b \cup a^X b^2 \cup a^Y b^3)$ is a DSRCG with parameters $(4n, 2(|X| + |Y|), \mu, \lambda, t)$ if and only if*

(i) $t = \mu$;
(ii) $\mathbf{t}(\mathbf{r} + \bar{\mathbf{r}}) = \frac{\mu n}{2}\Delta_0 + \frac{\lambda - \mu}{2}\mathbf{t}$;
(iii) $\mathbf{r}^2 + |\mathbf{t}|^2 = \frac{\mu n}{2}\Delta_0 + \frac{\lambda - \mu}{2}\mathbf{r}$.

Proof. By Equation (1), we have

$$\begin{aligned} 2\overline{a^Y}\Lambda_1 &= 2\overline{a^Y}\overline{a^{X \uplus (-X)}} \\ &= 2\sum_{j \in \mathbb{Z}_n} \Delta_Y(j)x^j \cdot \sum_{k \in \mathbb{Z}_n} \Delta_{X \uplus (-X)}(k)x^k \\ &= 2\sum_{j,k \in \mathbb{Z}_n} \Delta_Y(j)x^j \cdot \Delta_{X \uplus (-X)}(k)x^{j+k} \\ &= 2\sum_{j \in \mathbb{Z}_n} \Delta_Y(j)x^j \cdot \Delta_{X \uplus (-X)}(i - j)x^i \\ &= 2(\Delta_Y * \Delta_{X \uplus (-X)})(i)x^i. \end{aligned}$$

and

$$\begin{aligned} (\lambda - \mu)\overline{a^Y} + \mu\overline{C_n} &= \sum_{i \in \mathbb{Z}_n} (\lambda - \mu)\Delta_Y(i)x^i + \sum_{i \in \mathbb{Z}_n} \mu\Delta_{\mathbb{Z}_n}(i)x^i \\ &= \sum_{i \in \mathbb{Z}_n} ((\lambda - \mu)\Delta_Y(i) + \mu\Delta_{\mathbb{Z}_n}(i))x^i. \end{aligned}$$

From the two equations above and (ii) of Lemma 10, we have

$$\left(\Delta_Y * \Delta_{X \uplus (-X)}\right)(i) = \frac{\lambda - \mu}{2}\Delta_Y(i) + \frac{\mu}{2}\Delta_{\mathbb{Z}_n}(i),$$

for $i \in \mathbb{Z}_n$. By Equation (2), we have

$$\mathcal{F}\left(\Delta_Y * \Delta_{X \uplus (-X)}\right)(i) = \mathcal{F}(\Delta_Y)\mathcal{F}(\Delta_{X \uplus (-X)})(i) = \mathbf{t}(\mathbf{r} + \bar{\mathbf{r}})(i),$$

and

$$\mathcal{F}(\frac{\lambda - \mu}{2}\Delta_Y)(i) + \mathcal{F}(\frac{\mu}{2}\Delta_{\mathbb{Z}_n})(i) = \frac{\lambda - \mu}{2}\mathcal{F}\Delta_Y(i) + \frac{\mu}{2}\mathcal{F}\Delta_{\mathbb{Z}_n}(i) = (\frac{\lambda - \mu}{2}\mathbf{t} + \frac{\mu}{2}n\Delta_0)(i).$$

Then by the two equations above, we have

$$\mathbf{t}(\mathbf{r}+\bar{\mathbf{r}}) = \frac{\mu n}{2}\Delta_0 + \frac{\lambda-\mu}{2}\mathbf{t}.$$

Using the same method, by (iii) of Lemma 10, we have

$$\left(\Delta_X * \Delta_X\right)^2(i) + \left(\Delta_Y * \Delta_{-Y}\right)(i) = \frac{\lambda-\mu}{2}\Delta_X(i) + \frac{\mu}{2}\Delta_{\mathbb{Z}_n}(i),$$

for $i \in \mathbb{Z}_n$. Moveover, by applying the Fourier transformation, we have

$$\mathbf{r}^2 + |\mathbf{t}|^2 = \frac{\mu n}{2}\Delta_0 + \frac{\lambda-\mu}{2}\mathbf{r}.$$

□

When $X = Y$, we have the following lemma.

Lemma 13. *The Cayley graph $\mathcal{C}(M_{4n}, a^X \cup a^X b \cup a^X b^2 \cup a^X b^3)$ is a DSRCG with parameters $(4n, 4|X|, \mu, \lambda, t)$ if and only if $t = \mu$ and*

$$\mathbf{r}(\mathbf{r}+\bar{\mathbf{r}}) = \frac{\mu n}{2}\Delta_0 + \frac{\lambda-\mu}{2}\mathbf{r}. \tag{6}$$

Let $\mathbf{q} \stackrel{\text{def}}{=} \mathbf{r}+\bar{\mathbf{r}} = \mathcal{F}\Delta_{X\uplus(-X)}$. Then we have

Lemma 14. *The Cayley graph $\mathcal{C}(M_{4p^\alpha}, a^X \cup a^X b \cup a^X b^2 \cup a^X b^3)$ is a DSRCG with parameters $(4p^\alpha, 4|X|, \mu, \lambda, t)$ then*

(1) The function $\mathbf{q}$ satisfies

$$\mathbf{q}(z) = (\mathbf{r}+\bar{\mathbf{r}})(z) = \left(\mathcal{F}\Delta_{X\uplus(-X)}\right)(z) \in \{0, \frac{\lambda-\mu}{2}\},$$

for any $0 \neq z \in \mathbb{Z}_{p^\alpha}$.

(2) There are some integers $r_1, r_2, \cdots, r_s$ with $\beta+1 \leqslant r_1 < r_2 < \cdots < r_s \leqslant \alpha$ satisfy

$$X \uplus (-X) = (O_{r_1} \cup O_{r_2} \cup \cdots \cup O_{r_s}) \uplus \left(\mathbb{Z}_{p^\alpha} \backslash p^{\alpha-\beta}\mathbb{Z}_{p^\alpha}\right),$$

where $0 \leqslant \beta \leqslant \alpha - 1$.

(3) If p is an odd prime, then

$$X \uplus (-X) = \mathbb{Z}_{p^\alpha} \backslash p^{\alpha-\beta}\mathbb{Z}_{p^\alpha}.$$

(4) If $p = 2$ and $X \cap (-X) \neq \emptyset$, then

$$X \uplus (-X) = O_{\beta+1} \uplus (\mathbb{Z}_{2^\alpha} \backslash 2^{\alpha-\beta}\mathbb{Z}_{p^\alpha}).$$

Proof. Taking conjugate on Equation (6), we have

$$\bar{\mathbf{r}}(\mathbf{r}+\bar{\mathbf{r}}) = \frac{\mu p^\alpha}{2}\Delta_0 + \frac{\lambda-\mu}{2}\bar{\mathbf{r}}. \tag{7}$$

Then the sum of Equations (6) and (7) leads to

$$(\mathbf{r}+\bar{\mathbf{r}})^2 = \mu p^\alpha \Delta_0 + \frac{\lambda-\mu}{2}(\mathbf{r}+\bar{\mathbf{r}}).$$

Thus we have

$$(\mathbf{r}+\bar{\mathbf{r}})(z)=\begin{cases} 2|X|, & z=0; \\ 0 \text{ or } \frac{\lambda-\mu}{2}, & z\neq 0. \end{cases}$$

By Lemmas 6, 7 and 8, we can prove Equations (2), (3) and (4) respectively. □

Next we will present several classes of DSRCGs when $X=Y$ or $X\subset Y$ from the above results. In the remainder of this section, v is always assumed to be a positive divisor of n and $l=\frac{n}{v}$.

Theorem 1. *Let T be a subset of $\{1,\cdots,v-1\}\subseteq\mathbb{Z}_n$, where v is an odd positive divisor of n, and X be a subset of $\mathbb{Z}_n$ satisfy the following conditions:*

(i) $X=T+v\mathbb{Z}_n$;
(ii) $X\cup(-X)=\mathbb{Z}_n\backslash v\mathbb{Z}_n$.

Then the Cayley graph $\mathcal{C}(T_{4n},a^X\cup a^Xb\cup a^Xb^2\cup a^Xb^3)$ is a DSRCG with parameters $(4n,2(n-l),n-l,n-3l,n-l)$.

Proof. From (ii), note that $|X|=|-X|=\frac{n-l}{2}$. Thus, we have $2\overline{a^X}\Lambda_1=2\overline{a^X}\,\overline{a^{X\uplus(-X)}}=2\overline{a^X}\,\overline{a^{\mathbb{Z}_n\backslash v\mathbb{Z}_n}}=2\overline{a^X}(\overline{a^{\mathbb{Z}_n}}-\overline{a^{v\mathbb{Z}_n}})=2\overline{a^X}(\overline{C_n}-\overline{a^{v\mathbb{Z}_n}})=-2l\overline{a^X}+(n-l)\overline{C_n}$. Therefore, by Lemma 11, we get the desired result. □

Example 1. *For $n=6$, we have $T_{24}=\{a,b\mid a^6=b^4=1,b^{-1}ab=a^{-1}\}$. Let $T=\{1\}$ and $v=3$. Then we have $X=T+3\mathbb{Z}_6=\{1,4\}$ and $X\cup(-X)=\{1,4,2,5\}=\mathbb{Z}_6\setminus 3\mathbb{Z}_6$, where $3\mathbb{Z}_6=\{0,3\}$. Thus X satisfies the conditions (i) and (ii) of the Theorem 1. So we have the Cayley graph $\mathcal{C}(T_{24},a^X\cup a^Xb\cup a^Xb^2\cup a^Xb^3)$ is a DSRCG with parameters $(24,8,4,0,4)$, where $X=\{1,4\}$.*

Theorem 2. *Let T be a subset of $\{1,\cdots,v-1\}\subseteq\mathbb{Z}_n$, where $v>2$ is an even positive divisor of n. The subset $X\subseteq\mathbb{Z}_n$ satisfies the following conditions:*

(i) $X=T+v\mathbb{Z}_n$;
(ii) $X\cup(-X)=(\mathbb{Z}_n\setminus v\mathbb{Z}_n)\uplus\left(\frac{v}{2}+v\mathbb{Z}_n\right)$;
(iii) $X\cup\left(\frac{v}{2}+X\right)=\mathbb{Z}_n$.

Then the Cayley graph $\Gamma=\mathcal{C}(M_{4n},a^X\cup a^Xb\cup a^Xb^2\cup a^Xb^3)$ is a DSRCG with parameters $(4n,2n,n+2l,n-2l,n+2l)$.

Proof. By (ii) and (iii), we have $|X|=\frac{n}{2}$ and $\Lambda_1=\overline{C_n}-\overline{x^{v\mathbb{Z}_n}}+\overline{x^{\frac{v}{2}+v\mathbb{Z}_n}}$. Therefore, $2\overline{a^X}\Delta_1=-2l\overline{a^X}+n\overline{C_n}+2\overline{a^X}\,\overline{a^{\frac{v}{2}+v\mathbb{Z}_n}}=-2l\overline{a^X}+n\overline{C_n}+2l\overline{a^{\frac{v}{2}+X}}=-2l\overline{a^X}+n\overline{C_n}+2l\overline{C_n}-2l\overline{a^X}=(n+2l)\,\overline{C_n}-4l\overline{a^X}$. The result follows from Lemma 11 directly. □

Example 2. *For $n=12$, we have $T_{48}=\{a,b\mid a^{12}=b^4=1,b^{-1}ab=a^{-1}\}$. Let $T=\{1,2\}$ and $v=4$. Then we have $X=T+4\mathbb{Z}_{12}=\{1,2,5,6,9,10\}$. Thus $X\cup(-X)=\{1,2,5,6,9,10,2,3,6,7,10,11\}=(\mathbb{Z}_{12}\setminus 4\mathbb{Z}_{12})\uplus(2+3\mathbb{Z}_{12})$, where $\mathbb{Z}_{12}\setminus 4\mathbb{Z}_{12}=\{1,2,3,5,6,7,9,10,11\}$, $4\mathbb{Z}_{12}=\{2,6,10\}$, and $X\cup(2+X)=\mathbb{Z}_{12}$. The set X satisfies the conditions (i) (ii) and (iii) of Theorem 2. So we have the Cayley graph $\mathcal{C}(T_{48},a^X\cup a^Xb\cup a^Xb^2\cup a^Xb^3)$ is a DSRCG with parameters $(48,24,18,6,18)$, where $X=\{1,2,5,6,9,10\}$.*

Theorem 3. *Let T be a subset of $\{0,1,\cdots,v-1\}\subseteq\mathbb{Z}_n$, where v is an odd positive divisor of n, with $0\in T$. The two subsets $X,Y\subseteq\mathbb{Z}_n$ satisfy the following conditions:*

(i) $Y=T+v\mathbb{Z}_n=X\cup v\mathbb{Z}_n$;
(ii) $Y\cup(-Y)=\mathbb{Z}_n\uplus v\mathbb{Z}_n$.

Then the Cayley graph $\Gamma=\mathcal{C}(T_{4n},a^X\cup a^Yb\cup a^Xb^2\cup a^Yb^3)$ is a DSRCG with parameters $(4n,2n,n+l,n-l,n+l)$.

Proof. Notice that $|X| = |Y| - l = \frac{n-l}{2}$, $\Lambda_1 = \overline{C_n} - \overline{a^{v\mathbb{Z}_n}}$ and $\Lambda_2 = \overline{a^{v\mathbb{Z}_n}}\Lambda_1 + l\overline{a^{v\mathbb{Z}_n}} = l\overline{C_n}$. Therefore, $2\overline{a^Y}\Lambda_1 = -2l\overline{a^Y} + (n+l)\overline{C_n}$ and $2(\overline{a^X}\Lambda_1 + \Lambda_2) = -2l\overline{a^X} + (n+l)\overline{C_n}$. Thus the result follows from Lemma 10 directly. □

Example 3. *For $n = 6$, we have $T_{24} = \{a, b \mid a^6 = b^4 = 1, b^{-1}ab = a^{-1}\}$. Let $T = \{0, 1\}$ and $v = 3$. Then we have $Y = T + 3\mathbb{Z}_6 = \{0, 1, 3, 4\}$ and $X = \{1, 4\}$. Thus we have $Y \cup (-Y) = \{0, 1, 3, 4, 0, 5, 3, 2\} = \mathbb{Z}_6 \uplus 3\mathbb{Z}_6$, where $3\mathbb{Z}_6 = \{0, 3\}$. Thus the set X satisfies the conditions (i) and (ii) of Theorem 3. So we have the Cayley graph $\mathcal{C}(T_{24}, a^X \cup a^Y b \cup a^X b^2 \cup a^Y b^3))$ is a DSRCG with parameters $(24, 12, 8, 4, 8)$, where $X = \{1, 4\}$ and $Y = \{0, 1, 3, 4\}$.*

4. Characterization of DSRG $\mathcal{C}(T_{4p^\alpha}, a^X \cup a^X b \cup a^X b^2 \cup a^X b^3)$

Firstly, we characterize the DSRCGs $\mathcal{C}(T_{4p^\alpha}, a^X \cup a^X b \cup a^X b^2 \cup a^X b^3)$ with $p > 2$.

Theorem 4. *Let p be an odd prime and α be a positive integer. Then the Cayley graph $\mathcal{C}(T_{4n}, a^X \cup a^X b \cup a^X b^2 \cup a^X b^3)$ is a DSRCG if and only if there is one β with $0 \le \beta \le \alpha - 1$ and a subset $T \subseteq \{1, \cdots, p^{\alpha-\beta} - 1\}$ satisfying the following conditions:*

(i) $X = T + p^{\alpha-\beta}\mathbb{Z}_{p^\alpha}$;
(ii) $X \cup (-X) = \mathbb{Z}_{p^\alpha} \backslash p^{\alpha-\beta}\mathbb{Z}_{p^\alpha}$.

Proof. By Theorem 1, we have that $\mathcal{C}(T_{4p^\alpha}, a^X \cup a^X b \cup a^X b^2 \cup a^X b^3)$ satisfies conditions (i) and (ii) is a DSRCG.

Conversely, suppose that the Cayley graph $\mathcal{C}(T_{4n}, a^X \cup a^X b \cup a^X b^2 \cup a^X b^3)$ is a DSRCG with parameters $(4n, k, \mu, \lambda, t)$, where $k = 4|X|$. From (3) of Lemma 14, we have

$$X \uplus (-X) = \mathbb{Z}_{p^\alpha} \backslash p^{\alpha-\beta}\mathbb{Z}_{p^\alpha},$$

for some $0 \le \beta \le \alpha - 1$, proving (ii).

Therefore,

$$\mathbf{r}(z) + \bar{\mathbf{r}}(z) = p^\alpha \Delta_0(z) - p^\beta \Delta_{p^\beta \mathbb{Z}_{p^\alpha}}(z).$$

Thus Equation (6) becomes

$$p^\alpha \Delta_0(z)\mathbf{r}(z) - p^\beta \Delta_{p^\beta \mathbb{Z}_{p^\alpha}}(z)\mathbf{r}(z) = \frac{\mu}{2} n \Delta_0(z) + \frac{\lambda - \mu}{2}\mathbf{r}(z).$$

This implies that

$$\mathbf{r}(z) = 0, \forall z \notin p^\beta \mathbb{Z}_{p^\alpha}.$$

By Lemma 9, we have $X = T + p^{\alpha-\beta}\mathbb{Z}_{p^\alpha}$, where T is a subset of $\{1, \cdots, p^{\alpha-\beta} - 1\}$, proving (i). □

We now focus on the directed strongly regular Cayley graphs $\mathcal{C}(T_{2^{\alpha+2}}, a^X \cup a^X b \cup a^X b^2 \cup a^X b^3)$.

Lemma 15. *A DSRCG cannot be a Cayley graphs of the form $\mathcal{C}(T_{2^{\alpha+2}}, a^X \cup a^X b \cup a^X b^2 \cup a^X b^3)$ with $X \cap (-X) = \emptyset$.*

Proof. Suppose $X \cap (-X) = \emptyset$. By Lemma 14 (2), we have

$$X \uplus (-X) = \mathbb{Z}_{2^\alpha} \backslash 2^{\alpha-\beta}\mathbb{Z}_{2^\alpha}.$$

Similar to the proof of Theorem 4, there is β with $0 \le \beta \le \alpha - 1$ and a subset $T \subseteq \{1, \cdots, 2^{\alpha-\beta} - 1\}$ such that $X = T + 2^{\alpha-\beta}\mathbb{Z}_{2^\alpha}$ and $X \cup (-X) = \mathbb{Z}_{2^\alpha} \backslash 2^{\alpha-\beta}\mathbb{Z}_{2^\alpha}$. Thus we have that $2|X| = 2|H|2^\beta = 2^\alpha - 2^\beta$. Then $2|H| = 2^{\alpha-\beta} - 1$, this is impossible. □

Theorem 5. *The Cayley graphs $\mathcal{C}(T_{2^{\alpha+2}}, a^X \cup a^X b \cup a^X b^2 \cup a^X b^3)$ is a DSRCG if and only if there exist one β with $0 \leq \beta \leq \alpha - 1$ and a subset $T \subseteq \{1, \cdots, 2^{\alpha-\beta} - 1\}$ satisfying the following conditions:*

(i) $X = T + 2^{\alpha-\beta}\mathbb{Z}_{2^\alpha}$;
(ii) $X \cup (-X) = (\mathbb{Z}_{2^\alpha} \backslash 2^{\alpha-\beta}\mathbb{Z}_{2^\alpha}) \uplus (2^{\alpha-\beta-1} + 2^{\alpha-\beta}\mathbb{Z}_{2^\alpha})$;
(iii) $X \cup (2^{\alpha-\beta-1} + X) = \mathbb{Z}_{2^\alpha}$.

Proof. It follows from Theorem 2 that the Cayley graph $\mathcal{C}(T_{2^{\alpha+2}}, a^X \cup a^X b \cup a^X b^2 \cup a^X b^3)$ satisfying the conditions (i), (ii) and (iii) is a DSRCG.

Conversely, suppose that the Cayley graph $\mathcal{C}(T_{2^{\alpha+2}}, a^X \cup a^X b \cup a^X b^2 \cup a^X b^3)$ is a DSRCG with parameters $(4n, k = 4|X|, \mu, \lambda, t)$, then we have $X \cap (-X) \neq \emptyset$ by Lemma 15. By Lemma 14 (4) and Equation (3), we have

$$\mathbf{q}(z) = (\mathbf{r} + \bar{\mathbf{r}})(z) \left(\mathcal{F}\Delta_{O_{\beta+1}}\right)(z) + \sum_{i=\beta+1} \left(\mathcal{F}\Delta_{O_i}\right)(z) \in \{0, \frac{\lambda-\mu}{2}\}$$

for some $0 \leq \beta \leq \alpha - 1$. Hence $k = \mathbf{q}(0) = 2^\alpha$ and $\mathbf{q}(2^\beta) = -2^{\beta+1} = \frac{\lambda-\mu}{2}$. Since $k(k + (\mu - \lambda)) = t + (n-1)\mu$ and $t = \mu$ by Lemma 11, we have $\mu = 2^{\alpha-2} + 2^\beta$. Since $\mu < k$, we have $\beta \leq \alpha - 1$. Thus, by Lemma 11 and Equation (6), we have

$$\mathbf{r}\left(\mathcal{F}\Delta_{O_{\beta+1}} + 2^\alpha\Delta_0 - 2^\beta\Delta_{2^\beta\mathbb{Z}_{2^\alpha}}\right) = \mu 2^\alpha \Delta_0 - 2^{\beta+1}\mathbf{r}. \tag{8}$$

Since $(\mathcal{F}\Delta_{O_{\beta+1}})(z) = 0$, we have $\mathbf{r}(z) = (\mathcal{F}\Delta_X)(z) = 0$ for $z \notin 2^\beta\mathbb{Z}_{2^\alpha}$. Thus, by Lemma 9, we have $X = T + 2^{\alpha-\beta}\mathbb{Z}_{2^\alpha}$, where $T \subseteq \{1, \cdots, 2^{\alpha-\beta} - 1\}$, proving (i). Thus we have

$$\mathbf{r}(z) = (\mathcal{F}\Delta_X)(z) = 2^\beta \Delta_{2^\beta\mathbb{Z}_{2^\alpha}} (\mathcal{F}\Delta_T).$$

So by Equation (8), we have, for $z \in 2^\beta\mathbb{Z}_{2\alpha}$,

$$(\mathcal{F}\Delta_H)(z)\left((\mathcal{F}\Delta_T)(z) + \overline{(\mathcal{F}\Delta_T)(z)}\right) = 2^{\alpha-\beta}\left(1 + 2^{\alpha-\beta-1}\right)\Delta_0(z) - 2(\mathcal{F}\Delta_T)(z). \tag{9}$$

Let $T' = \psi_{2^{\alpha-\beta}}(T)$ and $\Delta_{T'}^{(2^{\alpha-\beta})} = \widetilde{\Delta}_{T'}$. Then $T' \subseteq \mathbb{Z}_{2^{\alpha-\beta}}$. Thus, by Lemmas 4, 8 and Equation (9), we have

$$T' \uplus (-T') = (\mathbb{Z}_{2^{\alpha-\beta}} \uplus O_1') \backslash \{0\} = \left(\mathbb{Z}_{2^{\alpha-\beta}} \uplus \left\{2^{\alpha-\beta-1}\right\}\right) \backslash \{0\}.$$

Since $\ker \psi_{2^{\alpha-\beta}} = 2^{\alpha-\beta}\mathbb{Z}_{p^\alpha}$, we have

$$T \uplus (-T) = \left\{i + x_i : 1 \leqslant i \leqslant 2^{\alpha-\beta} - 1\right\} \uplus \left\{2^{\alpha-\beta-1} + y\right\},$$

for $x_1, x_2, \cdots, x_{2^{\alpha-\beta}-1}, y \in 2^{\alpha-\beta}\mathbb{Z}_{p^\alpha}$. So

$$X \cup (-X) = \left(T + 2^{\alpha-\beta}\mathbb{Z}_{p^\alpha}\right) \uplus \left(-T + 2^{\alpha-\beta}\mathbb{Z}_{p^\alpha}\right) = \left(\mathbb{Z}_{2^\alpha} \backslash 2^{\alpha-\beta}\mathbb{Z}_{2^\alpha}\right) \uplus \left(2^{\alpha-\beta-1} + 2^{\alpha-\beta}\mathbb{Z}_{2^\alpha}\right),$$

proving (ii). By Lemma 11, we have

$$\overline{a^{\mathbb{Z}_{2^\alpha}}} = \overline{a^X} + \overline{x^{2^{\alpha-\beta-1}+X}},$$

then $X \cup (2^{\alpha-\beta-1} + X) = \mathbb{Z}_{2^\alpha}$, proving (iii). $\square$

5. Characterization of DSRCG $\mathcal{C}(T_{4p^\alpha}, a^X \cup a^Y b \cup a^X b^2 \cup a^Y b^3)$ with $X \subseteq Y$

Throughout this section, p is assumed to be an odd prime. Let $\mathbf{w} = \mathbf{r} - \mathbf{t}$. Then we have

Lemma 16. *Suppose the Cayley graph $\mathcal{C}(T_{4p^\alpha}, a^X \cup a^Y b \cup a^X b^2 \cup a^Y b^3)$ is a DSRCG with $X \subseteq Y$. Then $\mathrm{Im}(\mathbf{w}) \in \mathbb{R}$ if and only if $Y \setminus X$ is the union of some $\mathbb{Z}_{p^\alpha}^*$-orbits in $\mathbb{Z}_{p^\alpha}$. Moreover, if $\mathcal{C}(T_{4p^\alpha}, a^X \cup a^Y b \cup a^X b^2 \cup a^Y b^3)$ is a DSRCG with $X \subseteq Y$ and $\mathrm{Im}(\mathbf{w}) \in \mathbb{R}$, then $\mathrm{Im}(\mathbf{w}) \in \{0, \frac{\lambda-\mu}{2}\}$ and $Y \backslash X = O_{r_1} \cup O_{r_1} \cup \cdots \cup O_{r_s}$, for some $0 = r_1 < r_2 < \cdots < r_s \leqslant \alpha$.*

Proof. Suppose the Cayley graph $\mathcal{C}(T_{4p^\alpha}, a^X \cup a^Y b \cup a^X b^2 \cup a^Y b^3)$ is a DSRG with $X \subseteq Y$. If $Y \setminus X$ is a union of some $\mathbb{Z}_{p^\alpha}^*$-orbits in $\mathbb{Z}_{p^\alpha}$, then $\mathrm{Im}(\mathbf{w}) \in \mathbb{R}$ clearly. If $\mathrm{Im}(\mathbf{w}) \in \mathbb{R}$, by Lemma 12, we have

$$\mathbf{w}^2 = \frac{\lambda-\mu}{2}\mathbf{w}.$$

Thus we have the two eigenvalues $0, \frac{\lambda-\mu}{2}$ are two roots of the quadratic Equation $x^2 = \frac{\lambda-\mu}{2}x$, so we can get $\mathrm{Im}(\mathbf{w}) \in \{0, \frac{\lambda-\mu}{2}\} \subseteq \mathbb{Q}$. Therefore, by Lemma 3, we have $\Delta_X - \Delta_Y = \sum_{r=0}^{\alpha} \alpha_r \Delta_{O_r}$, for some $\alpha_r \in \{0, -1\}$ and $\alpha_0 = -1$. Thus we have $Y \backslash X = O_{r_1} \cup O_{r_1} \cup \cdots \cup O_{r_s}$, for some $0 = r_1 < r_2 < \cdots < r_s \leqslant \alpha$. $\square$

In the following theorem, we characterize certain DSRCG $\mathcal{C}(T_{4p^\alpha}, a^X \cup a^Y b \cup a^X b^2 \cup a^Y b^3)$ with $X \subseteq Y$.

Theorem 6. *Let X, Y be subsets of $\mathbb{Z}_{p^\alpha}$ with $X \subseteq Y$ and $Y \setminus X$ is a union of some $\mathbb{Z}_{p^\alpha}^*$-orbits with $Y \setminus X \neq \{0\}$. Then the Cayley graph $\mathcal{C}(T_{4p^\alpha}, a^X \cup a^Y b \cup a^X b^2 \cup a^Y b^3)$ is a DSRCG if and only if the following conditions holds:*

(i) $Y = T + p^\beta \mathbb{Z}_{p^\alpha} = X \cup p^\beta \mathbb{Z}_{p^\alpha}$,
(ii) $Y \uplus (-Y) = \mathbb{Z}_{p^\alpha} \uplus p^\beta \mathbb{Z}_{p^\alpha}$,

where $0 < \beta < \alpha$ and T is a subset of $\{0, 1, \cdots, p^\beta - 1\}$.

Proof. By Construction 3, we have that the Cayley graph $\mathcal{C}(T_{4p^\alpha}, a^X \cup a^Y b \cup a^X b^2 \cup a^Y b^3)$ with conditions (i) and (ii) is a DSRCG.

Conversely, suppose that the Cayley graph $\mathcal{C}(T_{4p^\alpha}, a^X \cup a^Y b \cup a^X b^2 \cup a^Y b^3)$ is a DSRG with $X \subseteq Y$ and $Y \setminus X$ is a union of some $\mathbb{Z}_{p^\alpha}^*$-orbits with $Y \setminus X \neq \{0\}$. By Lemma 16, we have $\mathrm{Im}(\mathbf{w}) \in \{0, \frac{\lambda-\mu}{2}\}$ and $Y \backslash X = O_{r_1} \cup O_{r_1} \cup \cdots \cup O_{r_s}$, for some $0 = r_1 < r_2 < \cdots < r_s \leqslant \alpha$. Thus $\mathbf{w} = -\sum_{i=1}^{s} \mathcal{F}\Delta_{O_i}$.

We claim that $\{0 = r_1, r_2, \cdots, r_s\} = \{0, 1, \cdots, s-1\}$. To prove this claim, we first assume $s > 1$ since this claim holds for $s = 1$. In fact, if there is an integer u such that $r_{u+1} > r_u + 1$ for some $1 \leq u \leq s-1$, by Equation (3), we have

$$\mathbf{w}\left(p^{r_u}\right) = -\sum_{i=1}^{u} \mu\left(\frac{p^{r_i}}{(p^{r_i}, p^{r_u})}\right) \frac{\varphi\left(p^{r_i}\right)}{\varphi\left(\frac{p^{r_i}}{(p^{r_i}, p^{r_u})}\right)} = -\sum_{i=1}^{u} \varphi\left(p^{r_i}\right) < 0,$$

but

$$\begin{aligned} \mathbf{w}\left(p^{r_s}\right) &= -\sum_{i=1}^{s} \mu\left(\frac{p^{r_i}}{(p^{r_i}, p^{r_s})}\right) \frac{\varphi\left(p^{r_i}\right)}{\varphi\left(\frac{p^{r_i}}{(p^{r_i}, p^{r_s})}\right)} = -\sum_{i=1}^{u} \varphi\left(p^{r_i}\right) \\ &= -\sum_{i=1}^{s} \varphi\left(p^{r_i}\right) < -\sum_{i=1}^{u} \varphi\left(p^{r_i}\right) = \mathbf{w}\left(p^{r_u}\right) < 0, \end{aligned}$$

a contradiction.

Thus $\mathbf{r} = \mathbf{t} + \mathbf{w} = \mathbf{t} - \left(\mathcal{F}\Delta_{p^\beta \mathbb{Z}_{p^\alpha}}\right) = \mathbf{t} - p^{\alpha-\beta}\Delta_{p^{\alpha-\beta}\mathbb{Z}_{p^\alpha}}$.

By Lemma 12, we have

$$\begin{aligned}
&\mathbf{t}^2 + |\mathbf{t}|^2 - 2p^{\alpha-\beta}\mathbf{t}\Delta_{p^{\alpha-\beta}\mathbb{Z}_{p^\alpha}} = \tfrac{\mu}{2}p^{\alpha}\Delta_0 + \tfrac{\lambda-\mu}{2}\mathbf{t},\\
&\left(\mathbf{t} - p^{\alpha-\beta}\Delta_{p^{\alpha-\beta}Z_{p^\alpha}}\right)^2 + |\mathbf{t}|^2 = \tfrac{\mu}{2}p^{\alpha}\Delta_0 + \tfrac{\lambda-\mu}{2}\left(\mathbf{t} - p^{\alpha-\beta}\Delta_{p^{\alpha}-\beta\mathbb{Z}_{p^\alpha}}\right).
\end{aligned}$$

The difference of these two equations gives

$$p^{2(\alpha-\beta)}\Delta_{p^{\alpha-\beta}Z_{p^\alpha}} = \frac{\lambda-\mu}{2}p^{\alpha-\beta}\Delta_{p^{\alpha-\beta}\mathbb{Z}_{p^\alpha}}.$$

Since $Y \setminus X \neq \{0\}$, then we have $0 < \beta < \alpha$. Thus we have $\frac{\lambda-\mu}{2} = p^{\alpha-\beta}$. Similar to the proof about Case 2 of Theorem 7.2 in [6], we can get the conditions (i) and (ii). $\square$

Author Contributions: Conceptualization, T.C., L.T. and L.F.; investigation, L.F. and T.C.; data curation, L.F. and T.C.; writing original draft preparation, L.F. and W.L.; writing review and editing, T.C., L.F. and W.L.; supervision, T.C. and L.F.; project administration, L.F.

Funding: This research was supported by NSFC (Nos. 11671402 and 11871479) (L.F. and W.L.), Hunan Provincial Natural Science Foundation (2016JJ2138 and 2018JJ2479) (L.F. and W.L.) and Mathematics and Interdisciplinary Sciences Project of CSU (L.F. and W.L.), NSFC (No. 11701339) (T.C.), Hunan Provincial Key Laboratory of Mathematical Modeling and Analysis in Engineering (Changsha University of Science & Technology) (L.F.), the Scientific Research Fund of Hunan Provincial Education Department (No. 16A005)(L.F.).

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Duval, A.M. A directed graph version of strongly regular graphs. *J. Combin. Theory Ser. A* **1988**, *47*, 71–100. [CrossRef]
2. Duval, A.M.; Iourinski, D. Semidirect product constructions of directed strongly regular graphs. *J. Combin. Theory Ser. A* **2003**, *104*, 157–167. [CrossRef]
3. Godsil, C.D.; Hobart, S.A.; Martin, W.J. Representations of directed strongly regular graphs. *Eur. J. Combin.* **2007**, *28*, 1980–1993. [CrossRef]
4. Gyürki, Š.; Mikhail, K. Sporadic Examples of Directed Strongly Regular Graphs Obtained by Computer Algebra Experimentation (Extended version). In Proceedings of the 16th International Workshop on Computer Algebra in Scientific Computing, Warsaw, Poland, 8–12 September 2014.
5. Klin, M.; Munemasa, A.; Muzychuk, M.; Zieschang, P.H. Directed strongly regular graphs obtained from coherent algebras. *Linear Algebra Appl.* **2004**, *377*, 83–109. [CrossRef]
6. He, Y.; Zhang, B.; Feng, R. Directed Strongly Regular Cayley Graphs on Dihedral Groups. *arXiv* **2018**, arXiv:1801.04748v3.
7. Cheng, T.; Feng, L.; Huang, H. Integral Cayley graphs over dicyclic group. *Linear Algebra Appl.* **2019**, *566*, 121–137. [CrossRef]
8. Lu, L.; Huang, Q.; Huang, X. Integral Cayley graphs over dihedral groups. *J. Algebr. Combin.* **2018**, *47*, 585–601. [CrossRef]
9. James, G.; Liebeck, M. *Representations and Characters of Groups*, 2nd ed.; Cambridge University Press: Cambridge, UK, 2001.
10. Kohl, T. Groups of order $4p$, twisted wreath products and Hopf-Galois theory. *J. Algebra* **2007**, *314*, 42–74. [CrossRef]
11. Miklavič, Š.; Potočnik, P. Distance-regular Cayley graphs on dihedral groups. *J. Combin. Theory Ser. B* **2007**, *97*, 14–33. [CrossRef]