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Article

Foldness of Bipolar Fuzzy Sets and Its Application in BCK/BCI-Algebras

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Abstract: Recent trends in modern information processing have focused on polarizing information, and and bipolar fuzzy sets can be useful. Bipolar fuzzy sets are one of the important tools that can be used to distinguish between positive information and negative information. Positive information, for example, already observed or experienced, indicates what is guaranteed to be possible, and negative information indicates that it is impossible, prohibited, or certainly false. The purpose of this paper is to apply the bipolar fuzzy set to BCK/BCI-algebras. The notion of (translated) k-fold bipolar fuzzy sets is introduced, and its application in BCK/BCI-algebras is discussed. The concepts of k-fold bipolar fuzzy subalgebra and k-fold bipolar fuzzy ideal are introduced, and related properties are investigated. Characterizations of k-fold bipolar fuzzy subalgebra/ideal are considered, and relations between k-fold bipolar fuzzy subalgebra and k-fold bipolar fuzzy ideal are displayed. Extension of k-fold bipolar fuzzy subalgebra is discussed.

Keywords: k-fold bipolar fuzzy set; k-fold bipolar fuzzy subalgebra; k-fold bipolar fuzzy ideal

MSC: 06F35; 03G25; 08A72

1. Introduction

A fuzzy set is introduced in [1], and it deals with uncertainty connected with perceptions, preferences, and imprecision of states. Fuzzy logic and many-valued logic, etc. are contained in nonclassical logic, and it takes the advantage of the classical logic to handle information with various facets of uncertainty (see [2] for a generalized theory of uncertainty), such as randomness, fuzziness, and so on. Nonclassical logic has become a formal and useful tool for computer science to deal with fuzzy and uncertain information. The bipolar fuzzy set introduced by Lee [3] was built on the space $\{(x, y) \mid x \in [-1, 0], y \in [0, 1]\}$. A bipolar fuzzy set is presented for cognitive modeling and multiagent decision analysis. In many areas of information processing, bipolarity is a key feature to be considered, indicating that positive information is possible or preferred and that negative information is forbidden or certainly false. A bipolar fuzzy set is an extension of a fuzzy set in which the membership degree range is $[-1, 1]$. The notion of a bipolar fuzzy finite state machine, which is a generalization of a fuzzy finite state machine, was introduced by Jun et al. in [4]. They introduced the concepts of a bipolar exchange property, a bipolar (immediate) successor, and a bipolar subsystem. They also considered a condition for a bipolar fuzzy finite state machine to satisfy the bipolar exchange property and established a characterization of a bipolar subsystem. Lee [5] introduced the concept of bipolar fuzzy subalgebras/ideals of a BCK/BCI-algebra, and investigated several properties. She gave relations between a bipolar fuzzy subalgebra and a bipolar fuzzy ideal and provided a condition for a bipolar fuzzy subalgebra to be a bipolar fuzzy ideal. She also gave characterizations of a bipolar fuzzy ideal and considered the concept of equivalence relations on the family of all bipolar fuzzy ideals of a BCK/BCI-algebra. Lee and Jun [6] introduced the notion of bipolar fuzzy a-ideals of BCI-algebras and investigated their properties. They discussed relations between bipolar fuzzy subalgebras, bipolar fuzzy ideals, and bipolar fuzzy a-ideals. They gave conditions for a bipolar fuzzy ideal to be a bipolar fuzzy a-ideal and considered characterizations of bipolar fuzzy a-ideals. Jun et al. [7–10] discussed several types of bipolar fuzzy ideals in hyper BCK-algebras. Kang [11] introduced the concepts of bipolar fuzzy hyper MV-subalgebras, (weak) bipolar fuzzy hyper MV-deductive systems and previously weak bipolar fuzzy hyper MV-deductive systems and investigated their relations/properties. He provided characterizations of bipolar fuzzy hyper MV-subalgebras and weak bipolar fuzzy hyper MV-deductive systems. Akram et al. [12] introduced certain notions of bipolar fuzzy soft graphs and investigated some of their properties. They presented several applications of the bipolar fuzzy soft graphs in a multiple criteria decision-making problem. Akram et al. [13] introduced certain notions including bipolar fuzzy graph structure (BFGS), strong bipolar fuzzy graph structure, bipolar fuzzy N_i-cycle, bipolar fuzzy N_i-tree, bipolar fuzzy N_i-cut vertex, and bipolar fuzzy N_i-bridge and illustrated these notions by several examples. Yang et al. [14] used the bipolar multi-fuzzy soft set in analyzing a decision-making problem.

In this paper, we think of the problem of folding bipolar fuzzy set. We fold the bipolar fuzzy set k times and introduce the (translated) k-fold bipolar fuzzy set. Then, we apply it to BCK/BCI-algebras. We introduce the notion of k-fold bipolar fuzzy subalgebra and k-fold bipolar fuzzy ideal and investigate several properties. We investigate relations between k-fold bipolar fuzzy subalgebra and k-fold bipolar fuzzy ideal. We provide conditions for k-fold bipolar fuzzy subalgebra to be k-fold bipolar fuzzy ideal. We consider characterizations of k-fold bipolar fuzzy subalgebra/ideal. We also introduce the extension of k-fold bipolar fuzzy set and discuss their properties.

2. Preliminaries

A BCK/BCI-algebra is a class of logical algebras introduced by K. Iséki (see [15,16]) and was extensively studied by several researchers.

A BCI-algebra is defined to be a set W with a binary operation $*$ and a special element 0 that satisfies the next conditions:

(I) $(\forall w,u,v \in W)\ (((w*u)*(w*v))*(v*u)=0)$,
(II) $(\forall w,u \in W)\ ((w*(w*u))*u=0)$,
(III) $(\forall w \in W)\ (w*w=0)$,
(IV) $(\forall w,u \in W)\ (w*u=0,\ u*w=0 \ \Rightarrow\ w=u)$.

If a BCI-algebra W satisfies the following identity:

(V) $(\forall w \in W)\ (0*w=0)$,

then we say W is a BCK-algebra. Any BCK/BCI-algebra W has the next conditions:

$$(\forall w \in W)\,(w*0=w)\,, \tag{1}$$

$$(\forall w,u,v \in W)\,(w \le u \ \Rightarrow\ w*v \le u*v,\ v*u \le v*w)\,, \tag{2}$$

$$(\forall w,u,v \in W)\,((w*u)*v=(w*v)*u) \tag{3}$$

where $w \leq u$ if and only if $w * u = 0$. A subset S of a BCK/BCI-algebra W is called a subalgebra of W if $w * u \in S$ for all $w, u \in S$. A subset D of a BCK/BCI-algebra W is called an ideal of W if it satisfies the following:

$$0 \in D, \tag{4}$$
$$(\forall w \in W)\,(\forall u \in D)\,(w * u \in D \;\Rightarrow\; w \in D)\,. \tag{5}$$

See the books in [17,18] for more information on BCK/BCI-algebras.

3. Foldness of Bipolar Fuzzy Sets

In what follows, let $\pi_i : ([-1,0] \times [0,1])^k \to [-1,0] \times [0,1]$ be an ith projection for $i = 1, 2, \cdots, k$, where k is a natural number unless otherwise specified. Every element $\langle (t_1^-, t_1^+), (t_2^-, t_2^+), \cdots, (t_k^-, t_k^+) \rangle$ of $([-1,0] \times [0,1])^k$ is denoted by $\{(t_i^-, t_i^+)\}_{i=1}^k$. We define an order relation "$\ll$" on $([-1,0] \times [0,1])^k$ by

$$\{(t_i^-, t_i^+)\}_{i=1}^k \ll \{(s_i^-, s_i^+)\}_{i=1}^k \;\Leftrightarrow\; t_i^- \leq s_i^- \text{ and } t_i^+ \geq s_i^+ \tag{6}$$

for all $\{(t_i^-, t_i^+)\}_{i=1}^k, \{(s_i^-, s_i^+)\}_{i=1}^k \in ([-1,0] \times [0,1])^k$. If

$$\{(t_i^-, t_i^+)\}_{i=1}^k \ll \{(s_i^-, s_i^+)\}_{i=1}^k$$

is not true, we denote it by $\{(t_i^-, t_i^+)\}_{i=1}^k \neg \ll \{(s_i^-, s_i^+)\}_{i=1}^k$.

Definition 1. *Let W be a universal set. By a k-fold bipolar fuzzy set over W, we mean a pair (W, ℓ) in which*

$$\ell : W \to ([-1,0] \times [0,1])^k,\; x \mapsto \{(\ell_i^-(z), \ell_i^+(z))\}_{i=1}^k, \tag{7}$$

that is, a k-fold bipolar fuzzy set over W is an object having the form

$$(W, \ell) = \left\{ \left\langle z, \{(\ell_i^-(z), \ell_i^+(z))\}_{i=1}^k \right\rangle \mid z \in W \right\}, \tag{8}$$

where $\ell_i^- = (\pi_i \circ \ell)^- : W \to [-1,0]$ and $\ell_i^+ = (\pi_i \circ \ell)^+ : W \to [0,1]$ are mappings.

For a k-fold bipolar fuzzy set (W, ℓ) over W and $\{(t_i^-, t_i^+)\}_{i=1}^k \in ([-1,0] \times [0,1])^k$, we define the following:

$$N(\ell, t_i^-) = \{z \in W \mid \ell_i^-(z) \leq t_i^-\},$$
$$P(\ell, t_i^+) = \{z \in W \mid \ell_i^+(z) \geq t_i^+\}.$$

They are called the negative cut and positive cut of (W, ℓ), respectively, for $i = 1, 2, \cdots, k$. For every $t_i \in [0,1]$, the set

$$W_{t_i} := N(\ell, -t_i) \cap P(\ell, t_i)$$

is called the t_i-cut of (W, ℓ) for $i = 1, 2, \cdots, k$.

For two k-fold bipolar fuzzy sets (W, ℓ) and $(W, \jmath)$ over W, the *union* of (W, ℓ) and $(W, \jmath)$ is a k-fold bipolar fuzzy set $(W, \ell \cup \jmath)$ over W in which

$$\ell \cup \jmath : W \to ([-1,0] \times [0,1])^k,\; x \mapsto \{((\ell \cup \jmath)_i^-(z), (\ell \cup \jmath)_i^+(z))\}_{i=1}^k, \tag{9}$$

where

$$(\ell \cup \jmath)_i^- : W \to [-1, 0],\ z \mapsto \min\{\ell_i^-(z), \jmath_i^-(z)\}$$

and

$$(\ell \cup \jmath)_i^+ : W \to [-1, 0],\ z \mapsto \max\{\ell_i^+(z), \jmath_i^+(z)\}.$$

The intersection of (W, ℓ) and $(W, \jmath)$ is a k-fold bipolar fuzzy set $(W, \ell \cap \jmath)$ over W, in which

$$\ell \cap \jmath : W \to ([-1, 0] \times [0, 1])^k,\ x \mapsto \{((\ell \cap \jmath)_i^-(z), (\ell \cap \jmath)_i^+(z))\}_{i=1}^k, \tag{10}$$

where

$$(\ell \cap \jmath)_i^- : W \to [-1, 0],\ z \mapsto \max\{\ell_i^-(z), \jmath_i^-(z)\}$$

and

$$(\ell \cap \jmath)_i^+ : W \to [-1, 0],\ z \mapsto \min\{\ell_i^+(z), \jmath_i^+(z)\}.$$

4. *k*-Fold Bipolar Fuzzy Subalgebras

Definition 2. *Let W be a BCK/BCI-algebra. A k-fold bipolar fuzzy set (W, ℓ) over W is called a k-fold bipolar fuzzy subalgebra of W if*

$$\begin{aligned}&\{((\pi_i \circ \ell)^-(a * b), (\pi_i \circ \ell)^+(z * x))\}_{i=1}^k \\ &\ll \{(\max\{(\pi_i \circ \ell)^-(a), (\pi_i \circ \ell)^-(b)\}, \min\{(\pi_i \circ \ell)^+(z), (\pi_i \circ \ell)^+(x)\})\}_{i=1}^k\end{aligned}$$

for all $a, b, z, x \in W$.

Example 1. *Consider a BCK-algebra $W = \{0, 1, 2, 3\}$ with the binary operation $*$ in Table 1.*
Let (W, ℓ) be a 4-fold bipolar fuzzy set over W given by Table 2.
It is routine to verify that (W, ℓ) is a 4-fold bipolar fuzzy subalgebra of W.

Table 1. Cayley table for the binary operation "$*$".

$*$	**0**	**1**	**2**	**3**
0	0	0	0	0
1	1	0	0	1
2	2	1	0	2
3	3	3	3	0

Table 2. Tabular representation of "ℓ".

W	**0**	**1**	**2**	**3**
$(\ell_1^-(z), \ell_1^+(z))$	$(-0.7, 0.6)$	$(-0.7, 0.6)$	$(-0.2, 0.3)$	$(-0.7, 0.6)$
$(\ell_2^-(z), \ell_2^+(z))$	$(-0.5, 0.6)$	$(-0.3, 0.6)$	$(-0.1, 0.2)$	$(-0.4, 0.4)$
$(\ell_3^-(z), \ell_3^+(z))$	$(-0.6, 0.6)$	$(-0.2, 0.4)$	$(-0.5, 0.3)$	$(-0.3, 0.5)$
$(\ell_4^-(z), \ell_4^+(z))$	$(-0.9, 0.8)$	$(-0.7, 0.7)$	$(-0.4, 0.3)$	$(-0.6, 0.5)$

Example 2. *Consider a BCI-algebra $W = \{0, 1, a, b, c\}$ with the binary operation $*$ in Table 3.*
Let (W, ℓ) be a 2-fold bipolar fuzzy set over W given by Table 4.
It is routine to verify that (W, ℓ) is a 2-fold bipolar fuzzy subalgebra of W.

Table 3. Cayley table for the binary operation "$*$".

$*$	**0**	**1**	$\boldsymbol{a}$	$\boldsymbol{b}$	$\boldsymbol{c}$
0	0	0	c	b	a
1	1	0	c	b	a
a	a	a	0	c	b
b	b	b	a	0	c
c	c	c	b	a	0

Table 4. Tabular representation of "ℓ".

$\boldsymbol{W}$	**0**	**1**	$\boldsymbol{a}$	$\boldsymbol{b}$	$\boldsymbol{c}$
$(\ell_1^-(z), \ell_1^+(z))$	$(-0.7, 0.8)$	$(-0.5, 0.6)$	$(-0.2, 0.3)$	$(-0.2, 0.3)$	$(-0.2, 0.3)$
$(\ell_2^-(z), \ell_2^+(z))$	$(-0.5, 0.6)$	$(-0.1, 0.5)$	$(-0.3, 0.2)$	$(-0.4, 0.4)$	$(-0.3, 0.2)$

Proposition 1. *If (W, ℓ) is a k-fold bipolar fuzzy subalgebra of a BCK/BCI-algebra W, then $\ell(0) \ll \ell(z)$ for all $z \in W$. Moreover, if (W, ℓ) satisfies the following condition:*

$$\{(\pi \circ \ell)^-(x), (\pi \circ \ell)^+(x)\}_{i=1}^k \gg \{(\pi \circ \ell)^-(z * x), (\pi \circ \ell)^+(z * x)\}_{i=1}^k \tag{11}$$

for all $z, x \in W$, then $\ell(0) = \ell(z)$ for all $z \in W$.

Proof. For any $z \in W$, we have

$$\begin{aligned}
\ell(0) &= \{((\pi_i \circ \ell)^-(0), (\pi_i \circ \ell)^+(0))\}_{i=1}^k \\
&= \{((\pi_i \circ \ell)^-(z * z), (\pi_i \circ \ell)^+(z * z))\}_{i=1}^k \\
&\ll \left\{\left(\max\{(\pi_i \circ \ell)^-(z), (\pi_i \circ \ell)^-(z)\}, \min\{(\pi_i \circ \ell)^+(z), (\pi_i \circ \ell)^+(z)\}\right)\right\}_{i=1}^k \\
&= \{((\pi_i \circ \ell)^-(z), (\pi_i \circ \ell)^+(z))\}_{i=1}^k = \ell(z).
\end{aligned}$$

Now, since $z * 0 = z$, we get from Equation (11) the following:

$$\begin{aligned}
\ell(z) &= \{((\pi_i \circ \ell)^-(z), (\pi_i \circ \ell)^+(z))\}_{i=1}^k \\
&= \{((\pi_i \circ \ell)^-(z * 0), (\pi_i \circ \ell)^+(z * 0))\}_{i=1}^k \\
&\ll \{((\pi_i \circ \ell)^-(0), (\pi_i \circ \ell)^+(0))\}_{i=1}^k = \ell(0).
\end{aligned}$$

This has completed the proof. □

Theorem 1. *Given a k-fold bipolar fuzzy set (W, ℓ) over a BCK/BCI-algebra W, the following assertions are equivalent:*

(1) *(W, ℓ) is a k-fold bipolar fuzzy subalgebra of W.*
(2) *The positive cut $P(\ell, t_i^+)$ and negative cut $N(\ell, t_i^-)$ of (W, ℓ) are subalgebras of W for all $\{(t_i^-, t_i^+)\}_{i=1}^k \in ([-1, 0] \times [0, 1])^k$.*

Proof. Assume that (W, ℓ) is a k-fold bipolar subalgebra of W, and let $a, b, z, x \in W$ be such that $a, b \in N(\ell, t_i^-)$ and $z, x \in P(\ell, t_i^+)$ for all $i = 1, 2, \cdots, k$. Then $\ell_i^-(a) \leq t_i^-$, $\ell_i^-(b) \leq t_i^-$, $\ell_i^+(z) \geq t_i^+$ and $\ell_i^+(x) \geq t_i^+$. It follows that

$$\begin{aligned}
&\{\ell_i^-(a*b), \ell_i^+(z*x)\}_{i=1}^k = \{(\pi_i \circ \ell)^-(a*b), (\pi_i \circ \ell)^+(z*x)\}_{i=1}^k \\
&\ll \{(\max\{(\pi_i \circ \ell)^-(a), (\pi_i \circ \ell)^-(b)\}, \min\{(\pi_i \circ \ell)^+(z), (\pi_i \circ \ell)^+(x)\})\}_{i=1}^k \\
&\ll \{(\max\{\ell_i^-(a), \ell_i^-(b)\}, \min\{\ell_i^+(z), \ell_i^+(x)\})\}_{i=1}^k \\
&\ll \{(t_i^-, t_i^+)\}_{i=1}^k,
\end{aligned}$$

i.e., $a * b \in N(\ell, t_i^-)$ and $z * x \in P(\ell, t_i^+)$ for all $i = 1, 2, \cdots, k$. Therefore, $P(\ell, t_i^+)$ and $N(\ell, t_i^-)$ are subalgebras of W for all $\{(t_i^-, t_i^+)\}_{i=1}^k \in ([-1, 0] \times [0, 1])^k$.

Conversely, suppose that Assertion (2) is valid. If (W, ℓ) is not a k-fold bipolar subalgebra of W, then there exists $i \in \{1, 2, \cdots, k\}$ such that

$$\ell_i^-(a*b) > \max\{\ell_i^-(a), \ell_i^-(b)\} \text{ or } \ell_i^+(z*x) < \min\{\ell_i^+(z), \ell_i^+(x)\}$$

for some $a, b, z, x \in W$. If we take $t_i^- := \max\{\ell_i^-(a), \ell_i^-(b)\}$, and $t_i^+ := \min\{\ell_i^+(z), \ell_i^+(x)\}$, then

$$a, b \in N(\ell, t_i^-) \text{ and } a * b \notin N(\ell, t_i^-)$$

or

$$z, x \in P(\ell, t_i^+) \text{ and } z * x \notin P(\ell, t_i^+).$$

This is a contradiction. Hence, (W, ℓ) is a k-fold bipolar fuzzy subalgebra of W. □

Corollary 1. *If (W, ℓ) is a k-fold bipolar fuzzy subalgebra of a BCK/BCI-algebra W, then $P(\ell, t_i^+) \cap N(\ell, t_i^-)$ is a subalgebra of W for all $\{(t_i^-, t_i^+)\}_{i=1}^k \in ([-1, 0] \times [0, 1])^k$.*

5. *k*-Fold Bipolar Fuzzy Ideals

Definition 3. *Let W be a BCK/BCI-algebra. A k-fold bipolar fuzzy set (W, ℓ) over W is called a k-fold bipolar fuzzy ideal of W if it satisfies the following conditions:*

$$\{((\pi_i \circ \ell)^-(0), (\pi_i \circ \ell)^+(0))\}_{i=1}^k \ll \{((\pi_i \circ \ell)^-(z), (\pi_i \circ \ell)^+(z))\}_{i=1}^k \tag{12}$$

and

$$\begin{aligned}
&\{((\pi_i \circ \ell)^-(a), (\pi_i \circ \ell)^+(z))\}_{i=1}^k \\
&\ll \{(\max\{(\pi_i \circ \ell)^-(a*b), (\pi_i \circ \ell)^-(b)\}, \min\{(\pi_i \circ \ell)^+(z*x), (\pi_i \circ \ell)^+(x)\})\}_{i=1}^k
\end{aligned} \tag{13}$$

for all $a, b, z, x \in W$.

Example 3. *Consider a BCK-algebra $W = \{0, 1, 2, 3, 4\}$ with the binary operation $*$ in Table 5.*

Let (W, ℓ) be a 3-fold bipolar fuzzy set over W given by Table 6.

It is routine to verify that (W, ℓ) is a 3-fold bipolar fuzzy ideal of W.

Table 5. Cayley table for the binary operation "$*$".

$*$	**0**	**1**	**2**	**3**	**4**
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	0	0
3	3	3	3	0	0
4	4	3	4	1	0

Table 6. Tabular representation of "ℓ".

W	**0**	**1**	**2**	**3**	**4**
$(\ell_1^-(z), \ell_1^+(z))$	$(-0.8, 0.7)$	$(-0.7, 0.2)$	$(-0.8, 0.7)$	$(-0.7, 0.2)$	$(-0.7, 0.2)$
$(\ell_2^-(z), \ell_2^+(z))$	$(-0.7, 0.8)$	$(-0.6, 0.5)$	$(-0.5, 0.6)$	$(-0.3, 0.4)$	$(-0.3, 0.4)$
$(\ell_3^-(z), \ell_3^+(z))$	$(-0.5, 0.6)$	$(-0.4, 0.4)$	$(-0.4, 0.3)$	$(-0.3, 0.2)$	$(-0.3, 0.2)$

Example 4. *Let (W, ℓ) be a k-fold bipolar fuzzy set over a BCI-algebra W in which ℓ is given by*

$$\ell : W \to ([-1, 0] \times [0, 1])^k,\ x \mapsto \begin{cases} \{(t_i^-, t_i^+)\}_{i=1}^k & \text{if } x \geq 0, \\ \{(s_i^-, s_i^+)\}_{i=1}^k & \text{otherwise,} \end{cases} \tag{14}$$

where $\{(t_i^-, t_i^+)\}_{i=1}^k \ll \{(s_i^-, s_i^+)\}_{i=1}^k$. It is routine to verify that (W, ℓ) is a k-fold bipolar fuzzy ideal of W.

Proposition 2. *Every k-fold bipolar fuzzy ideal (W, ℓ) of a BCK/BCI-algebra W satisfies the following:*

$$\begin{aligned} &\{((\pi_i \circ \ell)^-(z), (\pi_i \circ \ell)^+(z))\}_{i=1}^k \\ &\ll \left\{\left(\max\{(\pi_i \circ \ell)^-(x), (\pi_i \circ \ell)^-(y)\}, \min\{(\pi_i \circ \ell)^+(x), (\pi_i \circ \ell)^+(y)\}\right)\right\}_{i=1}^k \end{aligned} \tag{15}$$

for all $z, x, y \in W$ with $z * x \leq y$.

Proof. Let $z, x, y \in W$ be such that $z * x \leq y$. Then $(z * x) * y = 0$, and thus,

$$\begin{aligned} &\{((\pi_i \circ \ell)^-(z), (\pi_i \circ \ell)^+(z))\}_{i=1}^k \\ &\ll \left\{\left(\max\{(\pi_i \circ \ell)^-(z * x), (\pi_i \circ \ell)^-(x)\}, \min\{(\pi_i \circ \ell)^+(z * x), (\pi_i \circ \ell)^+(x)\}\right)\right\}_{i=1}^k \\ &= \Big\{\big(\max\{\max\{(\pi_i \circ \ell)^-((z * x) * y), (\pi_i \circ \ell)^-(y)\}, (\pi_i \circ \ell)^-(x)\}, \\ &\qquad \min\{\min\{(\pi_i \circ \ell)^+((z * x) * y), (\pi_i \circ \ell)^+(y)\}, (\pi_i \circ \ell)^+(x)\}\big)\Big\}_{i=1}^k \\ &= \Big\{\big(\max\{\max\{(\pi_i \circ \ell)^-(0), (\pi_i \circ \ell)^-(y)\}, (\pi_i \circ \ell)^-(x)\}, \\ &\qquad \min\{\min\{(\pi_i \circ \ell)^+(0), (\pi_i \circ \ell)^+(y)\}, (\pi_i \circ \ell)^+(x)\}\big)\Big\}_{i=1}^k \\ &= \left\{\left(\max\{(\pi_i \circ \ell)^-(x), (\pi_i \circ \ell)^-(y)\}, \min\{(\pi_i \circ \ell)^+(x), (\pi_i \circ \ell)^+(y)\}\right)\right\}_{i=1}^k \end{aligned}$$

This has completed the proof. $\square$

Proposition 3. *Every k-fold bipolar fuzzy ideal (W,ℓ) of a BCK/BCI-algebra W satisfies the following assertion:*

$$\{((\pi_i \circ \ell)^-(a), (\pi_i \circ \ell)^+(z))\}_{i=1}^k \ll \{((\pi_i \circ \ell)^-(b), (\pi_i \circ \ell)^+(x))\}_{i=1}^k. \tag{16}$$

for all $a, b, z, x \in W$ with $a \leq b$ and $z \leq x$.

Proof. Let $a, b, z, x \in W$ be such that $a \leq b$ and $z \leq x$. Then, $a * b = 0$ and $z * x = 0$, and thus,

$$\begin{aligned}
&\{((\pi_i \circ \ell)^-(a), (\pi_i \circ \ell)^+(z))\}_{i=1}^k \\
&\ll \left\{\left(\max\{(\pi_i \circ \ell)^-(a * b), (\pi_i \circ \ell)^-(b)\}, \min\{(\pi_i \circ \ell)^+(z * x), (\pi_i \circ \ell)^+(x)\}\right)\right\}_{i=1}^k \\
&= \left\{\left(\max\{(\pi_i \circ \ell)^-(0), (\pi_i \circ \ell)^-(b)\}, \min\{(\pi_i \circ \ell)^+(0), (\pi_i \circ \ell)^+(x)\}\right)\right\}_{i=1}^k \\
&= \{((\pi_i \circ \ell)^-(b), (\pi_i \circ \ell)^+(x))\}_{i=1}^k.
\end{aligned}$$

This has completed the proof. □

Proposition 4. *Given a k-fold bipolar fuzzy ideal (W,ℓ) of a BCK/BCI-algebra W, the following are equivalent.*

(1) $\{(\ell_i^-(z * x), \ell_i^+(z * x))\}_{i=1}^k \ll \{(\ell_i^-((z * x) * x), \ell_i^+((z * x) * x))\}_{i=1}^k$

(2) $\{(\ell_i^-((z * y) * (x * y)), \ell_i^+((z * y) * (x * y)))\}_{i=1}^k \ll \{(\ell_i^-((z * x) * y), \ell_i^+((z * x) * y))\}_{i=1}^k$

for all $z, x, y \in W$.

Proof. Suppose that Assertion (1) is valid, and let $z, x, y \in W$. Since $((z * (x * y)) * y) * y \leq (z * x) * y$, it follows from Equation (3), Assertion (1), and Proposition 3 that

$$\begin{aligned}
&\{(\ell_i^-((z * y) * (x * y)), \ell_i^+((z * y) * (x * y))\}_{i=1}^k \\
&= \{(\ell_i^-((z * (x * y)) * y), \ell_i^+((z * (x * y)) * y)\}_{i=1}^k \\
&\ll \{(\ell_i^-(((z * (x * y)) * y) * y), \ell_i^+(((z * (x * y)) * y) * y)\}_{i=1}^k \\
&\ll \{(\ell_i^-((z * x) * y), \ell_i^+((z * x) * y))\}_{i=1}^k
\end{aligned}$$

Conversely, Assertion (1) is obtained by taking $x = y$ in Assertion (1) and using Condition (III) and Equation (1). □

Theorem 2. *In a BCK-algebra, every k-fold bipolar fuzzy ideal is a k-fold bipolar fuzzy subalgebra.*

Proof. Let (W,ℓ) be a k-fold bipolar fuzzy ideal of a BCK-algebra W, and let $a, b, z, x \in W$. Since $a * b \leq a$ and $z * x \leq z$, it follows from Proposition 3 and Equation (13) that

$$\begin{aligned}
&\{((\pi_i \circ \ell)^-(a * b), (\pi_i \circ \ell)^+(z * x))\}_{i=1}^k \ll \{((\pi_i \circ \ell)^-(a), (\pi_i \circ \ell)^+(z))\}_{i=1}^k \\
&\ll \left\{\left(\max\{(\pi_i \circ \ell)^-(a * b), (\pi_i \circ \ell)^-(b)\}, \min\{(\pi_i \circ \ell)^+(z * x), (\pi_i \circ \ell)^+(x)\}\right)\right\}_{i=1}^k \\
&\ll \left\{\left(\max\{(\pi_i \circ \ell)^-(a), (\pi_i \circ \ell)^-(b)\}, \min\{(\pi_i \circ \ell)^+(z), (\pi_i \circ \ell)^+(x)\}\right)\right\}_{i=1}^k.
\end{aligned}$$

Therefore, (W,ℓ) is a k-fold bipolar fuzzy subalgebra of W. □

The converse of Theorem 2 is not true in general. In fact, the 4-fold bipolar fuzzy subalgebra (W,ℓ) of W in Example 1 is not a 4-fold bipolar fuzzy ideal of W since

$$\begin{aligned}&\{(\pi_i\circ\ell)^-(2),(\pi_i\circ\ell)^+(2)\}_{i=1}^k\\&\neg\ll\{(\max\{(\pi_i\circ\ell)^-(2*1),(\pi_i\circ\ell)^-(1)\},\min\{(\pi_i\circ\ell)^+(2*1),(\pi_i\circ\ell)^+(1)\})\}_{i=1}^k.\end{aligned}$$

A condition for a k-fold bipolar subalgebra to be a k-fold bipolar ideal is given in the following theorem.

Theorem 3. *Let (W,ℓ) be a k-fold bipolar fuzzy subalgebra of a BCK-algebra W. If (W,ℓ) satisfies the condition of Equation (15) for all $z,x,y\in W$ with $z*x\le y$, then (W,ℓ) is a k-fold bipolar fuzzy ideal of W.*

Proof. By Proposition 1, we know that

$$\{((\pi_i\circ\ell)^-(0),(\pi_i\circ\ell)^+(0))\}_{i=1}^k\ll\{((\pi_i\circ\ell)^-(z),(\pi_i\circ\ell)^+(z))\}_{i=1}^k$$

for all $z\in W$. Note that $a*(a*b)\le b$ and $z*(z*x)\le x$ for all $a,b,z,x\in W$. Thus,

$$\begin{aligned}&\{((\pi_i\circ\ell)^-(a),(\pi_i\circ\ell)^+(z))\}_{i=1}^k\\&\ll\big\{\big(\max\{(\pi_i\circ\ell)^-(a*b),(\pi_i\circ\ell)^-(b)\},\min\{(\pi_i\circ\ell)^+(z*x),(\pi_i\circ\ell)^+(x)\}\big)\big\}_{i=1}^k\end{aligned}$$

for all $a,b,z,x\in W$. Therefore, (W,ℓ) is a k-fold bipolar fuzzy ideal of W. □

The following example shows that Theorem 2 is not true in BCI-algebras.

Example 5. *Let $(Y,*,0)$ be a BCI-algebra and $(\mathbb{Z},-,0)$ be the adjoint BCI-algebra of the additive group $(\mathbb{Z},+,0)$ of integers. Then, the cartesian product $W:=Y\times\mathbb{Z}$ of Y and $\mathbb{Z}$ is a BCI-algebra (see [17]). Let $A:=Y\times\mathbb{N}$, where $\mathbb{N}$ is the set of nonnegative integers, and define a k-fold bipolar fuzzy (W,ℓ) over W by*

$$\ell:W\to([-1,0]\times[0,1])^k,\ x\mapsto\{(\ell_i^-(z),\ell_i^+(z))\}_{i=1}^k,$$

where

$$\ell_i^-=(\pi_i\circ\ell)^-:W\to[-1,0],z\mapsto\begin{cases}-0.6 & \text{if } x\in A,\\ -0.3 & \text{if } x\notin A,\end{cases}$$

and

$$\ell_i^+=(\pi_i\circ\ell)^+:W\to[0,1],z\mapsto\begin{cases}0.8 & \text{if } x\in A,\\ 0.4 & \text{if } x\notin A,\end{cases}$$

for $i=1,2,\cdots,k$. Then, (W,ℓ) is a k-fold bipolar fuzzy ideal of W. If we take $a=(0,0)$ and $b=(0,2)$, then $a*b=(0,0)*(0,2)=(0,-2)\notin A$. Hence,

$$(\pi_i\circ\ell)^-(a*b)=-0.3>-0.6=\max\{(\pi_i\circ\ell)^-(a),(\pi_i\circ\ell)^-(b)\}$$

and/or

$$(\pi_i\circ\ell)^+(a*b)=0.4<0.8=\min\{(\pi_i\circ\ell)^+(a),(\pi_i\circ\ell)^+(b)\}.$$

Therefore, (W, ℓ) is not a k-fold bipolar fuzzy subalgebra of W.

We provide a condition for which Theorem 2 is valid in BCI-algebras.

Theorem 4. *If a k-fold bipolar fuzzy ideal (W, ℓ) of a BCI-algebra W satisfies the following condition:*

$$\{((\pi_i \circ \ell)^-(0 * a), (\pi_i \circ \ell)^+(0 * z))\}_{i=1}^k \ll \{((\pi_i \circ \ell)^-(a), (\pi_i \circ \ell)^+(z))\}_{i=1}^k \tag{17}$$

for all $a, z \in W$, then (W, ℓ) is a k-fold bipolar fuzzy subalgebra of W.

Proof. Using Condition (III) and Equations (3) and (13), we have the following:

$$\begin{aligned}
&\{((\pi_i \circ \ell)^-(a * b), (\pi_i \circ \ell)^+(z * x))\}_{i=1}^k \\
&\ll \big\{\big(\max\{(\pi_i \circ \ell)^-((a * b) * a), (\pi_i \circ \ell)^-(a)\}, \min\{(\pi_i \circ \ell)^+((z * x) * z), (\pi_i \circ \ell)^+(z)\}\big)\big\}_{i=1}^k \\
&= \big\{\big(\max\{(\pi_i \circ \ell)^-((a * a) * b), (\pi_i \circ \ell)^-(a)\}, \min\{(\pi_i \circ \ell)^+((z * z) * x), (\pi_i \circ \ell)^+(z)\}\big)\big\}_{i=1}^k \\
&= \big\{\big(\max\{(\pi_i \circ \ell)^-(0 * b), (\pi_i \circ \ell)^-(a)\}, \min\{(\pi_i \circ \ell)^+(0 * x), (\pi_i \circ \ell)^+(z)\}\big)\big\}_{i=1}^k \\
&\ll \big\{\big(\max\{(\pi_i \circ \ell)^-(b), (\pi_i \circ \ell)^-(a)\}, \min\{(\pi_i \circ \ell)^+(x), (\pi_i \circ \ell)^+(z)\}\big)\big\}_{i=1}^k
\end{aligned}$$

for all $a, b, z, x \in W$. Therefore, (W, ℓ) is a k-fold bipolar fuzzy subalgebra of W. □

Theorem 5. *A k-fold bipolar fuzzy set (W, ℓ) over a BCK/BCI-algebra W is a k-fold bipolar fuzzy ideal of W if and only if the positive cut $P(\ell, t_i^+)$ and negative cut $N(\ell, t_i^-)$ of (W, ℓ) are ideals of W for all $\{(t_i^-, t_i^+)\}_{i=1}^k \in ([-1, 0] \times [0, 1])^k$.*

Proof. Assume that (W, ℓ) is a k-fold bipolar fuzzy ideal of W. Let $\{(t_i^-, t_i^+)\}_{i=1}^k \in ([-1, 0] \times [0, 1])^k$ be such that the positive cut $P(\ell, t_i^+)$ and negative cut $N(\ell, t_i^-)$ of (W, ℓ) are non-empty. Then, there exists $z \in P(\ell, t_i^+)$ and $a \in N(\ell, t_i^-)$, and therefore, $(\pi_i \circ \ell)^-(a) = \ell_i^-(a) \leq t_i^-$ and $(\pi_i \circ \ell)^+(z) = \ell_i^+(z) \geq t_i^+$. It follows from Equation (12) that

$$\{((\pi_i \circ \ell)^-(0), (\pi_i \circ \ell)^+(0))\}_{i=1}^k \ll \{((\pi_i \circ \ell)^-(a), (\pi_i \circ \ell)^+(z))\}_{i=1}^k = \{(t_i^{-1}, t_i^+)\}_{i=1}^k.$$

Hence, $0 \in P(\ell, t_i^+)$ and $0 \in N(\ell, t_i^-)$ for all $i = 1, 2, \cdots, k$. Let $a, b, z, x \in W$ be such that $a * b \in N(\ell, t_i^-)$, $b \in N(\ell, t_i^-)$, $z * x \in P(\ell, t_i^+)$ and $x \in P(\ell, t_i^+)$ for all $i = 1, 2, \cdots, k$. Then, $\ell_i^-(a * b) \leq t_i^-$, $\ell_i^-(b) \leq t_i^-$, $\ell_i^+(z * x) \geq t_i^+$ and $\ell_i^+(x) \geq t_i^+$. It follows from Equation (13) that

$$\begin{aligned}
&\{\ell_i^-(a), \ell_i^+(z)\}_{i=1}^k = \{((\pi_i \circ \ell)^-(a), (\pi_i \circ \ell)^+(z))\}_{i=1}^k \\
&\ll \big\{\big(\max\{(\pi_i \circ \ell)^-(a * b), (\pi_i \circ \ell)^-(b)\}, \min\{(\pi_i \circ \ell)^+(z * x), (\pi_i \circ \ell)^+(x)\}\big)\big\}_{i=1}^k \\
&= \big\{\big(\max\{\ell_i^-(a * b), \ell_i^-(b)\}, \min\{\ell_i^+(z * x), \ell_i^+(x)\}\big)\big\}_{i=1}^k \\
&\ll \big\{\big(\max\{t_i^-, t_i^-\}, \min\{t_i^+, t_i^+\}\big)\big\}_{i=1}^k = \{t_i^-, t_i^+\}_{i=1}^k.
\end{aligned}$$

Hence, $a \in N(\ell, t_i^-)$ and $z \in P(\ell, t_i^+)$. Therefore, $P(\ell, t_i^+)$ and $N(\ell, t_i^-)$ are ideals of W for all $\{(t_i^-, t_i^+)\}_{i=1}^k \in ([-1, 0] \times [0, 1])^k$.

Conversely, suppose that the positive cut $P(\ell, t_i^+)$ and negative cut $N(\ell, t_i^-)$ of (W, ℓ) are ideals of W for all $\{(t_i^-, t_i^+)\}_{i=1}^k \in ([-1,0] \times [0,1])^k$. If (W, ℓ) is not a k-fold bipolar ideal of W, then Equation (12) is false or Equation (13) is false. Assume that Equation (12) is false, that is,

$$\{((\pi_i \circ \ell)^-(0), (\pi_i \circ \ell)^+(0))\}_{i=1}^k \neg \ll \{((\pi_i \circ \ell)^-(a), (\pi_i \circ \ell)^+(z))\}_{i=1}^k$$

for some $a, z \in W$. Then, $\ell_i^-(0) = (\pi_i \circ \ell)^-(0) > (\pi_i \circ \ell)^-(a) = \ell_i^-(a)$ or $\ell_i^+(0) = (\pi_i \circ \ell)^+(0) < (\pi_i \circ \ell)^+(z) = \ell_i^+(z)$, which imply that $0 \notin N(\ell, t_i^-)$ or $0 \notin P(\ell, t_i^+)$, where $t_i^- = \ell_i^-(a)$ and $t_i^+ = \ell_i^+(z)$. This is a contradiction, and so, Equation (12) is valid.

If Equation (13) is false, then there exists $i \in \{1, 2, \cdots, k\}$ such that

$$\ell_i^-(a) > \max\{\ell_i^-(a * b), \ell_i^-(b)\} \text{ or } \ell_i^+(z * x) < \min\{\ell_i^+(z), \ell_i^+(x)\}$$

for some $a, b, z, x \in W$. Taking $t_i^- := \max\{\ell_i^-(a * b), \ell_i^-(b)\}$, and $t_i^+ := \min\{\ell_i^+(z), \ell_i^+(x)\}$ induces the following:

$$a * b \in N(\ell, t_i^-), b \in N(\ell, t_i^-) \text{ and } a \notin N(\ell, t_i^-)$$

or

$$z * x \in P(\ell, t_i^+), x \in P(\ell, t_i^+) \text{ and } z \notin P(\ell, t_i^+).$$

This is a contradiction, and hence, Equation (13) is valid. Consequently, (W, ℓ) is a k-fold bipolar fuzzy ideal of W. □

Corollary 2. *If (W, ℓ) is a k-fold bipolar fuzzy ideal of a BCK/BCI-algebra W, then the intersection of the positive cut $P(\ell, t_i^+)$ and negative cut $N(\ell, t_i^-)$ of (W, ℓ) is an ideal of W for all $\{(t_i^-, t_i^+)\}_{i=1}^k \in ([-1,0] \times [0,1])^k$. In particular, the t_i-cut of (W, ℓ) is an ideal of W for all $t_i \in [0,1]$.*

The following example shows that there exists $\{(t_i^-, t_i^+)\}_{i=1}^k \in ([-1,0] \times [0,1])^k$ such that, if (W, ℓ) is a k-fold bipolar fuzzy ideal of a BCK/BCI-algebra W, then the union of positive cut $P(\ell, t_i^+)$ and negative cut $N(\ell, t_i^-)$ of (W, ℓ) may not be an ideal of W.

Example 6. *Consider a BCI-algebra $W = \{0, a, b, c\}$ with the binary operation $*$ in Table 7.*

Let (W, ℓ) be a 5-fold bipolar fuzzy set over W given by Table 8.

It is routine to verify that (W, ℓ) is a 5-fold bipolar fuzzy ideal of W and

$$N(\ell, t_i^-) \cup P(\ell, t_i^+) = \begin{cases} \{0, a, b\} & \text{if } (t_i^-, t_i^+) = (-0.6, 0.5) \text{ for } i = 1, \\ \{0, a, c\} & \text{if } (t_i^-, t_i^+) = (-0.4, 0.4) \text{ for } i = 2, \\ \{0, b, c\} & \text{if } (t_i^-, t_i^+) = (-0.2, 0.3) \text{ for } i = 3, \\ \{0, a, b\} & \text{if } (t_i^-, t_i^+) = (-0.5, 0.6) \text{ for } i = 4, \\ \{0, b, c\} & \text{if } (t_i^-, t_i^+) = (-0.3, 0.4) \text{ for } i = 5, \end{cases}$$

which are not ideals of W for $i = 1, 2, 3, 4, 5$.

Table 7. Cayley table for the binary operation "$*$".

$*$	**0**	***a***	***b***	***c***
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

Table 8. Tabular representation of "ℓ".

W	**0**	***a***	***b***	***c***
$(\ell_1^-(z), \ell_1^+(z))$	$(-0.8, 0.7)$	$(-0.3, 0.6)$	$(-0.7, 0.4)$	$(-0.3, 0.4)$
$(\ell_2^-(z), \ell_2^+(z))$	$(-0.5, 0.6)$	$(-0.5, 0.3)$	$(-0.2, 0.3)$	$(-0.2, 0.5)$
$(\ell_3^-(z), \ell_3^+(z))$	$(-0.7, 0.6)$	$(-0.1, 0.2)$	$(-0.6, 0.2)$	$(-0.1, 0.5)$
$(\ell_4^-(z), \ell_4^+(z))$	$(-0.9, 0.8)$	$(-0.4, 0.7)$	$(-0.6, 0.5)$	$(-0.4, 0.5)$
$(\ell_5^-(z), \ell_5^+(z))$	$(-0.6, 0.5)$	$(-0.2, 0.2)$	$(-0.5, 0.2)$	$(-0.2, 0.5)$

We provide conditions for the union of negative cut and positive cut of a k-fold bipolar fuzzy ideal (W, ℓ) to be an ideal of W.

Theorem 6. *If (W, ℓ) is a k-fold bipolar fuzzy ideal of W such that*

$$(\forall z \in W)(\ell_i^-(z) + \ell_i^+(z) \geq 0) \tag{18}$$

for $i = 1, 2, \cdots, k$, then the union of negative cut $N(\ell, t_i^-)$ and positive cut $P(\ell, t_i^+)$ of (W, ℓ) is an ideal of W for all $\{(t_i^-, t_i^+)\}_{i=1}^k \in ([-1, 0] \times [0, 1])^k$ with $t_i^- = -t_i^+$.

Proof. Let $\{(t_i^-, t_i^+)\}_{i=1}^k \in ([-1, 0] \times [0, 1])^k$ with $t_i^- = -t_i^+$. If (W, ℓ) is a k-fold bipolar fuzzy ideal of W, then $N(\ell, t_i^-)$ and $P(\ell, t_i^+)$ are ideals of W by Theorem 5. Hence, $0 \in N(\ell, t_i^-) \cup P(\ell, t_i^+)$. Let $z, x \in W$ be such that $z * x \in N(\ell, t_i^-) \cup P(\ell, t_i^+)$ and $x \in N(\ell, t_i^-) \cup P(\ell, t_i^+)$. Then, we can consider the following four cases:

(i) $z * x \in P(\ell, t_i^+)$ and $x \in P(\ell, t_i^+)$,
(ii) $z * x \in N(\ell, t_i^-)$ and $x \in N(\ell, t_i^-)$,
(iii) $z * x \in P(\ell, t_i^+)$ and $x \in N(\ell, t_i^-)$,
(iv) $z * x \in N(\ell, t_i^-)$ and $x \in P(\ell, t_i^+)$.

From cases (i) and (ii), we have $z \in P(\ell, t_i^+) \subseteq N(\ell, t_i^-) \cup P(\ell, t_i^+)$ and $z \in N(\ell, t_i^-) \subseteq N(\ell, t_i^-) \cup P(\ell, t_i^+)$, respectively. For the third case, we get $\ell_i^+(z * x) \geq t_i^+$ and $\ell_i^-(x) \leq t_i^- = -t_i^+$. For the final case, we have $\ell_i^-(z * x) \leq t_i^- = -t_i^+$ and $\ell_i^+(x) \geq t_i^+$. It follows from Equations (13) and (18) that

$$\begin{aligned}
\{(\ell_i^-(z), \ell_i^+(z))\}_{i=1}^k &= \{((\pi_i \circ \ell)^-(z), (\pi_i \circ \ell)^+(z))\}_{i=1}^k \\
&\ll \left\{\left(\max\{(\pi_i \circ \ell)^-(z * x), (\pi_i \circ \ell)^-(x)\}, \min\{(\pi_i \circ \ell)^+(z * x), (\pi_i \circ \ell)^+(x)\}\right)\right\}_{i=1}^k \\
&= \left\{\left(\max\{\ell_i^-(z * x), \ell_i^-(x)\}, \min\{\ell_i^+(z * x), \ell_i^+(x)\}\right)\right\}_{i=1}^k \\
&\ll \left\{\left(\max\{\ell_i^-(z * x), \ell_i^-(x)\}, t_i^+\right)\right\}_{i=1}^k.
\end{aligned}$$

Hence, $\ell_i^+(z) \geq t_i^+$, and so, $z \in P(\ell, t_i^+) \subseteq N(\ell, t_i^-) \cup P(\ell, t_i^+)$. Therefore, $N(\ell, t_i^-) \cup P(\ell, t_i^+)$ is an ideal of W. □

6. Translated *k*-Fold Bipolar Fuzzy Sets

For any k-fold bipolar fuzzy set (W, ℓ) over W, we denote

$$\perp_i := -1 - \inf\{\ell_i^-(z) \mid z \in W\} \text{ and } \top_i := 1 - \sup\{\ell_i^+(z) \mid z \in W\}$$

for $i = 1, 2, \cdots, k$.

Definition 4. *Let (W, ℓ) be a k-fold bipolar fuzzy set over W and $\{(\alpha_i, \beta_i)\}_{i=1}^k \in ([\perp_i, 0] \times [0, \top_i])^k$. By a $\{(\alpha_i, \beta_i)\}_{i=1}^k$-translated k-fold bipolar fuzzy set of (W, ℓ), we mean a k-fold bipolar fuzzy set $(W, T(\ell))$ over W in which*

$$T(\ell) : W \to ([-1, 0] \times [0, 1])^k, \ x \mapsto \{(\ell_{\alpha_i}^-(z), \ell_{\beta_i}^+(z))\}_{i=1}^k, \tag{19}$$

where

$$\begin{aligned} &\ell_{\alpha_i}^- : W \to [-1, 0], \ z \mapsto \ell_i^-(z) + \alpha_i = (\pi_i \circ \ell)^-(x) + \alpha_i \\ &\ell_{\beta_i}^+ : W \to [0, 1], \ z \mapsto \ell_i^+(z) + \beta_i = (\pi_i \circ \ell)^+(x) + \beta_i. \end{aligned}$$

Theorem 7. *Every $\{(\alpha_i, \beta_i)\}_{i=1}^k$-translated k-fold bipolar fuzzy set of a k-fold bipolar fuzzy subalgebra is a k-fold bipolar fuzzy subalgebra.*

Proof. Let $(W, T(\ell))$ be a $\{(\alpha_i, \beta_i)\}_{i=1}^k$-translated k-fold bipolar fuzzy set of a k-fold bipolar fuzzy subalgebra (W, ℓ). For any $a, b, z, x \in W$, we have

$$\begin{aligned} &\{(\ell_{\alpha_i}^-(a * b), \ell_{\beta_i}^+(z * x))\}_{i=1}^k = \{((\pi_i \circ \ell)^-(a * b) + \alpha_i, (\pi_i \circ \ell)^+(z * x) + \beta_i)\}_{i=1}^k \\ &\ll \left\{\left(\max\{(\pi_i \circ \ell)^-(a), (\pi_i \circ \ell)^-(b)\} + \alpha_i, \min\{(\pi_i \circ \ell)^+(z), (\pi_i \circ \ell)^+(x)\} + \beta_i\right)\right\}_{i=1}^k \\ &= \left\{\left(\max\{(\pi_i \circ \ell)^-(a) + \alpha_i, (\pi_i \circ \ell)^-(b) + \alpha_i\}, \min\{(\pi_i \circ \ell)^+(z) + \beta_i, (\pi_i \circ \ell)^+(x) + \beta_i\}\right)\right\}_{i=1}^k \\ &= \left\{\left(\max\{\ell_{\alpha_i}^-(a), \ell_{\alpha_i}^-(b)\}, \min\{\ell_{\beta_i}^+(z), \ell_{\beta_i}^+(x)\}\right)\right\}_{i=1}^k. \end{aligned}$$

Therefore, $(W, T(\ell))$ is a k-fold bipolar fuzzy subalgebra of W. □

Theorem 8. *Let (W, ℓ) be a k-fold bipolar fuzzy set over W such that its $\{(\alpha_i, \beta_i)\}_{i=1}^k$-translated k-fold bipolar fuzzy set is a k-fold bipolar fuzzy subalgebra of W for some $\{(\alpha_i, \beta_i)\}_{i=1}^k \in ([\perp_i, 0] \times [0, \top_i])^k$. Then, (W, ℓ) is a k-fold bipolar fuzzy subalgebra of W.*

Proof. Let $(W, T(\ell))$ be a $\{(\alpha_i, \beta_i)\}_{i=1}^k$-translated k-fold bipolar fuzzy set which is a k-fold bipolar fuzzy subalgebra of W for some $\{(\alpha_i, \beta_i)\}_{i=1}^k \in ([\perp_i, 0] \times [0, \top_i])^k$. For any $a, b, z, x \in W$, we have

$$\begin{aligned} &\{((\pi_i \circ \ell)^-(a * b) + \alpha_i, (\pi_i \circ \ell)^+(z * x) + \beta_i)\}_{i=1}^k \\ &= \{(\ell_{\alpha_i}^-(a * b), \ell_{\beta_i}^+(z * x))\}_{i=1}^k \\ &\ll \left\{\left(\max\{\ell_{\alpha_i}^-(a), \ell_{\alpha_i}^-(b)\}, \min\{\ell_{\beta_i}^+(z), \ell_{\beta_i}^+(x)\}\right)\right\}_{i=1}^k \\ &= \left\{\left(\max\{(\pi_i \circ \ell)^-(a) + \alpha_i, (\pi_i \circ \ell)^-(b) + \alpha_i\}, \min\{(\pi_i \circ \ell)^+(z) + \beta_i, (\pi_i \circ \ell)^+(x) + \beta_i\}\right)\right\}_{i=1}^k \\ &= \left\{\left(\max\{(\pi_i \circ \ell)^-(a), (\pi_i \circ \ell)^-(b)\} + \alpha_i, \min\{(\pi_i \circ \ell)^+(z), (\pi_i \circ \ell)^+(x)\} + \beta_i\right)\right\}_{i=1}^k. \end{aligned}$$

It follows that

$$\{((\pi_i \circ \ell)^-(a * b), (\pi_i \circ \ell)^+(z * x))\}_{i=1}^k$$
$$\ll \{(\max\{(\pi_i \circ \ell)^-(a), (\pi_i \circ \ell)^-(b)\}, \min\{(\pi_i \circ \ell)^+(z), (\pi_i \circ \ell)^+(x)\})\}_{i=1}^k.$$

Therefore, (W, ℓ) is a k-fold bipolar fuzzy subalgebra of W. □

Definition 5. *Let (W, ℓ) and $(W, \jmath)$ be k-fold bipolar fuzzy sets over W. Then, $(W, \jmath)$ is called an extension of (W, ℓ) based on a subalgebra (briefly, S-extension of (W, ℓ)) if the following assertions are valid.*

(i) $\{((\pi_i \circ \jmath)^-(a), (\pi_i \circ \jmath)^+(z))\}_{i=1}^k \ll \{((\pi_i \circ \ell)^-(a), (\pi_i \circ \ell)^+(z))\}_{i=1}^k$ *for all $a, z \in W$*
(ii) *If (W, ℓ) is a k-fold bipolar fuzzy subalgebra of W, then $(W, \jmath)$ is a k-fold bipolar fuzzy subalgebra of W.*

Theorem 9. *Let (W, ℓ) be a k-fold bipolar fuzzy subalgebra of W, and let $\{(\alpha_i, \beta_i)\}_{i=1}^k \in ([\perp_i, 0] \times [0, \top_i])^k$. Then, the $\{(\alpha_i, \beta_i)\}_{i=1}^k$-translated k-fold bipolar fuzzy set $(W, T(\ell))$ of (W, ℓ) is an S-extension of (W, ℓ).*

Proof. For any $a, z \in W$, we have

$$\{(\ell_{\alpha_i}^-(a), \ell_{\beta_i}^+(z))\}_{i=1}^k = \{((\pi_i \circ \ell)^-(a) + \alpha_i, (\pi_i \circ \ell)^+(z) + \beta_i)\}_{i=1}^k$$
$$\ll \{((\pi_i \circ \ell)^-(a), (\pi_i \circ \ell)^+(z))\}_{i=1}^k.$$

Now, if (W, ℓ) is a k-fold bipolar fuzzy subalgebra of W, then $(W, T(\ell))$ is a k-fold bipolar fuzzy subalgebra of W by Theorem 7. Therefore, the $\{(\alpha_i, \beta_i)\}_{i=1}^k$-translated k-fold bipolar fuzzy set $(W, T(\ell))$ of (W, ℓ) is an S-extension of (W, ℓ). □

The converse of Theorem 9 is not true in general, as seen in the following example.

Example 7. *Consider a BCK-algebra $W = \{0, 1, 2, 3, 4\}$ with the binary operation $*$ in Table 9.*
Let (W, ℓ) be a 2-fold bipolar fuzzy set over W given by Table 10.
Then, (W, ℓ) is a 2-fold bipolar fuzzy subalgebra of W.
Let $(W, \jmath)$ be a 2-fold bipolar fuzzy set over W given by Table 11.
Then, $(W, \jmath)$ is a 2-fold bipolar fuzzy subalgebra of W, which is an S-extension of (W, ℓ). However, it is not the $\{(\alpha_i, \beta_i)\}_{i=1}^k$-translated one.

Table 9. Cayley table for the binary operation "$*$".

$*$	**0**	**1**	**2**	**3**	**4**
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	2	0
3	3	1	3	0	1
4	4	4	4	4	0

Table 10. Tabular representation of "ℓ".

W	0	1	2	3	4
$(\ell_1^-(z), \ell_1^+(z))$	$(-0.6, 0.8)$	$(-0.4, 0.5)$	$(-0.3, 0.3)$	$(-0.5, 0.6)$	$(-0.1, 0.2)$
$(\ell_2^-(z), \ell_2^+(z))$	$(-0.9, 0.7)$	$(-0.5, 0.6)$	$(-0.5, 0.4)$	$(-0.8, 0.6)$	$(-0.2, 0.2)$

Table 11. Tabular representation of "$\jmath$".

W	0	1	2	3	4
$(\jmath_1^-(z), \jmath_1^+(z))$	$(-0.63, 0.84)$	$(-0.42, 0.56)$	$(-0.37, 0.38)$	$(-0.54, 0.67)$	$(-0.16, 0.21)$
$(\jmath_2^-(z), \jmath_2^+(z))$	$(-0.93, 0.73)$	$(-0.56, 0.66)$	$(-0.57, 0.47)$	$(-0.88, 0.68)$	$(-0.25, 0.25)$

Theorem 10. *Let (W, ℓ) be a k-fold bipolar fuzzy subalgebra of W. If $(W, \jmath)$ and (W, κ) are S-extensions of (W, ℓ), then the intersection of $(W, \jmath)$ and (W, κ) is also an S-extension of (W, ℓ).*

Proof. For any $a, b, z, x \in W$, we have

$$\begin{aligned}
&\left\{\left((\jmath \cap \kappa)_i^-(a * b), (\jmath \cap \kappa)_i^+(z * x)\right)\right\}_{i=1}^k \\
&= \left\{\left(\max\{\jmath_i^-(a * b), \kappa_i^-(a * b)\}, \min\{\jmath_i^+(z * x), \kappa_i^+(z * x)\}\right)\right\}_{i=1}^k \\
&\ll \left\{\left(\max\{\max\{\jmath_i^-(a), \jmath_i^-(b)\}, \max\{\kappa_i^-(a), \kappa_i^-(b)\}\}, \min\{\min\{\jmath_i^+(z), \jmath_i^+(x)\}, \min\{\kappa_i^+(z), \kappa_i^+(x)\}\}\right)\right\}_{i=1}^k \\
&= \left\{\left(\max\{\max\{\jmath_i^-(a), \kappa_i^-(a)\}, \max\{\jmath_i^-(b), \kappa_i^-(b)\}\}, \min\{\min\{\jmath_i^+(z), \kappa_i^+(z)\}, \min\{\jmath_i^+(x), \kappa_i^+(x)\}\}\right)\right\}_{i=1}^k \\
&= \left\{\left(\max\{(\jmath \cap \kappa)_i^-(a), (\jmath \cap \kappa)_i^-(b)\}, \min\{(\jmath \cap \kappa)_i^+(z), (\jmath \cap \kappa)_i^+(x)\}\right)\right\}_{i=1}^k
\end{aligned}$$

Therefore, $(W, \jmath \cap \kappa)$ is an S-extension of (W, ℓ). □

The following example shows that the union of two S-extensions of a k-fold bipolar fuzzy subalgebra (W, ℓ) of W may not be an S-extension of (W, ℓ).

Example 8. *Consider a BCK-algebra $W = \{0, 1, 2, 3, 4\}$ with the binary operation $*$ in Table 12.*

Let (W, ℓ) be a 3-fold bipolar fuzzy set over W given by Table 13.

Then, (W, ℓ) is a 3-fold bipolar fuzzy subalgebra of W.

Let $(W, \jmath)$ and (W, κ) be 3-fold bipolar fuzzy sets over W given by Tables 14 and 15, respectively.

Then, $(W, \jmath)$ and (W, κ) are S-extensions of (W, ℓ). The union $(W, \jmath \cup \kappa)$ of $(W, \jmath)$ and (W, κ) is given by Table 16.

We know that

$$\begin{aligned}
&\left\{\left((\jmath \cup \kappa)_i^-(4 * 2), (\jmath \cup \kappa)_i^+(3 * 4)\right)\right\}_{i=1}^3 \\
&\neg \ll \left\{\left(\max\{(\jmath \cup \kappa)_i^-(4), (\jmath \cup \kappa)_i^-(2)\}, \min\{(\jmath \cup \kappa)_i^+(3), (\jmath \cup \kappa)_i^+(4)\}\right)\right\}_{i=1}^3
\end{aligned}$$

since $(\jmath \cup \kappa)_1^+(4 * 2) = (\jmath \cup \kappa)_1^+(3) = \max\{\jmath_1^+(3), \kappa_1^+(3)\} = 0.6 \ngeq 0.7 = \min\{(\jmath \cup \kappa)_1^+(4), (\jmath \cup \kappa)_1^+(2)\}$. *Therefore $(W, \jmath \cup \kappa)$ is not S-extension of (W, ℓ).*

Table 12. Cayley table for the binary operation "$*$".

$*$	**0**	**1**	**2**	**3**	**4**
0	0	0	0	0	0
1	1	0	0	0	0
2	2	1	0	0	0
3	3	1	1	0	0
4	4	3	3	1	0

Table 13. Tabular representation of "ℓ".

W	**0**	**1**	**2**	**3**	**4**
$(\ell_1^-(z), \ell_1^+(z))$	$(-0.6, 0.7)$	$(-0.3, 0.4)$	$(-0.5, 0.6)$	$(-0.2, 0.3)$	$(-0.1, 0.3)$
$(\ell_2^-(z), \ell_2^+(z))$	$(-0.8, 0.6)$	$(-0.7, 0.5)$	$(-0.5, 0.6)$	$(-0.4, 0.3)$	$(-0.2, 0.1)$
$(\ell_3^-(z), \ell_3^+(z))$	$(-0.5, 0.5)$	$(-0.3, 0.4)$	$(-0.1, 0.3)$	$(-0.3, 0.5)$	$(-0.5, 0.2)$

Table 14. Tabular representation of "$\jmath$".

W	**0**	**1**	**2**	**3**	**4**
$(\jmath_1^-(z), \jmath_1^+(z))$	$(-0.66, 0.8)$	$(-0.33, 0.6)$	$(-0.55, 0.8)$	$(-0.22, 0.4)$	$(-0.2, 0.4)$
$(\jmath_2^-(z), \jmath_2^+(z))$	$(-0.87, 0.67)$	$(-0.76, 0.56)$	$(-0.58, 0.68)$	$(-0.45, 0.35)$	$(-0.24, 0.14)$
$(\jmath_3^-(z), \jmath_3^+(z))$	$(-0.56, 0.53)$	$(-0.36, 0.43)$	$(-0.16, 0.33)$	$(-0.36, 0.53)$	$(-0.56, 0.23)$

Table 15. Tabular representation of "κ".

W	**0**	**1**	**2**	**3**	**4**
$(\kappa_1^-(z), \kappa_1^+(z))$	$(-0.7, 0.9)$	$(-0.4, 0.6)$	$(-0.6, 0.6)$	$(-0.4, 0.6)$	$(-0.3, 0.7)$
$(\kappa_2^-(z), \kappa_2^+(z))$	$(-0.89, 0.67)$	$(-0.76, 0.58)$	$(-0.59, 0.68)$	$(-0.43, 0.35)$	$(-0.28, 0.14)$
$(\kappa_3^-(z), \kappa_3^+(z))$	$(-0.60, 0.53)$	$(-0.39, 0.45)$	$(-0.18, 0.33)$	$(-0.36, 0.57)$	$(-0.56, 0.29)$

Table 16. Tabular representation of "$\jmath \cup \kappa$".

W	**0**	**1**	**2**	**3**	**4**
$((\jmath \cup \kappa)_1^-(z), (\jmath \cup \kappa)_1^+(z))$	$(-0.7, 0.9)$	$(-0.4, 0.6)$	$(-0.6, 0.8)$	$(-0.4, 0.6)$	$(-0.3, 0.7)$
$((\jmath \cup \kappa)_2^-(z), (\jmath \cup \kappa)_2^+(z))$	$(-0.89, 0.67)$	$(-0.76, 0.58)$	$(-0.59, 0.68)$	$(-0.45, 0.35)$	$(-0.28, 0.14)$
$((\jmath \cup \kappa)_3^-(z), (\jmath \cup \kappa)_3^+(z))$	$(-0.60, 0.53)$	$(-0.39, 0.45)$	$(-0.18, 0.33)$	$(-0.36, 0.57)$	$(-0.56, 0.29)$

Let (W, ℓ) be a k-fold bipolar fuzzy sets over W, and consider the following sets:

$$N_{\alpha_i}(\ell, t_i^-) = \{z \in W \mid \ell_i^-(z) \leq t_i^- - \alpha_i\},$$
$$P_{\beta_i}(\ell, t_i^+) = \{z \in W \mid \ell_i^+(z) \geq t_i^+ - \beta_i\},$$

where $\{(\alpha_i, \beta_i)\}_{i=1}^k \in ([\perp_i, 0] \times [0, \top_i])^k$ and $(t_i^-, t_i^+) \in [-1, \alpha_i] \times [\beta_i, 1]$ for $i = 1, 2, \cdots, k$.

If (W, ℓ) is a k-fold bipolar fuzzy subalgebra of W, then it is clear that $N_{\alpha_i}(\ell, t_i^-)$ and $P_{\beta_i}(\ell, t_i^+)$ are subalgebras of W for all $(t_i^-, t_i^+) \in \mathrm{Im}(\ell_i^-) \times \mathrm{Im}(\ell_i^+)$ with $t_i^- \leq \alpha_i$ and $t_i^+ \geq \beta_i$ for $i = 1, 2, \cdots, k$. However, if we do not give a condition that (W,ℓ) is a k-fold bipolar fuzzy subalgebra of W, then $N_{\alpha_i}(\ell, t_i^-)$ and/or $P_{\beta_i}(\ell, t_i^+)$ are not subalgebras of W, as seen in the following example.

Example 9. *Consider the BCK-algebra $W = \{0,1,2,3,4\}$ which is given in Example 8. Let (W,ℓ) be a 2-fold bipolar fuzzy set over W given by Table 17.*

Then, (W,ℓ) is not a 2-fold bipolar fuzzy subalgebra of W since

$$\begin{aligned}&\{((\pi_i\circ\ell)^-(4*2),(\pi_i\circ\ell)^+(3*4))\}_{i=1}^2\\ &\neg\ll\left\{\left(\max\{(\pi_i\circ\ell)^-(4),(\pi_i\circ\ell)^-(2)\},\min\{(\pi_i\circ\ell)^+(3),(\pi_i\circ\ell)^+(4)\}\right)\right\}_{i=1}^2.\end{aligned}$$

If we take $(\alpha_1,\beta_1)=(-0.15,0.1)$ and $(t_1^-,t_1^+)\in(-0.5,0.5)$, then $N_{\alpha_1}(\ell,t_1^-)=\{0,1,2\}$ is a subalgebra of W, but $P_{\beta_1}(\ell,t_1^+)=\{0,1,2,4\}$ is not a subalgebra of W.

Table 17. Tabular representation of "ℓ".

W	0	1	2	3	4
$(\ell_1^-(z),\ell_1^+(z))$	$(-0.8,0.7)$	$(-0.4,0.4)$	$(-0.7,0.6)$	$(-0.2,0.3)$	$(-0.2,0.5)$
$(\ell_2^-(z),\ell_2^+(z))$	$(-0.6,0.5)$	$(-0.3,0.4)$	$(-0.3,0.2)$	$(-0.6,0.4)$	$(-0.1,0.1)$

Theorem 11. *Let (W,ℓ) be a k-fold bipolar fuzzy set over W and $\{(\alpha_i,\beta_i)\}_{i=1}^k\in([\bot_i,0]\times[0,\top_i])^k$. Then, the $\{(\alpha_i,\beta_i)\}_{i=1}^k$-translated k-fold bipolar fuzzy set $(W,T(\ell))$ of (W,ℓ) is a k-fold bipolar fuzzy subalgebra of W if and only if $N_{\alpha_i}(\ell,t_i^-)$ and $P_{\beta_i}(\ell,t_i^+)$ are subalgebras of W for all $(t_i^-,t_i^+)\in\mathrm{Im}(\ell_i^-)\times\mathrm{Im}(\ell_i^+)$ with $t_i^-\le\alpha_i$ and $t_i^+\ge\beta_i$ for $i=1,2,\cdots,k$.*

Proof. Necessity is clear. To prove the sufficiency, assume that there exist $a,b,z,x\in W$ such that

$$\begin{aligned}&\{(\ell_{\alpha_i}^-(a*b),\ell_{\beta_i}^+(z*x))\}_{i=1}^k\\ &\neg\ll\left\{\left(\max\{\ell_{\alpha_i}^-(a),\ell_{\alpha_i}^-(b)\},\min\{\ell_{\beta_i}^+(z),\ell_{\beta_i}^+(x)\}\right)\right\}_{i=1}^k.\end{aligned}$$

Then, $\ell_{\alpha_i}^-(a*b)>\max\{\ell_{\alpha_i}^-(a),\ell_{\alpha_i}^-(b)\}$ or $\ell_{\beta_i}^+(z*x)<\min\{\ell_{\beta_i}^+(z),\ell_{\beta_i}^+(x)\}$. It follows that

$$\ell_{\alpha_i}^-(a*b)>t_i^-\ge\max\{\ell_{\alpha_i}^-(a),\ell_{\alpha_i}^-(b)\}$$

or

$$\ell_{\beta_i}^+(z*x)<t_i^+\le\min\{\ell_{\beta_i}^+(z),\ell_{\beta_i}^+(x)\}$$

for some $(t_i^-,t_i^+)\in[-1,\alpha_i)\times(\beta_i,1]$ for $i=1,2,\cdots,k$. Hence, $a,b\in N_{\alpha_i}(\ell,t_i^-)$ or $z,x\in P_{\beta_i}(\ell,t_i^+)$, but $a*b\notin N_{\alpha_i}(\ell,t_i^-)$ or $z*x\notin P_{\beta_i}(\ell,t_i^+)$. This is a contradiction. Thus,

$$\{(\ell_{\alpha_i}^-(a*b),\ell_{\beta_i}^+(z*x))\}_{i=1}^k\ll\left\{\left(\max\{\ell_{\alpha_i}^-(a),\ell_{\alpha_i}^-(b)\},\min\{\ell_{\beta_i}^+(z),\ell_{\beta_i}^+(x)\}\right)\right\}_{i=1}^k.$$

Therefore, $(W,T(\ell))$ is a k-fold bipolar fuzzy subalgebra of W. □

Theorem 12. *Let (W,ℓ) be a k-fold bipolar fuzzy subalgebra of W and let $\{(\alpha_i,\beta_i)\}_{i=1}^k\in([\bot_i,0]\times[0,\top_i])^k$ and $\{(\varepsilon_i,\delta_i)\}_{i=1}^k\in([\bot_i,0]\times[0,\top_i])^k$. If $\{(\alpha_i,\beta_i)\}_{i=1}^k\ll\{(\varepsilon_i,\delta_i)\}_{i=1}^k$, then the $\{(\alpha_i,\beta_i)\}_{i=1}^k$-translated k-fold* *bipolar fuzzy subalgebra of* (W, ℓ) *is an S-extension of the the* $\{(\varepsilon_i, \delta_i)\}_{i=1}^k$*-translated k-fold bipolar fuzzy subalgebra of* (W, ℓ).

Proof. Let $(W, T(\ell))$ be the $\{(\alpha_i, \beta_i)\}_{i=1}^k$-translated k-fold bipolar fuzzy set of (W, ℓ), and let $(W, Q(\ell))$ be the $\{(\varepsilon_i, \delta_i)\}_{i=1}^k$-translated k-fold bipolar fuzzy set of (W, ℓ). Since (W, ℓ) is a k-fold bipolar fuzzy subalgebra of W, it follows from Theorem 7 that $(W, T(\ell))$ and $(W, Q(\ell))$ are k-fold bipolar fuzzy subalgebras of W. For any $a, z \in W$, we have

$$\begin{aligned} \{(\ell^-_{\alpha_i}(a), \ell^+_{\beta_i}(z))\}_{i=1}^k &= \{((\pi \circ \ell)^-(a) + \alpha_i, (\pi \circ \ell)^+(z) + \beta_i)\}_{i=1}^k \\ &\ll \{((\pi \circ \ell)^-(a) + \varepsilon_i, (\pi \circ \ell)^+(z) + \delta_i)\}_{i=1}^k \\ &= \{(\ell^-_{\varepsilon_i}(a), \ell^+_{\delta_i}(z))\}_{i=1}^k. \end{aligned}$$

This has completed the proof. □

7. Conclusions

The traditional fuzzy set expression cannot distinguish between elements unrelated to the opposite. It is difficult to express differences in components unrelated to the opposing elements in the fuzzy set only if the membership extends over the interval [0, 1]. If a set expression can express this kind of difference, it will be more beneficial than a traditional fuzzy set expression. Based on these observations, Lee introduced an extension of the fuzzy set called the bipolar value fuzzy set in his paper [3]. This concept is being applied from various angles to algebraic structure and applied science, etc. The purpose of this paper is to study the folding of bipolar value fuzzy sets in BCK/BCI-algebras. We have first introduced the notion of k-fold bipolar fuzzy set over a universe and have applied it BCK/BCI-algebras. We have introduced the k-fold bipolar fuzzy subalgebra and the k-fold bipolar fuzzy ideal of BCK/BCI-algebras and have investigated several properties. We have discussed relations between k-fold bipolar fuzzy subalgebra and k-fold bipolar fuzzy ideal and have provided conditions for k-fold bipolar fuzzy subalgebra to be k-fold bipolar fuzzy ideal. We have considered characterizations of k-fold bipolar fuzzy subalgebra/ideal and have introduced the extension of k-fold bipolar fuzzy set. In upcoming research and papers, we will continue to think of these and to define new concepts in some algebraic structures.

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