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Existence, Uniqueness and Exponential Stability of Periodic Solution for Discrete-Time Delayed BAM Neural Networks Based on Coincidence Degree Theory and Graph Theoretic Method

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Abstract: In this work, a general class of discrete time bidirectional associative memory (BAM) neural networks (NNs) is investigated. In this model, discrete and continuously distributed time delays are taken into account. By utilizing this novel method, which incorporates the approach of Kirchhoff's matrix tree theorem in graph theory, Continuation theorem in coincidence degree theory and Lyapunov function, we derive a few sufficient conditions to ensure the existence, uniqueness and exponential stability of the periodic solution of the considered model. At the end of this work, we give a numerical simulation that shows the effectiveness of this work.

Keywords: discrete-time BAMNNs; periodic solution; coincidence degree theory; exponential stability; Krichhoff's matrix tree theorem; time-varying delays

1. Introduction

Differential and difference dynamic models have been intensively investigated because of their significance and applications in areas such as physics, mathematical biology and artificial neural networks [1–12]. In 1988, Bart Kosko first proposed the Bidirectional associative memory neural networks (BAMNNs) [13], a class of Recurrent neural networks which consist of two layers of neurons, namely U-layer and V-layer; neurons in the U-layer are fully interconnected with the neuron in the V-layer, moreover the connections between the neurons are no more when they are in the same layer. Due to its potential applications in optimization [14,15], associative memories [16,17], signal processing [18], pattern recognition [19], and so forth, BAM type NNs are attractive to many researchers [20–22]. Furthermore, time delays are unavoidable due to various reasons such as finite switching speed of amplifier circuit in electrical analog, sudden transmission of signals in NNs and so on [23–25]. Time delays are often encountered in different types of NNs like Hopfield neural networks [26], BAMNNs [27–31], inertial neural networks [32,33], cellular neural networks [34,35], complex neural networks, and so forth [36–38]. It may generate unwanted dynamical response such as stability, instability, oscillation, chaotic, periodic and so forth. Moreover, convergence analysis of BAMNNs have been a recent hot topic for research [21,39].

In practice, time delays are not mandatorily a constant; they may change over time and/or depending on system parameters [40–44]. It is pointed out that most of the studies on delayed neural networks (DNNs) have dealt with the stability issue of discrete time delay [21]. Moreover, NNs have a spatial nature because their parallel pathways have various sizes and lengths of axons, which lead the signal propagation to be no longer instantaneous but distributed during a certain period of time. This behavior can be modeled as distributed delay [39,45]. Furthermore, NNs are the mathematical analog of the human brain or biological NNs. Due to the potential application, NNs were applied in the field of speech recognition, optimization, characteristic recognition, control, time series prediction and so on. In the past few decades, NNs have attracted the considerable attention of researchers, among them most of the considered results and works are in the sense of continuous time and few of them are in discrete time [46]. Moreover, discrete time NNs are described in the application perspective in the fields of engineering and more specifically, numerical simulation.

In many real-world phenomena, periodic motion is common, such as in the biological system, the human brain oscillates periodically, the changing of climates in four seasons, waves and vibrations, and so on. Recently, there has been an increasing number of researchers working on this periodicity of NNs [22,47–50]. Until now, in the study of stability analysis of the NNs, the Lyapunov functional approach played a vital role [51,52]. Moreover, the construction of Lyapunov functional for a large scale system is not an easy job. To overcome this computational complexity, Li et al. [53] proposed the novel graph-theoretic approach. The main advantage of this work is to construct a global Lyapunov function for the large scale systems that are more related to the topological structure. Utilizing the achievements of the pioneering works, a few researchers have initiated their work and applied this approach to their research. For example, in Reference [54], the boundedness of the stochastic van der Pol oscillators was studied by using graph theory and the Lyapunov functional method; in Reference [55], the boundedness of the stochastic differential equations were studied by utilizing the novel approach; in Reference [56] the stability of neural networks was studied with the help of the graph-theoretic approach. However, the graph-theoretic approach is frequently used in the study of stability.

Motivated by the aforementioned works, we investigated the discrete time BAMNNs with mixed time-varying delays which are exponentially stable and have a unique periodic solution. The main contributions of this work are given below:

- According to the survey, there are few works on the exponential stability of periodic solution for discrete time delayed BAMNNs(DDBAMNNs).
- In this manuscript, together with the discrete and continuously distributed delay, the existence and periodic solution of discrete time BAMNNs is firstly proposed in the base of the graph theoretic approach, which generalizes and improves on the existing literature.
- To avoid the complication of finding the Lyapunov function, we construct a suitable Lyapunov function for a vertex system by using the results from graph theory.
- With the help of Lyapunov-Krasovkii functional and coincidence degree theory two types of sufficient criteria are derived that are different from the various techniques in previous works [22,49,57,58].

The rest of this work is organized as follows: in the next section we give the model description of the proposed DDBAMNNs after which we provide the lemma, definition, and assumptions which are used throughout this work. In Section 3, the existence of periodic solutions for DDBAMNNs is achieved by employing the combination of degree theory and Kirchhoff's matrix tree theorem. In Section 4, by constructing the suitable Lyapunov function and the periodic solution conditions, the exponential stability of discrete-time NNs is derived. At the end of this work, to show the exactness of this work, we present a numerical simulation.

2. Preliminaries and Model Description

In this work, the set of all positive integers, non-negative integers and n-dimensional Euclidean space are respectively indicated by $\mathbb{Z}^+$, $\mathbb{Z}_0^+$ and $\mathbb{R}^n$, $\mathbb{R}^{m\times n}$ be denote the set of all $m \times n$ real matrices. The sets $S_T = \{0, 1, 2, ..., T-1\}$ and $\mathbb{S} = \{1, 2, ..., n\}$. The difference operator of $f(x)$ be defined as $\Delta f(w) = f(w+1) - f(w)$. Denote $k, T \in \mathbb{Z}_0^+$, $[a, b]_{\mathbb{Z}} = \{a, a+1, ..., b-1, b\}$, $|.|$ denotes the Euclidean norm.

Graph theory [59]. A digraph $\mathbf{G} = (\mathbf{V}, \mathbf{A})$ with a set of all vertices or nodes $\mathbf{V} = \{1, 2, ..., l\}$ as a neuron and directed edges $\mathbf{A} \subseteq \mathbf{V} \times \mathbf{V}$ the connection between them. A subgraph $\mathbf{H} = (\mathbf{V}_H, \mathbf{A}_H)$ of a digraph $\mathbf{G}$, if it has a same number of vertex set, that is, $\mathbf{V} = \mathbf{V}_H$ then it is known as a spanning subgraph of $\mathbf{G}$. If each arc (k, h) is assigned by a positive weight p_{kh} then the digraph $\mathbf{G}$ is said to be weighted digraph $\mathbf{W}(\mathbf{G})$. A set of all arcs of the digraph $\mathbf{G}$ with distinct vertices $\{P_1, P_2, ..., P_k\}$ is said to be a directed path $\mathbf{P}$ if $\{(P_k, P_{k+1}) : \ k = 1, 2, 3, ..., l-1\}$. The directed path $\mathbf{P}$ is said to be a directed cycle $\mathbf{Q}$ if $P_k = P_1$. A unicyclic graph $\mathbf{U}$ in digraph $\mathbf{G}$ is a subgraph with a disjoint union of rooted trees whose roots form a dicycle. We define the weighted matrix $Z = (z_{kh})_{n\times n}$ of given weighted digraph $\mathbf{W}$ with l vertices whose entry $z_{kh} > 0$ if the weight of arc (k, h) exists, otherwise 0. $\mathbf{G}$ is said to be strongly connected whenever for any two of distinct vertices, there is a dipath to each other. The Laplacian matrix $\mathbf{L}_p$ of $(\mathbf{G}, \mathbf{A})$ is defined by $-l_{kh}$ for $k \neq h$ and if $k = h$ then it must be equal to $\sum\limits_{i\neq k} l_{ki}$.

Coincidence degree theory [60]. Let V and W be respectively the normed vector spaces, the linear mapping $\mathcal{L} : dom\mathcal{L} \to W$ satisfies $\dim\ker\mathcal{L} = co\dim\ker Im\mathcal{L} < \infty$ and $Im\mathcal{L}$ is closed in W, is called the fredholm mapping of index zero, which implies that there exist projection $\mathcal{P} : V \to V$ and $\mathcal{Q} : W \to W$ which are continuous such that $Im\mathcal{P} = \ker\mathcal{L}$ and $Im\mathcal{L} = \ker Q = Im(I - \mathcal{Q})$, which implies that $\mathcal{L}/dom\mathcal{L} \cup \ker\mathcal{P} : (I - \mathcal{P})V \to V$ is invertible, $K_p : Im\mathcal{L} \to \ker P$ and we denote $\mathcal{L}_p^{-1}$ by K_p. If Θ is an open bounded subset of V, the mapping $\mathcal{N}$ will be called $\mathcal{L}$-Compact on $\bar{\Theta}$ if $QN\bar{\Theta}$ is bounded and $K_p(\mathcal{I} - \mathcal{Q})\mathcal{N} : \bar{\Theta} \to V$ is compact, that is, $K_{\mathcal{P},\mathcal{Q}} = K_{\mathcal{P}}(I - Q)$. Since, $Im\mathcal{Q} \cong \ker\mathcal{L}$, there exists an isomorphism $\mathcal{J} : Im\mathcal{Q} \to \ker\mathcal{L}$.

A mathematical description of BAM Neural Networks are considered as follows:

$$\begin{aligned} u_k(i+1) &= -a_k u_k(i) + \sum_{h=1}^{l} b_{kh} f_h(v_h(i)) + I_k \qquad (1) \\ v_k(i+1) &= -p_k v_k(i) + \sum_{h=1}^{l} q_{kh} g_h(u_h(i)) + J_k, \ h = 1, 2, 3, ...l \end{aligned}$$

for $l \geq 2$ to be the number of neurons in the first and second layers respectively; here, for $k \in Z_0^+$, $u_k(i)$ and $v_k(i) \in \mathbb{R}^n$ denotes the position of kth neuron at time $i \in \{1, 2, ..., m\} \in \mathbb{Z}^+$ in both layers and $(u_k(i), v_k(i)) = w_k(i)\,; A = diag\{a_1, a_2, ..., a_n\} > 0\ and\ P = diag\{p_1, p_2, ..., p_n\} > 0$ be the self-feedback matrices; in both layers $B = (b_{kh})_{l\times l}$ and $Q = (q_{kh})_{l\times l}$ in $\mathbb{R}^{n\times m}$ which stands for the connection weight matrices between kth and hth neuron; $f_h : \mathbb{R}^n \times \mathbb{Z}^+ \to \mathbb{R}^n$ and $g_h : \mathbb{R}^n \times \mathbb{Z}^+ \to \mathbb{R}^n$ be denotes the neuronal activation functions and the output functions are represented by $I_k\ and\ J_k$.

As mentioned in the introduction, we introduce the discrete and continuously distributed delay in BAM Neural Networks:

$$\begin{aligned} u_k(i+1) &= -a_k u_k(i) + \sum_{h=1}^{l} b_{kh} f_1(v_h(i)) + \sum_{h=1}^{l} d_{kh} f_2(v_h(i - \tau_{kh}(i))) \\ &\quad + \sum_{h=1}^{l} e_{kh} f_3(\sum_{\nu=1}^{\infty} M_{kh}(\nu) v_h(i - \nu)) + I_k \end{aligned}$$

$$v_k(i+1) = -p_k v_k(i) + \sum_{h=1}^{l} q_{kh} g_1(u_h(i)) + \sum_{h=1}^{l} r_{kh} g_2(u_h(i-\varsigma_{kh}(i))) + \sum_{h=1}^{l} s_{kh} g_3(\sum_{\vartheta=1}^{\infty} O_{kh}(\vartheta) u_h(i-\vartheta)) + J_k; \quad k = 1,2,3,...,l \tag{2}$$

with Initial conditions,

$$u_k(j) = \psi(j), \quad \forall j \in [-\tau, 0]_{\mathbb{Z}}, \quad \tau = \max_{1\le h\le l}\{\tau_{kh}\}$$
$$v_k(j) = \chi(j), \quad \forall j \in [-\varsigma, 0]_{\mathbb{Z}}, \quad \varsigma = \max_{1\le h\le l}\{\varsigma_{kh}\}$$

where the set of real valued map $\psi : [-\tau, 0]_{\mathbb{Z}} \to \mathbb{R}^n$ and $\chi : [-\varsigma, 0]_{\mathbb{Z}} \to \mathbb{R}^n$. Here, $\tau(i)$, $\varsigma(i)$ and ν, ϑ represent respectively the discrete and distributed delay in both layers with $0 \le \tau_{kh} \le \tau$, $0 \le \varsigma_{kh} \le \varsigma$; $D = (d_{kh})_{l\times l}$ and $R = (r_{kh})_{l\times l} \in \mathbb{R}^{m\times n}$ denotes the connection weight matrices of the discrete activation function $f_{2h}(v_h(i-\tau_{kh}(i)))$ *and* $g_{2h}(v_h(i-\varsigma_{kh}(i)))$;$E = (e_{kh})_{l\times l}$ and $S = (s_{kh})_{l\times l}$ denotes the connection weight matrices of the infinitely distributed activation function $f_{3h}(\sum_{\nu=1}^{\infty} M_{kh}(\nu) v_h(i-\nu))$ *and* $g_{3h}(\sum_{\vartheta=1}^{\infty} O_{kh}(\vartheta) v_h(i-\vartheta))$; where M_{kh} and O_{kh} stands for the kernel function.

Assumption

(A_1) For any $h \in \mathbb{S}$ and a continuous and bounded function $f_h(v_h(.))$ and $g_h(u_h(.))$ there exist constants L_h, N_h such that

$$|f_h(v_h(.))| \le L_h |v_h(.)|$$
$$|g_h(u_h(.))| \le N_h |u_h(.)|, \quad h = 1,2,3.$$

(A_2) The Kernel function $M_{kh}(\nu), O_{kh}(\vartheta) \in \mathbb{R}^+, \forall \nu, \vartheta \in \mathbb{Z}$ is bounded.

(A_3) $\sum_{\nu=1}^{\infty} M_{kh}(\nu) = 1$ and $\sum_{\vartheta=1}^{\infty} N_{kh}(\vartheta) = 1$

Lemma 1. *See Reference [53] for $l \ge 2$ then the upcoming identity holds*

$$\sum_{k,h=1}^{l} c_k a_{kh} F_{kh}(w_k(i), w_h(i), i) = \sum_{\mathbf{U}\in\mathbf{U}} \mathbf{W}(\mathbf{U}) \sum_{(k,h)\in\mathbf{A}(\mathbf{Q_U})} F_{kh}(w_k(i), w_h(i), i).$$

Here, for any $k, h \in \mathbb{S}$, the cofactor of the kth diagonal element of the Laplacian matrix is expressed as c_k, $F_{kh}(w_k(i), w_h(i), i)$ is an arbitrary function, $\mathbf{U}$ is the collection of all spanning unicyclic graphs of $(\mathbf{G}, \mathbf{A})$, $\mathbf{W}(\mathbf{U})$ is the weight of $\mathbf{U}$ and $\mathbf{Q_U}$ denotes the directed cycle of $\mathbf{Q}$. In addition, $c_k > 0$ whenever $(\mathbf{G}, \mathbf{A})$ is strongly connected.

Lemma 2 ([60])**.** *The Fredholm mapping $\mathcal{L}$ of index zero and $\mathcal{L}$-Compact $\mathcal{N}$ on Θ. Suppose*

1. *$\mathcal{L}w \ne \Lambda\mathcal{L}w$, for all $w \in \partial\Theta \cap \ker\mathcal{L}$ and $\Lambda \in (0,1)$.*
2. *$\mathcal{QN}w \ne 0$, for all $w \in \partial\Theta \cap \ker\mathcal{L}$.*
3. *$deg_{\mathcal{B}}\mathcal{JQN}$, $\Theta \cap \ker L, 0 \ne 0$. where deg denotes the Brouwer degree.*

which implies that $\mathcal{L}w = \mathcal{N}w$ has at least one solution lying in $Dom\mathcal{L} \cap \bar{\Theta}$.

3. The Existence of Periodic Solution for DBAMNNs

In this section, for the given delayed BAM neural networks we derive the sufficient condition for the existence of a periodic solution by using the continuation theorem and Krichhoff's matrix tree theorem.

Theorem 1. *Let us consider the following assumptions are true:*

(P_1) *There exists a Lyapunov function $v_k(w_k(i), i)$ such that*

$$v_k(w_k(i), i) = v_k(w_k(i+T)), \quad \lim_{|w_k| \to \infty} v_k(w_k, i) = \infty \tag{3}$$

(P_2) *There exist the constants $\sigma_k, \gamma_k, \delta_k > 0$ and the matrix $(\alpha_{kh})_{l \times l}$, the arbitrary function $F_{kh}(w_k(i), w_h(i), i)$, for any $\lambda \in (0,1)$ such that*

$$\Delta v_k(w_k(i), i) \leq -\sigma_k v_k(w_k(i), i) + \gamma_k v_k(w_k(i - \varpi_{kh}(i)), i) + \sum_{h=1}^{l} \alpha_{kh} F_{kh}(w_k(i), w_h(i), i) + \delta_k \tag{4}$$

where $\varpi_{kh}(i) = \max_{k,h}\{\tau_{kh}(i), \zeta_{kh}(i)\}$.

(P_3) *Along with each dicycle $\mathbf{C}$ of a weighted strongly connected directed graph $(\mathbf{G}, \mathbf{A})$ such that*

$$\sum_{(p,q) \in E(\mathbf{C_Q})} F_{pq}(w_k(i), w_h(i), i) \leq 0 \tag{5}$$

(P_4) *Suppose that*

$$\begin{pmatrix} -a_1 u_1(i) + \sum\limits_{h=1}^{l} b_{1h} f_1(v_h(i)) + \sum\limits_{h=1}^{l} d_{1h} f_2(v_h(i - \tau_{kh}(i))) + \sum\limits_{h=1}^{l} e_{1h} f_3(\sum\limits_{\nu=1}^{\infty} M_{1h}(\nu) v_h(i-\nu)) + I_1(t) \\ \vdots \\ -a_n u_n(i) + \sum\limits_{h=1}^{l} b_{nh} f_1(v_h(i)) + \sum\limits_{h=1}^{l} d_{nh} f_2(v_h(i - \tau_{kh}(i))) + \sum\limits_{h=1}^{l} e_{nh} f_3(\sum\limits_{\nu=1}^{\infty} M_{nh}(\nu) v_h(i-\nu)) + I_n(t) \\ -p_1 v_1(i) + \sum\limits_{h=1}^{l} q_{1h} g_1(u_h(i)) + \sum\limits_{h=1}^{l} r_{1h} g_2(u_h(i - \varsigma_{kh}(i))) + \sum\limits_{h=1}^{l} e_{1h} g_3(\sum\limits_{\vartheta=1}^{\infty} M_{1h}(\vartheta) u_h(i-\vartheta)) + J_1(t) \\ \vdots \\ -p_n v_n(i) + \sum\limits_{h=1}^{l} q_{nh} g_1(u_h(i)) + \sum\limits_{h=1}^{l} r_{nh} g_2(u_h(i - \varsigma_{kh}(i))) + \sum\limits_{h=1}^{l} e_{nh} g_3(\sum\limits_{\vartheta=1}^{\infty} M_{nh}(\vartheta) u_h(i-\vartheta)) + J_n(t) \end{pmatrix} = \mathcal{S}(w) w \tag{6}$$

where $(\mathrm{S}(w))_{2n \times 2n}$ is a non-singular matrix. Then (2) has at least one T-Periodic solution.

Proof. For notation convenience denote

$$w(i) = (w_1(i), w_2(i), ..., w_n(i))^T = (u_1(i), u_2(i), ..., u_n(i), v_1(i), v_2(i), ..., v_n(i))^T \in \mathbb{R}^{2n}.$$

□

Let us consider

$$V = W = \{w = \{w(i)\} : w(i) \in \mathbb{R}^{2n}, n \in \mathbb{N}\} = p^m$$

and $p^T \subset p^m$ which denotes the subspace of all T-periodic sequence equipped with norm

$$\|w\| = \sum_{k=1}^{l} \left(\max_{i \in S_T} |u_k(i)| + \max_{i \in S_T} |v_k(i)| \right)$$

for any $w(i) \in p^T$. Clearly, p^T is a finite dimensional Banach space. Define a map $\mathcal{L} : Dom\mathcal{L} \subset V \to V$ and $\mathcal{N} : V \to V$ by $\mathcal{L} w(i) = \Delta \mathcal{N} w(i)$ and

$$\mathcal{N}w(i) = \mathcal{N}\begin{pmatrix} w_1(i) \\ w_2(i) \\ \vdots \\ w_n(i) \end{pmatrix}$$

$$= \begin{bmatrix} -a_1u_1(i)+\sum_{h=1}^{l} b_{1h}f_1(v_h(i))+\sum_{h=1}^{l} d_{1h}f_2(v_h(i-\tau_{kh}(i)))+\sum_{h=1}^{l} e_{1h}f_3(\sum_{\nu=1}^{\infty} M_{1h}(\nu)v_h(i-\nu))+I_1(t) \\ \vdots \\ -a_nu_n(i)+\sum_{h=1}^{l} b_{nh}f_1(v_h(i))+\sum_{h=1}^{l} d_{nh}f_2(v_h(i-\tau_{kh}(i)))+\sum_{h=1}^{l} e_{nh}f_3(\sum_{\nu=1}^{\infty} M_{nh}(\nu)v_h(i-\nu))+I_n(t) \\ -p_1v_1(i)+\sum_{h=1}^{l} q_{1h}g_1(u_h(i))+\sum_{h=1}^{l} r_{1h}g_2(u_h(i-\varsigma_{kh}(i)))+\sum_{h=1}^{l} e_{1h}g_3(\sum_{\vartheta=1}^{\infty} M_{1h}(\vartheta)u_h(i-\vartheta))+J_1(t) \\ \vdots \\ -p_nv_n(i)+\sum_{h=1}^{l} q_{nh}g_1(u_h(i))+\sum_{h=1}^{l} r_{nh}g_2(u_h(i-\varsigma_{kh}(i)))+\sum_{h=1}^{l} e_{nh}g_3(\sum_{\vartheta=1}^{\infty} M_{nh}(\vartheta)u_h(i-\vartheta))+J_n(t) \end{bmatrix} \quad (7)$$

Then,

$$\begin{aligned} Im\mathcal{L} &= \{w = \{w(i)\} \in p^T : \sum_{i=0}^{T-1} w(i) = 0\} \\ Ker\mathcal{L} &= \{w = \{w(i)\} \in p^T : w(i) = constant \in \mathbb{R}^{2n}, i \in \mathbb{Z}\} \end{aligned}$$

which is closed in W and $dimKer\mathcal{L} = 2n = \dim Im\mathcal{L} < +\infty$. *It is easy to verify that* $Im\mathcal{L}$ *and* $Ker\mathcal{L}$ *are a closed linear subspace of* p^T *and*

$$p^T = Ker\mathcal{L} \oplus Im\mathcal{L}$$

Since $Im\mathcal{L}$ *is closed in W and it has a finite dimensional, hence* $\mathcal{L}$ *is a Fredholm mapping of index zero. Let us define a projector* $\mathcal{P}$ *and* $\mathcal{Q}$ *as follows,* $\mathcal{P} : V \cap Dom\mathcal{L} \to ker\mathcal{L}$ *and* $\mathcal{Q} : V \to V/Im\mathcal{L}$.

$$\begin{aligned} \mathcal{P}w &= \frac{1}{T}\sum_{i=0}^{T-1} w(i), \quad \mathcal{Q}w = \frac{1}{T}\sum_{i=0}^{T-1} w(i), \quad \forall w \in V \\ \mathcal{P}w &= \mathcal{Q}w = \frac{1}{T}\begin{pmatrix} \sum_{i=0}^{T-1} w_1(i) \\ \sum_{i=0}^{T-1} w_2(i) \\ \vdots \\ \sum_{i=0}^{T-1} w_n(i) \end{pmatrix} \end{aligned}$$

Hence,

$$Im\mathcal{P} = Ker\mathcal{L}, \; Im\mathcal{L} = Ker\mathcal{Q} = Im(I - \mathcal{Q}).$$

Furthermore, the generalized inverse of L, $K_p : Im\mathcal{L} \to Ker\mathcal{P} \cap Dom\mathcal{L}$. *is defined as*

$$K_p(w) = \sum_{i=0}^{T-1} w(i) - \frac{1}{T}\sum_{i=0}^{T-1}(T-i)w(i)$$

Clearly,

$$\mathcal{QN}w = \begin{bmatrix} \frac{1}{T}\sum_{i=0}^{T-1}\Big[-a_1u_1(i)+\sum_{h=1}^{l} b_{1h}f_1(v_h(i))+\sum_{h=1}^{l} d_{1h}f_2(v_h(i-\tau_{kh}(i)))+\sum_{h=1}^{l} e_{1h}f_3(\sum_{\nu=1}^{\infty} M_{1h}(\nu)v_h(i-\nu))+I_1(t)\Big] \\ \vdots \\ \frac{1}{T}\sum_{i=0}^{T-1}\Big[-a_nu_n(i)+\sum_{h=1}^{l} b_{nh}f_1(v_h(i))+\sum_{h=1}^{l} d_{nh}f_2(v_h(i-\tau_{kh}(i)))+\sum_{h=1}^{l} e_{nh}f_3(\sum_{\nu=1}^{\infty} M_{nh}(\nu)v_h(i-\nu))+I_n(t)\Big] \\ \frac{1}{T}\sum_{i=0}^{T-1}\Big[-p_1v_1(i)+\sum_{h=1}^{l} q_{1h}g_1(u_h(i))+\sum_{h=1}^{l} r_{1h}g_2(u_h(i-\varsigma_{kh}(i)))+\sum_{h=1}^{l} e_{1h}g_3(\sum_{\vartheta=1}^{\infty} M_{1h}(\vartheta)u_h(i-\vartheta))+J_1(t)\Big] \\ \vdots \\ \frac{1}{T}\sum_{i=0}^{T-1}\Big[-p_nv_n(i)+\sum_{h=1}^{l} q_{nh}g_1(u_h(i))+\sum_{h=1}^{l} r_{nh}g_2(u_h(i-\varsigma_{kh}(i)))+\sum_{h=1}^{l} e_{nh}g_3(\sum_{\vartheta=1}^{\infty} M_{nh}(\vartheta)u_h(i-\vartheta))+J_n(t)\Big] \end{bmatrix}$$

and $K_p(I-\mathcal{Q})\mathcal{N}w$ *are continuous. Since* V *and* W *are finite dimensional Banach space and* $K_p(I-\mathcal{Q})\mathcal{N}$ *is continuous. By using Ascoli-Arzela's theorem we can show that* $\mathcal{QN}\overline{(\Theta)}$ *and* $K_p(I-\mathcal{Q})\mathcal{N}\overline{\Theta}$ *are relatively compact for any open bounded set* $\Theta \in V$*. Hence* $\mathcal{N}$ *is compact on* $\overline{\Theta}$*. In the view of (1) the operation equation*

$$\mathcal{L}w = \Lambda\mathcal{N}$$

for some $\Lambda \in (0,1)$*.*

$$\Delta w(i) = \Lambda \begin{bmatrix} -(1+a_1)u_1(i)+\sum_{h=1}^{l} b_{1h}f_1(v_h(i))+\sum_{h=1}^{l} d_{1h}f_2(v_h(i-\tau_{kh}(i)))+\sum_{h=1}^{l} e_{1h}f_3(\sum_{\nu=1}^{\infty} M_{1h}(\nu)v_h(i-\nu))+I_1(t) \\ \vdots \\ -(1+a_n)u_n(i)+\sum_{h=1}^{l} b_{nh}f_1(v_h(i))+\sum_{h=1}^{l} d_{nh}f_2(v_h(i-\tau_{kh}(i)))+\sum_{h=1}^{l} e_{nh}f_3(\sum_{\nu=1}^{\infty} M_{nh}(\nu)v_h(i-\nu))+I_n(t) \\ -(1+p_1)v_1(i)+\sum_{h=1}^{l} q_{1h}g_1(u_h(i))+\sum_{h=1}^{l} r_{1h}g_2(u_h(i-\varsigma_{kh}(i)))+\sum_{h=1}^{l} e_{1h}g_3(\sum_{\vartheta=1}^{\infty} M_{1h}(\vartheta)u_h(i-\vartheta))+J_1(t) \\ \vdots \\ -(1+p_n)v_n(i)+\sum_{h=1}^{l} q_{nh}g_1(u_h(i))+\sum_{h=1}^{l} r_{nh}g_2(u_h(i-\varsigma_{kh}(i)))+\sum_{h=1}^{l} e_{nh}g_3(\sum_{\vartheta=1}^{\infty} M_{nh}(\vartheta)u_h(i-\vartheta))+J_n(t) \end{bmatrix} \tag{8}$$

Let us consider the global Lyapunov function

$$V(w(i)) = \sum_{k=1}^{l} c_k v_k(w_k(i)) \tag{9}$$

where c_k *indicate the cofactor of kth diagonal element of* $\mathbf{L}_p$ *and*

$$\begin{aligned} \Delta V(w) &= \sum_{k=1}^{l} c_k \Delta v_k(w_k(i)) \\ &\leq \Lambda \sum_{k=1}^{l} c_k[-\sigma_k|w_k(i)|^2 + \alpha_k|w_k(i-\varpi_{kh}(i))|^2 + \sum_{h=1}^{l} F_{kh}(w_k(i), w_h(i)) + \delta_k] \\ &\leq \Lambda \sum_{k=1}^{l} c_k[-\frac{\sigma_k}{\eta_k}V_k(w_k(i)) + \frac{\alpha_k}{\gamma_k}V_k(w_k(i-\varpi_{kh}(i))) + \sum_{h=1}^{l} F_{kh}(w_k(i), w_h(i)) + \delta_k] \\ &\leq -\Lambda \sum_{k=1}^{l} c_k \frac{\sigma_k}{\eta_k} V_k(w_k(i)) + \Lambda \sum_{k,h=1}^{l} c_k \frac{\alpha_k}{\gamma_k} V_k(w_k(i-\varpi_{kh}(i))) + \Lambda \sum_{k=1}^{l} c_k F_{kh}(w_k(i), w_h(i)) \\ &\quad + \Lambda \sum_{k=1}^{l} c_k \delta_k. \end{aligned}$$

where, $w(i)$ is T-periodic solution which implies $V(w(i))$ is also a T-periodic function.

$$\begin{aligned} 0 &\leq -\Lambda \sum_{k=1}^{l} c_k \frac{\sigma_k}{\eta_k} \sum_{i=0}^{T-1} V_k(w_k(i)) + \Lambda \sum_{k=1}^{l} c_k \frac{\alpha_k}{\gamma_k} \sum_{i=0}^{T-1} V_k(w_k(i-\varpi_{kh}(i))) + \Lambda \sum_{k,h=1}^{l} c_k \\ &\quad \times \sum_{i=0}^{T-1} F_{kh}(w_k(i), w_h(i)) + \Lambda T \sum_{k=1}^{l} c_k \delta_k \end{aligned}$$

From assumption

$$\begin{aligned} &\leq -\Lambda \sum_{k=1}^{l} c_k \frac{\sigma_k}{\eta_k} \sum_{i=0}^{T-1} V_k(w_k(i)) + \Lambda \sum_{k=1}^{l} c_k \frac{\alpha_k}{\gamma_k} \sum_{i=0}^{T-1} V_k(w_k(i-\varpi_{kh}(i))) + \Lambda T \sum_{k=1}^{l} c_k \delta_k \\ &\leq -\Lambda\sigma \sum_{i=0}^{T-1} V(w(i)) + \Lambda\alpha \sum_{i=0}^{T-1} V(w(i-\varpi_{kh}(i))) + \Lambda T \sum_{k=1}^{l} c_k \delta_k \\ &< 0. \end{aligned}$$

Here,

$$c = \min_k \{c_k\}; \sigma = \min_k \{\frac{\sigma_k}{\eta_k}\}; \alpha = \min_k \{\frac{\alpha_k}{\gamma_k}\};$$

$$\lim_{w \to \infty} \Delta V(w) < 0$$

This contradicts a T-periodic function $V(w(i))$. Therefore, there exists $\mathcal{B} > 0$ which is separate from the selection of Λ, so that the solution of $\mathcal{L}w = \Lambda w$, which satisfies $\|w\| < \mathcal{B}$. Denotes

$$\Theta = \{w \in \mathcal{B} : \|w\| < \mathcal{B}\}.$$

Then, we know that $\mathcal{L}w \neq \Lambda \mathcal{N} w$, $\Lambda \in (0,1)$ for $w \in Ker\mathcal{L} \cap \partial\Theta$ So, we obtain

$$\mathcal{QN}w = \Lambda \begin{bmatrix} \frac{1}{T}\sum_{i=0}^{T-1}\Big\{-(1+a_1)u_1(i)+\sum_{h=1}^{l} b_{1h}f_1(v_h(i))+\sum_{h=1}^{l} d_{1h}f_2(v_h(i-\tau_{kh}(i)))+\sum_{h=1}^{l} e_{1h}f_3(\sum_{\nu=1}^{\infty} M_{1h}(\nu)v_h(i-\nu))+I_1(t)\Big\} \\ \vdots \\ \frac{1}{T}\sum_{i=0}^{T-1}\Big\{-(1+a_n)u_n(i)+\sum_{h=1}^{l} b_{nh}f_1(v_h(i))+\sum_{h=1}^{l} d_{nh}f_2(v_h(i-\tau_{kh}(i)))+\sum_{h=1}^{l} e_{nh}f_3(\sum_{\nu=1}^{\infty} M_{nh}(\nu)v_h(i-\nu))+I_n(t)\Big\} \\ \frac{1}{T}\sum_{i=0}^{T-1}\Big\{-(1+p_1)v_1(i)+\sum_{h=1}^{l} q_{1h}g_1(u_h(i))+\sum_{h=1}^{l} r_{1h}g_2(u_h(i-\varsigma_{kh}(i)))+\sum_{h=1}^{l} e_{1h}g_3(\sum_{\vartheta=1}^{\infty} M_{1h}(\vartheta)u_h(i-\vartheta))+J_1(t)\Big\} \\ \vdots \\ \frac{1}{T}\sum_{i=0}^{T-1}\Big\{-(1+p_n)v_n(i)+\sum_{h=1}^{l} q_{nh}g_1(u_h(i))+\sum_{h=1}^{l} r_{nh}g_2(u_h(i-\varsigma_{kh}(i)))+\sum_{h=1}^{l} e_{nh}g_3(\sum_{\vartheta=1}^{\infty} M_{nh}(\vartheta)u_h(i-\vartheta))+J_n(t)\Big\} \end{bmatrix} \neq 0$$

Let us define a identity mapping $\mathcal{I} : Im\mathcal{Q} \to Ker\mathcal{L}$. Then for any $(w, \mu) \in \Theta \in [0,1]$.

$$\mathcal{IQN}w = \begin{bmatrix} \frac{1}{T}\sum_{i=0}^{T-1}\Big\{-(1+a_1)u_1(i)+\sum_{h=1}^{l} b_{1h}f_1(v_h(i))+\sum_{h=1}^{l} d_{1h}f_2(v_h(i-\tau_{kh}(i)))+\sum_{h=1}^{l} e_{1h}f_3(\sum_{\nu=1}^{\infty} M_{1h}(\nu)v_h(i-\nu))+I_1(t)\Big\} \\ \vdots \\ \frac{1}{T}\sum_{i=0}^{T-1}\Big\{-(1+a_n)u_n(i)+\sum_{h=1}^{l} b_{nh}f_1(v_h(i))+\sum_{h=1}^{l} d_{nh}f_2(v_h(i-\tau_{kh}(i)))+\sum_{h=1}^{l} e_{nh}f_3(\sum_{\nu=1}^{\infty} M_{nh}(\nu)v_h(i-\nu))+I_n(t)\Big\} \\ \frac{1}{T}\sum_{i=0}^{T-1}\Big\{-(1+p_1)v_1(i)+\sum_{h=1}^{l} q_{1h}g_1(u_h(i))+\sum_{h=1}^{l} r_{1h}g_2(u_h(i-\varsigma_{kh}(i)))+\sum_{h=1}^{l} e_{1h}g_3(\sum_{\vartheta=1}^{\infty} M_{1h}(\vartheta)u_h(i-\vartheta))+J_1(t)\Big\} \\ \vdots \\ \frac{1}{T}\sum_{i=0}^{T-1}\Big\{-(1+p_n)v_n(i)+\sum_{h=1}^{l} q_{nh}g_1(u_h(i))+\sum_{h=1}^{l} r_{nh}g_2(u_h(i-\varsigma_{kh}(i)))+\sum_{h=1}^{l} e_{nh}g_3(\sum_{\vartheta=1}^{\infty} M_{nh}(\vartheta)u_h(i-\vartheta))+J_n(t)\Big\} \end{bmatrix}$$

Define,

$$\begin{aligned} F(w,\mu) &= \mu w - (1-\mu)\mathcal{IQN}w, \quad (w,\mu) \in \Theta \times [0,1]. \\ &= \mu \begin{pmatrix} w_1(i) \\ w_2(i) \\ \vdots \\ w_n(i) \end{pmatrix} - \frac{1-\mu}{T} \end{aligned}$$

$$\begin{bmatrix} \frac{1}{T}\sum_{i=0}^{T-1}\Big\{-(1+a_1)u_1(i)+\sum_{h=1}^{l} b_{1h}f_1(v_h(i))+\sum_{h=1}^{l} d_{1h}f_2(v_h(i-\tau_{kh}(i)))+\sum_{h=1}^{l} e_{1h}f_3(\sum_{\nu=1}^{\infty} M_{1h}(\nu)v_h(i-\nu))+I_1(t)\Big\} \\ \vdots \\ \frac{1}{T}\sum_{i=0}^{T-1}\Big\{-(1+a_n)u_n(i)+\sum_{h=1}^{l} b_{nh}f_1(v_h(i))+\sum_{h=1}^{l} d_{nh}f_2(v_h(i-\tau_{kh}(i)))+\sum_{h=1}^{l} e_{nh}f_3(\sum_{\nu=1}^{\infty} M_{nh}(\nu)v_h(i-\nu))+I_n(t)\Big\} \\ \frac{1}{T}\sum_{i=0}^{T-1}\Big\{-(1+p_1)v_1(i)+\sum_{h=1}^{l} q_{1h}g_1(u_h(i))+\sum_{h=1}^{l} r_{1h}g_2(u_h(i-\varsigma_{kh}(i)))+\sum_{h=1}^{l} e_{1h}g_3(\sum_{\vartheta=1}^{\infty} M_{1h}(\vartheta)u_h(i-\vartheta))+J_1(t)\Big\} \\ \vdots \\ \frac{1}{T}\sum_{i=0}^{T-1}\Big\{-(1+p_n)v_n(i)+\sum_{h=1}^{l} q_{nh}g_1(u_h(i))+\sum_{h=1}^{l} r_{nh}g_2(u_h(i-\varsigma_{kh}(i)))+\sum_{h=1}^{l} e_{nh}g_3(\sum_{\vartheta=1}^{\infty} M_{nh}(\vartheta)u_h(i-\vartheta))+J_n(t)\Big\} \end{bmatrix}$$

For $\mu \in (0,1)$.

$$deg_{\mathcal{B}}\{\mathcal{JQN}, \Theta \cap Ker\mathcal{L}, 0\} = Sgn(det(G(0))) \neq 0$$

Hence, all the conditions in the continuation theorem are satisfied. Then (2) *has T-periodic solution.*

Remark 1. *Suppose that the digraph* $(\mathbf{G}, \mathbf{A})$ *is balanced, then the following equation possess*

$$\sum_{k,h=1}^{l} a_{kh}F_{kh}(w_k(i), w_h(i)) = \frac{1}{2}\sum_{\mathbf{U}\in\mathbb{U}} \mathbf{W}(\mathbf{U}) \sum_{(k,h)\in\mathbf{A}(\mathbf{Q_U})} \Big[F_{kh}(w_k(i), w_h(i)) + F_{hk}(w_h(i), w_k(i))\Big]. \tag{10}$$

Extending the condition, (5) *into*

$$\sum_{(k,h)\in\mathbf{Q_U}} [F_{kh}(w_k(i), w_h(i)) + F_{hk}(w_h(i), w_k(i))] \leq 0. \tag{11}$$

Corollary 1. *Let* $(\mathbf{G}, \mathbf{A})$ *be a balanced digraph. If we put* (10) *instead of* (4) *in the place of* $a_{kh}F_{kh}(w_k(i), w_h(i))$ *then the system* (2) *has T-periodic solution.*

Remark 2. *For each k, h, there exists functions* $R_k(w_k)$ *and* $R_h(w_h)$ *such that*

$$F_{kh}(w_k(i), w_h(i)) \leq R_k(w_k(i)) - R_h(w_h(i)). \tag{12}$$

At that point

$$\sum_{(k,h)\in\mathbf{Q_U}} [R_k(w_k(i)) - R_h(w_h(i))] \leq 0. \tag{13}$$

Corollary 2. *The determination of existence theorem holds whenever we replace* (4) *by (11).*

Remark 3. *For DDBAMNNs* (2)*, we construct a global Lyapunov function* $V = \sum_{k=1}^{l} c_k v_i$*, and that is closely connected to the topological structure. However, the construction of Lyapunov function for delayed system is a difficult one. Hence, we design* $v_i = v_k^{(1)}(i) + v_k^{(2)}(i) + v_k^{(3)}(i)$*, which is useful for solving this type problem.*

4. Uniqueness and Exponential Stability of Periodic Solution

Definition 1. *The system* (2) *with T-periodic solution $u^*(i)$ and $v^*(i)$ is said to be exponentially stable if there is a constant $\kappa > 0$ and $\rho > 1$ such that*

$$|u(i) - u^*(i)|^2 + |v(i) - v^*(i)|^2 \leq \rho^i \kappa \left(\sup_{s \in \mathbb{Z}[-\tau,0]} \|u(i) - u^*(i)\|^2 + \sup_{s \in \mathbb{Z}[-\varsigma,0]} \|v(i) - v^*(i)\|^2 \right)$$

Moreover, exponentially stable implies the uniqueness of T-periodic solution $u^*(i)$ and $v^*(i)$.

Theorem 2. *Let us consider the condition $(A_1) - (A_3)$, if*

$$\max\Big\{\rho|a_k|^2 + \rho \sum_{h=1}^{l} e_{kh}^2 L_3^2 \sum_{\nu=1}^{\infty} M_{kh}(\nu) + \rho \sum_{k=1}^{l} |d_{hk}|^2 L_2^2 + \rho \sum_{k=1}^{l} |q_{hk}|^2 N_1^2, \rho|p_k|^2 + \rho \sum_{h=1}^{l} s_{kh}^2 N_3^2 \times \sum_{\vartheta=1}^{\infty} O_{kh}(\vartheta) + \rho \sum_{k=1}^{l} |r_{hk}|^2 N_2^2 + \rho \sum_{k=1}^{l} |b_{hk}|^2 L_1^2\Big\} > 1$$

then the system (2) *is exponential stability and it is unique.*

Proof. On the basis of Theorem 1, the system (2) has at least one T-periodic solution. Let

$$w^*(i) = (u_1^*(i), u_2^*(i), ..., u_n^*(i), v_1^*(i), v_2^*(i), ..., v_n^*(i))^T$$

be denotes the T-Periodic solution of (2). Then we have,

$$\begin{aligned} u_k^*(i+1) &= -a_k u_k^*(i) + \sum_{h=1}^{l} b_{kh} f_1(v_h^*(i)) + \sum_{h=1}^{l} d_{kh} f_2(v_h^*(i - \tau_{kh}(i))) \\ &\quad + \sum_{h=1}^{l} e_{kh} f_3(\sum_{\nu=1}^{\infty} M_{kh}(\nu) v_h^*(i-\nu)) + I_k \\ v_k^*(i+1) &= -p_k v_k^*(i) + \sum_{h=1}^{l} q_{kh} g_1(u_h^*(i)) + \sum_{h=1}^{l} r_{kh} g_2(u_h^*(i - \varsigma_{kh}(i))) \\ &\quad + \sum_{h=1}^{l} s_{kh} g_3(\sum_{\vartheta=1}^{\infty} O_{kh}(\vartheta) u_h^*(i-\vartheta)) + J_k \end{aligned} \tag{14}$$

for $i \in \{0, 1, ..., T\}$, $k \in \mathbb{L}$. □

Let

$$x_k(i) = u_k(i) - u_k^*(i) \qquad y_k(i) = v_k(i) - v_k^*(i)$$

Then, we have

$$\begin{aligned}
x_k^*(i+1) &= -a_k x_k^*(i) + \sum_{h=1}^{l} b_{kh}\big(f_1(v_h(i)) - f_1(v_h^*(i))\big) + \sum_{h=1}^{l} d_{kh}\big(f_2(v_h(i-\tau_{kh}(i)))\\
&\quad -f_2(v_h^*(i-\tau_{kh}(i)))\Big) + \sum_{h=1}^{l} e_{kh}\Big(f_3(\sum_{\nu=1}^{\infty} M_{kh}(\nu)v_h(i-\nu)) - f_3(\sum_{\nu=1}^{\infty} M_{kh}(\nu)\\
&\quad \times v_h^*(i-\nu))\Big) + I_k\\
y_k^*(i+1) &= -p_k y_k^*(i) + \sum_{h=1}^{l} q_{kh}\Big(g_1(u_h(i)) - g_1(u_h^*(i))\Big) + \sum_{h=1}^{l}\Big(r_{kh}g_2(u_h(i-\varsigma_{kh}(i)))\\
&\quad -r_{kh}g_2(u_h^*(i-\varsigma_{kh}(i)))\Big) + \sum_{h=1}^{l} s_{kh}\Big(g_3(\sum_{\vartheta=1}^{\infty} O_{kh}(\vartheta)u_h(i-\vartheta)) - g_3(\sum_{\vartheta=1}^{\infty} O_{kh}(\vartheta)\\
&\quad \times u_h^*(i-\vartheta))\Big) + J_k
\end{aligned} \tag{15}$$

Denote the Lyapunov function as $v_k(i) = v_k^{(1)}(i) + v_k^{(2)}(i) + v_k^{(3)}(i)$ *and the cofactor of kth diagonal element of* L_p *is* c_k*. Let us define a global Lyapunov function*

$$V(i) = \sum_{k=1}^{l} c_k v_k(i) \tag{16}$$

Here,

$$\begin{aligned}
v_k^{(1)}(i) &= \frac{1}{2}\rho^i|x_k(i)|^2 + \frac{1}{2}\rho^i|y_k(i)|^2\\
v_k^{(2)}(i) &= \frac{1}{2}\rho^i\sum_{h=1}^{l} d_{kh}^2 L_2^2 \sum_{\xi=i-\tau_{kh}(i)}^{i-1} |x_h(\xi)|^2 + \frac{1}{2}\rho^i\sum_{h=1}^{l} r_{kh}^2 N_2^2 \sum_{\xi=i-\varsigma_{kh}(i)}^{i-1} |y_h(\xi)|^2\\
v_k^{(3)}(i) &= \frac{1}{2}\rho^i\sum_{h=1}^{l} e_{kh}^2 L_3^2 \sum_{\nu=1}^{\infty} M_{kh}(\nu) \sum_{r=i-\nu}^{i-1} |x_h(r)|^2 + \frac{1}{2}\rho^i\sum_{h=1}^{l} s_{kh}^2 N_3^2 \sum_{\vartheta=1}^{\infty} O_{kh}(\vartheta) \sum_{r=i-\vartheta}^{i-1} |y_h(r)|^2
\end{aligned}$$

From Assumption I, we get

$$
\begin{aligned}
\Delta v_k^{(1)}(i) =& v_k^{(1)}(i+1) - v_k^{(1)}(i) \\
=& \frac{1}{2}\rho^i\{\rho x_k^2(i+1) - x_k^2(i)\} + \frac{1}{2}\rho^i\{\rho y_k^2(i+1) - y_k^2(i)\} \\
=& \frac{1}{2}\rho^i\Big\{\rho\big(| - a_k x_k(i) + \sum_{h=1}^{l} b_{kh} f_1(v_h(i)) - \sum_{h=1}^{l} b_{kh} f_1(v_h^*(i)) + \sum_{h=1}^{l} d_{kh} f_2(v_h(i-\tau_{kh}(i))) - \sum_{h=1}^{l} d_{kh} \\
& \times f_2(v_h^*(i-\tau_{kh}(i))) + \sum_{h=1}^{l} e_{kh} f_3 \sum_{\nu=1}^{\infty} M_{kh}(\nu) v_h(i-\nu) - \sum_{h=1}^{l} e_{kh} f_3(\sum_{\nu=1}^{\infty} M_{kh}(\nu) v_h^*(i-\nu)) + I_k|^2\big)\Big\} \\
& + \frac{1}{2}\rho^i\Big\{\rho\big(| - p_k y_k^*(i) + \sum_{h=1}^{l} q_{kh} g_1(u_h(i)) - \sum_{h=1}^{l} q_{kh} g_1(u_h^*(i)) + \sum_{h=1}^{l} r_{kh} g_2(u_h(i-\varsigma_{kh}(i))) - \sum_{h=1}^{l} r_{kh} \\
& \times g_2(u_h^*(i-\varsigma_{kh}(i))) + \sum_{h=1}^{l} s_{kh} g_3(\sum_{\vartheta=1}^{\infty} O_{kh}(\vartheta) u_h(i-\vartheta)) - \sum_{h=1}^{l} s_{kh} g_3(\sum_{\vartheta=1}^{\infty} O_{kh}(\vartheta) u_h^*(i-\vartheta)) + J_k|^2\big) \\
& - \frac{1}{2}|x_k|^2 - \frac{1}{2}|y_k|^2\Big\} \\
\le& \rho^i\Big\{\frac{1}{2}(\rho|a_k|^2 - 1)|x_k(i)|^2 + \frac{1}{2}\rho\sum_{h=1}^{l}|b_{kh}|^2 L_1^2 |y_h(i)|^2 + \rho\frac{1}{2}\sum_{h=1}^{l}|d_{kh}|^2 L_2^2 |y_h(i-\tau_{kh}(i))|^2 + \frac{1}{2}\rho\sum_{h=1}^{l}|e_{kh}|^2 \\
& \times L_3^2 \sum_{\nu=1}^{\infty}|M_{kh}(\nu)|^2 |y_h(i-\nu)|^2 + \frac{1}{2}\rho|I_k|^2 + \frac{1}{2}(\rho|p_k|^2 - 1)|y_k(i)|^2 + \frac{1}{2}\rho\sum_{h=1}^{l}|q_{kh}|^2 N_1^2 |x_h(i)|^2 + \frac{1}{2}\rho \\
& \times \sum_{h=1}^{l}|r_{kh}|^2 N_2 |x_h(i-\varsigma_{kh}(i))|^2 + \frac{1}{2}\rho\sum_{h=1}^{l}|s_{kh}|^2 N_3 \sum_{\vartheta=1}^{\infty}|O_{kh}(\vartheta)|^2 |x_h(i-\vartheta)|^2 + \frac{1}{2}\rho|J_k|^2\Big\} \\
\le& \rho^i\Big\{\frac{1}{2}\sum_{h=1}^{l}(\rho|a_k|^2 - 1)|x_k(i)|^2 + \frac{1}{2}\sum_{h=1}^{l}(\rho|p_k|^2 - 1)|y_k(i)|^2 + \frac{1}{2}\rho\sum_{h=1}^{l}|r_{kh}|^2 N_2 |x_h(i-\zeta_{kh}(i))|^2 + \frac{1}{2}\rho \\
& \times \sum_{h=1}^{l}|d_{kh}|^2 L_2^2 |y_h(i-\tau_{kh}(i))|^2 + \frac{1}{2}\rho\sum_{h=1}^{l}|e_{kh}|^2 L_3^2 \sum_{\nu=1}^{\infty}|M_{kh}(\nu)|^2 |y_h(i-\nu)|^2 + \frac{1}{2}\rho|I_k|^2 + \frac{1}{2}\rho\sum_{h=1}^{l}|s_{kh}|^2 \\
& \times N_3 \sum_{\vartheta=1}^{\infty}|O_{kh}(\vartheta)|^2 |x_h(i-\vartheta)|^2 + \frac{1}{2}\rho|J_k|^2 + \rho|q_{kh}|^2 N_1^2 |x_h(i)|^2 + \rho|b_{kh}|^2 L_1^2 |y_h(i)|^2\Big\}
\end{aligned}
\tag{17}
$$

$$
\begin{aligned}
\Delta v_k^{(2)}(i) =& v_k^{(2)}(i+1) - v_k^{(2)}(i) \\
=& \rho^i\Big\{\frac{1}{2}\rho\sum_{h=1}^{l}|d_{kh}|^2 L_2^2 \sum_{\xi=i+1-\tau_{kh}(i+1)}^{i}|x_h(\xi)|^2 + \frac{1}{2}\rho\sum_{h=1}^{l}|r_{kh}|^2 N_2^2 \sum_{\xi=i+1-\varsigma_{kh}(i+1)}^{i}|y_h(\xi)|^2 \\
& - \frac{1}{2}\sum_{h=1}^{l}|d_{kh}|^2 L_2^2 \sum_{\xi=i-\tau_{kh}(i)}^{i-1}|x_h(\xi)|^2 - \frac{1}{2}\sum_{h=1}^{l}|r_{kh}|^2 N_2^2 \sum_{\xi=i-\varsigma_{kh}(i)}^{i-1}|y_h(\xi)|^2\Big\}
\end{aligned}
$$

Since, $1 < \tau_{kh}(i+1) < \tau_{kh}(i) + 1$ *and* $1 < \varsigma_{kh}(i+1) < \varsigma_{kh}(i) + 1$

$$
\begin{aligned}
\le& \rho^i\Big\{\frac{1}{2}\rho\sum_{h=1}^{l}|d_{kh}|^2 L_2^2 \sum_{\xi=i-\tau_{kh}(i)}^{i}|x_h(\xi)|^2 + \frac{1}{2}\rho\sum_{h=1}^{l}|r_{kh}|^2 N_2^2 \sum_{\xi=i-\varsigma_{kh}(i)}^{i}|y_h(\xi)|^2 - \frac{1}{2}\sum_{h=1}^{l}|d_{kh}|^2 L_2^2 \\
& \times \sum_{\xi=i-\tau_{kh}(i)}^{i-1}|x_h(\xi)|^2 - \frac{1}{2}\sum_{h=1}^{l}|r_{kh}|^2 N_2^2 \sum_{\xi=i-\varsigma_{kh}(i)}^{i-1}|y_h(\xi)|^2\Big\} \\
=& \rho^i\Big\{\frac{1}{2}\rho\sum_{h=1}^{l}|d_{kh}|^2 L_2^2 |x_h(i)|^2 + \frac{1}{2}\rho\sum_{h=1}^{l}|d_{kh}|^2 L_2^2 \sum_{\xi=i-\tau_{kh}(i)}^{i-1}|x_h(\xi)|^2 + \frac{1}{2}\rho\sum_{h=1}^{l}|r_{kh}|^2 N_2^2 |y_h(i)|^2
\end{aligned}
$$

$$+\frac{1}{2}\rho\sum_{h=1}^{l}|r_{kh}|^2N_2^2\sum_{\xi=i-\varsigma_{kh}(i)}^{i-1}|y_h(\xi)|^2-\frac{1}{2}\sum_{h=1}^{l}|d_{kh}|^2L_2^2\sum_{\xi=i-\tau_{kh}(i)}^{i-1}|x_h(\xi)|^2-\frac{1}{2}\sum_{h=1}^{l}|r_{kh}|^2N_2^2$$

$$\times\sum_{\xi=i-\varsigma_{kh}(i)}^{i-1}|y_h(\xi)|^2\Big\}$$

$$\leq\rho^i\Big\{\frac{1}{2}\rho\sum_{h=1}^{l}|d_{kh}|^2L_2^2|x_h(i)|^2+\frac{1}{2}\rho\sum_{h=1}^{l}|r_{kh}|^2N_2^2|y_h(i)|^2\Big\} \quad (18)$$

$$v_3(i)=\frac{1}{2}\rho^i\sum_{h=1}^{l}e_{kh}^2L_3^2\sum_{\nu=1}^{\infty}M_{kh}(\nu)\sum_{r=i-\nu}^{i-1}|x_k(r)|^2+\frac{1}{2}\rho^i\sum_{h=1}^{l}s_{kh}^2N_3^2\sum_{\vartheta=1}^{\infty}O_{kh}(\vartheta)\sum_{r=i-\vartheta}^{i-1}|y_k(r)|^2$$

$$\Delta v_k^{(3)}(i)=\rho^i\Big\{\frac{1}{2}\rho\sum_{h=1}^{l}e_{kh}^2L_3^2\sum_{\nu=1}^{\infty}M_{kh}(\nu)\sum_{r=i-\nu+1}^{i}|x_k(r)|^2+\frac{1}{2}\rho\sum_{h=1}^{l}s_{kh}^2N_3^2\sum_{\vartheta=1}^{\infty}O_{kh}(\vartheta)\sum_{r=i-\vartheta+1}^{i}|y_k(r)|^2$$

$$-\frac{1}{2}\sum_{h=1}^{l}e_{kh}^2L_3^2\sum_{\nu=1}^{\infty}M_{kh}(\nu)\sum_{r=i-\nu}^{i-1}|x_k(r)|^2-\frac{1}{2}\sum_{h=1}^{l}s_{kh}^2N_3^2\sum_{\vartheta=1}^{\infty}O_{kh}(\vartheta)\sum_{r=i-\vartheta}^{i-1}|y_k(r)|^2\Big\} \quad (19)$$

Substitute (17)–(19) this in (16) , we obtain,

$$\Delta v_k(i)\leq\rho^i\Bigg\{\frac{1}{2}\sum_{h=1}^{l}(\rho|a_k|^2-1)|x_k(i)|^2+\frac{1}{2}\sum_{h=1}^{l}(\rho|p_k|^2-1)|y_k(i)|^2+\frac{1}{2}\rho\sum_{h=1}^{l}|r_{kh}|^2N_2^2|x_h(i-\zeta_{kh}(i))|^2$$

$$+\frac{1}{2}\rho\sum_{h=1}^{l}|d_{kh}|^2L_2^2|y_h(i-\tau_{kh}(i))|^2+\frac{1}{2}\rho\sum_{h=1}^{l}|e_{kh}|^2L_3^2\sum_{\nu=1}^{\infty}|M_{kh}(\nu)|^2|y_h(i-\nu)|^2+\frac{1}{2}\rho|I_k|^2$$

$$+\frac{1}{2}\rho\sum_{h=1}^{l}|s_{kh}|^2N_3^2\sum_{\vartheta=1}^{\infty}|O_{kh}(\vartheta)|^2|x_h(i-\vartheta)|^2+\frac{1}{2}\rho|J_k|^2+\frac{1}{2}\rho\sum_{h=1}^{l}|d_{kh}|^2L_2^2|x_h(i)|^2$$

$$+\frac{1}{2}\rho\sum_{h=1}^{l}|r_{kh}|^2N_2^2|y_h(i)|^2+\frac{1}{2}\rho\sum_{h=1}^{l}e_{kh}^2L_3^2\sum_{\nu=1}^{\infty}M_{kh}(\nu)\sum_{r=i-\nu+1}^{i}|x_k(r)|^2$$

$$+\frac{1}{2}\rho\sum_{h=1}^{l}s_{kh}^2N_3^2\sum_{\vartheta=1}^{\infty}O_{kh}(\vartheta)\sum_{r=i-\vartheta+1}^{i}|y_k(r)|^2-\frac{1}{2}\sum_{h=1}^{l}e_{kh}^2L_3^2\sum_{\nu=1}^{\infty}M_{kh}(\nu)\sum_{r=i-\nu}^{i-1}|x_k(r)|^2$$

$$-\frac{1}{2}\sum_{h=1}^{l}s_{kh}^2N_3^2\sum_{\vartheta=1}^{\infty}O_{kh}(\vartheta)\sum_{r=i-\vartheta}^{i-1}|y_k(r)|^2+\rho\sum_{h=1}^{l}|q_{kh}|^2N_1^2|x_h(i)|^2+\rho\sum_{h=1}^{l}|b_{kh}|^2L_1^2|y_h(i)|^2\Bigg\}$$

$$\leq\frac{1}{2}\rho^i\Bigg\{\sum_{h=1}^{l}(\rho|a_k|^2+\rho\sum_{h=1}^{l}e_{kh}^2L_3^2\sum_{\nu=1}^{\infty}M_{kh}(\nu)\rho\sum_{k=1}^{l}|d_{hk}|^2L_2^2+\rho\sum_{k=1}^{l}|q_{hk}|^2N_1^2-1)|x_k(i)|^2$$

$$+\sum_{h=1}^{l}(\rho|p_k|^2+\rho\sum_{h=1}^{l}s_{kh}^2N_3^2\sum_{\vartheta=1}^{\infty}O_{kh}(\vartheta)\rho\sum_{k=1}^{l}|r_{hk}|^2N_2^2+\rho\sum_{k=1}^{l}|b_{hk}|^2L_1^2-1)|y_k(i)|^2$$

$$+\rho\sum_{h=1}^{l}|r_{kh}|^2N_2^2|x_h(i-\zeta_{kh}(i))|^2+\rho\sum_{h=1}^{l}|d_{kh}|^2L_2^2|y_h(i-\tau_{kh}(i))|^2+\rho|I_k|^2+\rho|J_k|^2$$

$$+\{\rho\sum_{h=1}^{l}|d_{kh}|^2L_2^2+\rho\sum_{h=1}^{l}|q_{kh}|^2N_1^2\}|x_h(i)|^2-\{\rho\sum_{k=1}^{l}|d_{hk}|^2L_2^2+\rho\sum_{k=1}^{l}|q_{hk}|^2N_1^2\}|x_k(i)|^2$$
$$+\{\rho\sum_{h=1}^{l}|r_{kh}|^2N_2^2+\rho\sum_{h=1}^{l}|b_{kh}|^2L_1^2\}|y_h(i)|^2-\{\rho\sum_{k=1}^{l}|r_{hk}|^2N_2^2+\rho\sum_{k=1}^{l}|b_{hk}|^2L_1^2\}|y_k(i)|^2\Big\}$$
$$\leq\frac{1}{2}\rho^i\Big\{\Big[\rho|a_k|^2+\rho\sum_{h=1}^{l}e_{kh}^2L_3^2\sum_{\nu=1}^{\infty}M_{kh}(\nu)+\rho\sum_{k=1}^{l}|d_{hk}|^2L_2^2+\rho\sum_{k=1}^{l}|q_{hk}|^2N_1^2+\rho|p_k|^2$$
$$+\rho\sum_{h=1}^{l}s_{kh}^2N_3^2\sum_{\vartheta=1}^{\infty}O_{kh}(\vartheta)+\rho\sum_{k=1}^{l}|r_{hk}|^2N_2^2+\rho\sum_{k=1}^{l}|b_{hk}|^2L_1^2-2\Big](|x_k(i)|^2+|y_k(i)|^2)$$
$$+\rho\sum_{h=1}^{l}(|r_{kh}|^2N_2^2+|d_{kh}|^2L_2^2)(|x_h(i-\zeta_{kh}(i))|^2+|y_h(i-\tau_{kh}(i))|^2)$$
$$+\{\rho\sum_{h=1}^{l}|d_{kh}|^2L_2^2+\rho\sum_{h=1}^{l}|q_{kh}|^2N_1^2\}|x_h(i)|^2-\{\rho\sum_{k=1}^{l}|d_{hk}|^2L_2^2+\rho\sum_{k=1}^{l}|q_{hk}|^2N_1^2\}|x_k(i)|^2$$
$$+\{\rho\sum_{h=1}^{l}|r_{kh}|^2N_2^2+\rho\sum_{h=1}^{l}|b_{kh}|^2L_1^2\}|y_h(i)|^2-\{\rho\sum_{k=1}^{l}|r_{hk}|^2N_2^2+\rho\sum_{k=1}^{l}|b_{hk}|^2L_1^2\}|y_k(i)|^2+\rho|I_k|^2+\rho|J_k|^2\Big\}$$

From, Theorem 1, $\Delta v(i) < 0$.

$$\Delta v(i) < 0 \Rightarrow v(i)\leq v(0)$$
$$v(i) \geq \frac{1}{2}|x_k(i)|^2+\frac{1}{2}|y_k(i)|^2+\frac{1}{2}\sum_{h=1}^{l}d_{kh}^2L_2^2|x_h(i)|^2+\frac{1}{2}\sum_{h=1}^{l}r_{kh}^2N_2^2|y_h(i)|^2$$

and

$$v(0) \geq \frac{1}{2}|x_k(0)|^2+\frac{1}{2}|y_k(0)|^2+\frac{1}{2}\sum_{h=1}^{l}d_{kh}^2L_2^2|x_h(0)|^2+\frac{1}{2}\sum_{h=1}^{l}r_{kh}^2N_2^2|y_h(0)|^2+\frac{1}{2}\sum_{h=1}^{l}e_{kh}^2L_3^2\sum_{\nu=1}^{\infty}M_{kh}(\nu)$$
$$\times\sum_{\xi=0-\tau_{kh}(0)}^{-1}|x_h(0)|^2+\frac{1}{2}\sum_{h=1}^{l}s_{kh}^2N_3^2\sum_{\vartheta=1}^{\infty}O_{kh}(\vartheta)\sum_{\xi=0-\varsigma_{kh}(0)}^{-1}|y_h(0)|^2$$
$$v(i) \leq v(0)\leq\frac{1}{2}\sum_{h=1}^{l}(e_{kh}^2L_3^2)\sum_{-\tau_{kh}}^{0}|x_k|^2+\frac{1}{2}\sum_{h=1}^{l}(s_{kh}^2N_3^2)\sum_{-\varsigma_{kh}}^{0}|y_k|^2$$

$$|x_k(i)|^2+|y_k|^2 \leq \frac{1}{2}\sum_{h=1}^{l}\kappa\Big(\sup_{\gamma\in(-\tau,0)}|x_k(i)|^2+\sup_{\gamma\in(-\varsigma,0)}|y_k|^2\Big)$$

Here, $\kappa=\max\{e_{kh}^2L_3^2, s_{kh}^2N_3^2\}$. Hence, the periodic solution of given system (2) is exponential stable.

5. Illustrative Example

In this section, to show the exactness of this proposed work a numerical simulation is presented. In this example, we consider two-dimensional DBAMNNs with discrete and distributed delays of two neurons.

Example 1. *The discrete time BAMNNs with mixed time varying delays are considered as follows*

$$
\begin{aligned}
u_k(i+1) = &- Au_k(i) + \sum_{h=1}^{l} Bf_1(v_h(i)) + \sum_{h=1}^{l} Df_2(v_h(i - \tau_{kh}(i))) \\
&+ \sum_{h=1}^{l} Ef_3(\sum_{\nu=1}^{\infty} M(\nu)v_h(i-\nu)) + I_k \\
v_k(i+1) = &- Pv_k(i) + \sum_{h=1}^{l} Qg_1(u_h(i)) + \sum_{h=1}^{l} Rg_2(u_h(i - \varsigma_{kh}(i))) \\
&+ \sum_{h=1}^{l} Sg_3(\sum_{\vartheta=1}^{\infty} O(\vartheta)u_h(i-\vartheta)) + J_k
\end{aligned} \tag{20}
$$

Here,

$$
A = \begin{pmatrix} 0.5 & 0 \\ 0 & 0.01 \end{pmatrix}, B = \begin{pmatrix} -1.02 & 1.3 \\ 0.5 & 0.6 \end{pmatrix}, D = \begin{pmatrix} 0.03 & 0.3 \\ 0.02 & 0.4 \end{pmatrix} E = \begin{pmatrix} 1.03 & 0.01 \\ 0.5 & 0.6 \end{pmatrix}
$$

$$
P = \begin{pmatrix} 0.02 & 0 \\ 0 & 0.1 \end{pmatrix} Q = \begin{pmatrix} 0.03 & 0.5 \\ 0.1 & -0.6 \end{pmatrix}, R = \begin{pmatrix} -0.5 & 0.3 \\ 0.5 & 0.04 \end{pmatrix}, S = \begin{pmatrix} 0.3 & 0.3 \\ 0.2 & 0.4 \end{pmatrix}
$$

and

$$
\begin{aligned}
&f_1(v_h(i)) = -0.1 \sin v_h(i); f_2(v_h(i - \tau_{kh}(i))) = -0.8 \sin v_h(i - \tau_{kh}(i)); f_3(v_h(i-\nu)) = -0.1 \cos v_h(i-\nu); \\
&g_1(u_h(i)) = -0.1 \sin u_h(i); g_2(u_h(i - \varsigma_{kh}(i))) = 0.5 \sin(v_h(i - \varsigma_{kh}(i))); g_3(v_h(i-\vartheta)) = -0.1e^{u_h(i-\vartheta)}; \\
&\quad \tau_{kh}(i) = \varsigma_{kh}(i) = 0.25i + 2; M_{kh} = 0.05e^{i}; O_{kh} = 0.5e^{i};
\end{aligned}
$$

From our observations, all the conditions in Theorems 1 and 2 are satisfied for Lipschitz constant 1 and $\rho = 1$. *Hence the DDBAMNNs* (2) *has a unique* 2π*-periodic solution which is exponentially stable. These facts are also supported by the illustrative numerical simulations, see Figure 1.*

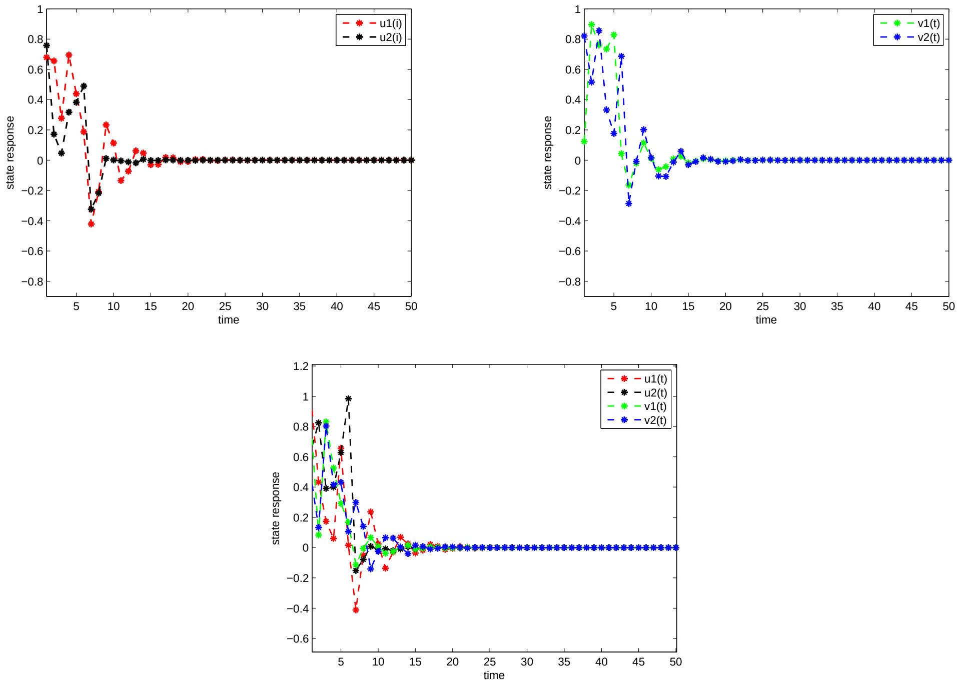

Figure 1. The state response $u(i)$ and $v(i)$ of (20).

6. Conclusions

In this manuscript, we have studied the existence, uniqueness and exponential stability of a periodic solution of discrete time BAMNNs with discrete and infinitely distributed delays. By using the global Lyapunov functional, coincidence degree theory combined with Krichhoff's matrix tree theorem, the sufficient conditions are derived. It should be noted that the method and techniques presented in this manuscript are more precisely variations on existing methods like the LMI approach, the method of the variation of parameters and so on. At the end of this work, we have given two examples to conclude and justify our main results.

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