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Smooth Counterexamples to Frame Bundle Freeness

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Abstract: We provide counterexamples to P. Olver's freeness conjecture for C^∞ actions. In fact, we show that a counterexample exists for any connected real Lie group with noncompact center, as well as for the additive group of the integers.

Keywords: prolongation; moving frame; dynamics

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1. Introduction

P. Olver's freeness conjecture (in his words) asserts: "If a Lie group acts effectively on a manifold, then, for some $n < \infty$, the action is free on [a nonempty] open subset of the jet bundle of order n." There is some ambiguity in this wording: No mention is made of connectedness of the group or manifold, the particular choice of jet bundle isn't made precise and the smoothness of the action is left unspecified.

In this note, we provide counterexamples to one interpretation of the freeness conjecture for C^∞ actions and higher order frame bundles. Those who know Olver's work will understand that there are a family of associated jet bundles to which he generally refers, and work in frame bundles then informs results in these jet bundles, through the associated bundle construction. In the C^∞ context, Olver has noted that, to avoid elementary counterexamples, "effective" must be strengthened to "fixpoint rare", which we define in Section 2 below. In Theorem 1 and Lemma 2, we show that a counterexample exists for any connected real Lie group with a noncompact center, as well as for the additive group of the integers. We also prove [1] the validity of the conjecture for connected real Lie groups with compact center. Finally, in [2], we describe a certain "meager" modification of the C^ω conjecture, and prove it holds for all connected real Lie groups.

For any fixed group, the C^∞ conjecture implies the C^ω conjecture, so the C^ω conjecture (on frame bundles) is now proved for connected real Lie groups with compact center. In [3], we offer a $\mathbb{Z}$-action on a manifold with an infinitely generated fundamental group, which, after induction of actions, provides a C^ω counterexample for any connected Lie group with a *noncompact* center. There is the possibility that the construction in Theorem 1 could be modified to make counterexamples to the C^ω conjecture on a contractible manifold, e.g., $\mathbb{R}^4$. The main difficulty in such an extension appears to be technical, and revolves around developing a good understanding of convergence of sequences in C^ω with respect to some well-chosen topology. For this, D. Morris' unpublished note [4] may be useful.

2. Miscellaneous Notation and Terminology

A subset of a topological space is meager (a.k.a. of first category) if it is a countable union of nowhere dense sets. A subset of a topological space is nonmeager (a.k.a. of second category) if it is not meager. A subset of a topological space is comeager (a.k.a. residual) if its complement is meager.

Let $\mathbb{N} := \{1, 2, 3, \ldots\}$. For all $d \in \mathbb{N}$, let $\mathrm{Id}_d : \mathbb{R}^d \to \mathbb{R}^d$ be the identity map, defined by $\mathrm{Id}_d(\sigma) = \sigma$. For every subset $S \subseteq \mathbb{R}$, for every $d \in \mathbb{N}$, we define $S^d := S \times S \times \cdots \times S \subseteq \mathbb{R}^d$.

Define $V_0 : \mathbb{R}^4 \to \mathbb{R}^4$ by: for all $\sigma \in \mathbb{R}^4$, $V_0(\sigma) = (0, 0, 0, 1)$.

Let $d \geq 1$ be an integer. A function $V : \mathbb{R}^d \to \mathbb{R}^d$ will be said to be complete if it is C^∞ and represents a complete vector field on $\mathbb{R}^d$. For any complete $V : \mathbb{R}^d \to \mathbb{R}^d$, we will use the notation $\Phi_t^V : \mathbb{R}^d \to \mathbb{R}^d$ to denote the time t flow of V, defined by the ODE $(d/dt)(\Phi_t^V(\sigma)) = V(\Phi_t^V(\sigma))$ and by the initial value condition $\Phi_0^V = \mathrm{Id}_d$.

Let $d \geq 1$ be an integer. Let $V : \mathbb{R}^d \to \mathbb{R}^d$ be complete. For any $A \subseteq \mathbb{R}$, for any $B \subseteq \mathbb{R}^d$, let $\Phi_A^V(B) := \{\Phi_a^V(b) \mid a \in A, b \in B\}$. For any $A \subseteq \mathbb{R}$, for any $b \in \mathbb{R}^d$, let $\Phi_A^V(b) := \{\Phi_a^V(b) \mid a \in A\}$. For any $a \in \mathbb{R}$, for any $B \subseteq \mathbb{R}^d$, let $\Phi_a^V(B) := \{\Phi_a^V(b) \mid b \in B\}$.

Let $d \geq 1$ be an integer. Let $V : \mathbb{R}^d \to \mathbb{R}^d$ be complete. Let $\sigma \in \mathbb{R}^d$. We say that (V, σ) is periodic if there exists an integer $n \neq 0$ such that $\Phi_n^V(\sigma) = \sigma$. For any integer $k \geq 0$, we say that (V, σ) is periodic to order k if there is an integer $n \neq 0$ such that the map $\Phi_n^V : \mathbb{R}^d \to \mathbb{R}^d$ agrees with the identity $\mathrm{Id}_d : \mathbb{R}^d \to \mathbb{R}^d$ to order k at σ. We say that (V, σ) is periodic to all orders if there is an integer $n \neq 0$ such that the map $\Phi_n^V : \mathbb{R}^d \to \mathbb{R}^d$ agrees with $\mathrm{Id}_d : \mathbb{R}^d \to \mathbb{R}^d$ to all orders at σ.

An action of a group G on a topological space X is fixpoint rare if, for any nonempty open subset U of X, for all $g \in G \backslash \{1_G\}$, there exists $u \in U$ such that $gu \neq u$. Any fixpoint rare action is effective. For a C^ω action on a connected manifold, fixpoint rare and effective are equivalent. Any continuous transitive action of a real Lie group preserves a C^ω structure, from which it follows that: By lemma 6.1 of [5], a continuous action of a connected real Lie group G on a topological space X is fixpoint rare iff, for every nonempty G-invariant open subset V of X, the G-action on V is effective.

3. Description of the Proof

To aid the reader, we give a broad description of how to construct a counterexample. Here, we focus only on flows in $\mathbb{R}^4$. We identify vector fields on $\mathbb{R}^4$ with smooth maps $\mathbb{R}^4 \to \mathbb{R}^4$.

Let $V : \mathbb{R}^4 \to \mathbb{R}^4$ be any complete vector field on $\mathbb{R}^4$. Then, PRP_V will denote the set of all $x \in \mathbb{R}^4$ such that $t \mapsto \Phi_t^V(x) : \mathbb{R} \to \mathbb{R}^4$ is proper. We say V is generically proper if PRP_V is comeager in $\mathbb{R}^4$.

Note that $\mathrm{PRP}_{V_0} = \mathbb{R}^4$, so V_0 is generically proper.

Let $\{\omega_1, \omega_2, \omega_3, \ldots\}$ be a countable dense subset of $\mathbb{R}^4$. For all $j \in \mathbb{N}$, let B_j denote the ball about $(0, 0, 0, 0)$ of radius j in $\mathbb{R}^4$ with its usual Euclidean metric. Let $\mathrm{Id} := \mathrm{Id}_4$, the identity map $\mathbb{R}^4 \to \mathbb{R}^4$.

Let $\sigma_1 := \omega_1$. Choose a complete vector field V_1 on $\mathbb{R}^4$ such that, for some $T_1 \in \mathbb{R} \backslash \{0\}$, $\Phi_{T_1}^{V_1}$ agrees with Id on a neighborhood of σ_1. (For example, $V_1 : R^4 \to \mathbb{R}^4$ could be the identically zero.) Note that V_1 is NOT generically proper because every point near σ_1 has a periodic (hence nonproper) orbit. Let $\Sigma_1 := \Phi_{\mathbb{R}}^{V_1}(\sigma_1)$ denote the orbit of σ_1 under the flow of V_1. Then, Σ_1 is compact. Let C_1 denote an open ball in $\mathbb{R}^4$ containing Σ_1.

Let $\mathcal{A}_1$ denote the set of all complete vector fields $V : \mathbb{R}^4 \to \mathbb{R}^4$ such that V agrees with V_1 to infinite order at every point of Σ_1.

Exercise for the reader: Find a generically proper vector field V_2 in $\mathcal{A}_1$. We expect that many (and perhaps MOST) vector fields in $\mathcal{A}_1$ are generically proper.

By density, choose $\sigma_2 \in \mathrm{PRP}_{V_2}$ such that $|\sigma_2 - \omega_2| < 1/2$.

Since $\sigma_2 \in \mathrm{PRP}_{V_2}$, for all sufficiently large $t > 0$, we have both $\Phi_t^{V_2}(\sigma_2) \notin C_1$ and $\Phi_{-t}^{V_2}(\sigma_2) \notin C_1$.

Choose a complete vector field V_2' on $\mathbb{R}^4$ such that V_2' agrees with V_2 on C_1 and for some $T_2 \in \mathbb{R} \backslash \{0\}$, $\Phi_{T_2}^{V_2'} = \mathrm{Id}$ on some neighborhood of σ_2.

Note that V_2' is NOT generically proper because every point near σ_2 has a closed (hence nonproper) orbit. Let $\Sigma_2 := \Phi_{\mathbb{R}}^{V_2'}(\sigma_2)$ denote the orbit of σ_2 under the flow of V_2'. Then, Σ_2 is compact. Let C_2 denote an open ball in $\mathbb{R}^4$ containing $B_1 \cup C_1 \cup \Sigma_1$.

Let $\mathcal{A}_2$ denote the set of all complete vector fields $V : \mathbb{R}^4 \to \mathbb{R}^4$ such that V agrees with V_2' to infinite order at every point of Σ_2 and such that V agrees with V_2 on C_1.

Exercise for the reader: Find a generically proper vector field V_3 in $\mathcal{A}_2$. We expect that many (and perhaps MOST) vector fields in $\mathcal{A}_2$ are generically proper.

By density, choose $\sigma_3 \in \mathrm{PRP}_{V_3}$ such that $|\sigma_3 - \omega_3| < 1/3$.

Since $\sigma_3 \in \mathrm{PRP}_{V_3}$, for all sufficiently large $t > 0$, we have both $\Phi_t^{V_3}(\sigma_3) \notin C_2$ and $\Phi_{-t}^{V_3}(\sigma_3) \notin C_2$.

Choose a complete vector field V_3' on $\mathbb{R}^4$ such that V_3' agrees with V_3 on C_2 and for some $T_3 \in \mathbb{R}\backslash\{0\}$, $\Phi_{T_3}^{V_3'} = \mathrm{Id}$ on a neighborhood of σ_3.

Note that V_3' is NOT generically proper because every point near σ_3 has a closed (hence nonproper) orbit. Let $\Sigma_3 := \Phi_{\mathbb{R}}^{V_3'}(\sigma_2)$ denote the orbit of σ_3 under the flow of V_3'. Then, Σ_3 is compact. Let C_3 denote an open ball in $\mathbb{R}^4$ containing $B_2 \cup C_2 \cup \Sigma_3$.

Let $\mathcal{A}_3$ denote the set of all complete vector fields $V : \mathbb{R}^4 \to \mathbb{R}^4$ such that V agrees with V_3' to infinite order at every point of Σ_3 and such that V agrees with V_2 on C_1.

Exercise for the reader: Find a generically proper vector field V_3 in $\mathcal{A}_3$. We expect that many (and perhaps MOST) vector fields in $\mathcal{A}_3$ are generically proper.

Continuing, we construct a sequence $V_1, V_2, \ldots$ of vector fields on $\mathbb{R}^4$. By construction, there is a unique vector field W on $\mathbb{R}^4$ such that:

$$\forall j \in \mathbb{N}, W = V_j \text{ on } C_j.$$

The flow of W agrees to infinite order with Id on the set $\{\sigma_1, \sigma_2, \ldots\}$. Since $\{\omega_1, \omega_2, \ldots\}$ is dense in $\mathbb{R}^4$, we see that $\{\sigma_1, \sigma_2, \ldots\}$ is dense in $\mathbb{R}^4$ as well. It remains to argue that the flow of W is fixpoint rare.

Given $t \in \mathbb{R}\backslash\{0\}$ and a nonempty open set U in $\mathbb{R}^4$, we want: $\exists u \in U$ s.t. $\Phi_t^W(u) \neq u$. Assume, for a contradiction that $\forall u \in U$, $\Phi_t^W(u) = u$.

For all $j \in \mathbb{N}$, let $Q_j := \{z \in \mathbb{R}^4 \,|\, \Phi_t^W(z) = \Phi_t^{V_j}(z)\}$. By construction, $\bigcup_{j\in\mathbb{N}} Q_j = \mathbb{R}^4$, so $\bigcup_{j\in\mathbb{N}} (Q_j \cap U) = U$. By the Baire Category Theorem, U is nonmeager in $\mathbb{R}^4$, so choose $j \in \mathbb{N}$ such that $Q_j \cap U$ is nonmeager in $\mathbb{R}^4$. Since V_j is generically proper, we get: PRP_{V_j} is comeager in $\mathbb{R}^4$. Then, $Q_j \cap U \cap \mathrm{PRP}_{V_j} \neq \emptyset$, and we fix a point $z \in Q_j \cap U \cap \mathrm{PRP}_{V_j}$. As $z \in Q_j$, we see that $\Phi_t^W(z) = \Phi_t^{V_j}(z)$. As $z \in U$, we see that $\Phi_t^W(z) = z$. Then, $\Phi_t^{V_j}(z) = z$. As $z \in \mathrm{PRP}_{V_j}$, we see that $s \mapsto \Phi_s^{V_j}(z) : \mathbb{R} \to \mathbb{R}^4$ is proper. Since $t \neq 0$ and $\Phi_t^{V_j}(z) = z$, we get: $s \mapsto \Phi_s^{V_j}(z) : \mathbb{R} \to \mathbb{R}^4$ is nonproper. Contradiction.

The exercises for the reader (given above) have not been completed by the author. They motivate the following OPEN PROBLEM:

Let $V : \mathbb{R}^4 \to \mathbb{R}^4$ be a complete, generically proper vector field. Let $C \subseteq \mathbb{R}^4$ have compact closure in $\mathbb{R}^4$, and let $p \in \mathrm{PRP}_V$. Prove: there is a complete, generically proper vector field $\widetilde{V} : \mathbb{R}^4 \to \mathbb{R}^4$ such that $\widetilde{V}$ agrees with V on C and for some $T \in \mathbb{R}\backslash\{0\}$, $\Phi_T^{\widetilde{V}}$ agrees with Id at p to infinite order.

To solve this open problem, one needs to take the V-orbit of p (which leaves compact sets, in both positive and negative time), and "bend" it, outside of C, so that it turns into an orbit that is periodic to infinite order. Moreover, this must be done in such a way as to preserve completeness and generic properness.

It is possible that this problem may admit a simple solution, in which case a relatively easy counterexample would be possible, as described above. In this paper, we replace "generically proper" by a stronger condition, "porous". We provide the iterative step in Lemma 1. Its proof is complicated and appears in [6].

4. The Iteration

Let $\mathcal{I} := \{(-a, a) \subseteq \mathbb{R} \,|\, a \in \mathbb{N}\}$. For every $I \in \mathcal{I}$, let $a_I := \sup I$, so $a_I \in \mathbb{N}$ and $I = (-a_I, a_I)$. For every $I \in \mathcal{I}$, for every integer $n \geq 1$, we define $nI := (-na_I, na_I) \in \mathcal{I}$; then, $a_{nI} = na_I$. For every $I \in \mathcal{I}$, let $\overline{I} := [-a_I, a_I]$ be the closure in $\mathbb{R}$ of I. We define $I_0 := (-1, 1) \in \mathcal{I}$; then, $a_{I_0} = 1$.

Let $\mathcal{C}$ be the set of C^∞ maps $V : \mathbb{R}^4 \to \mathbb{R}^4$ such that $V(\mathbb{R}^4) \subseteq \overline{I}_0^4$. Then, $V_0 \in \mathcal{C}$. For all $V \in \mathcal{C}$, the vector field $V : \mathbb{R}^4 \to \mathbb{R}^4$ is complete.

Let $V \in \mathcal{C}$. Let $\Pi : \mathbb{R}^4 \to \mathbb{R}$ be projection onto the 4th coordinate, defined by $\Pi(x_1, x_2, x_3, x_4) = x_j$. We say $\sigma \in \mathbb{R}^4$ is undeterred by V if $\Pi(\Phi^V_{\mathbb{R}}(\sigma)) = \mathbb{R}$. The undeterred set for V is the set $\mathcal{U}(V)$ of all $\sigma \in \mathbb{R}^4$ such that σ is undeterred by V. This set $\mathcal{U}(V)$ is V-invariant. We say V is porous if $\mathcal{U}(V)$ is dense in $\mathbb{R}^4$. For example, $\mathcal{U}(V_0) = \mathbb{R}^4$, so V_0 is porous. By a deterrence system, we mean an element $(V, I) \in \mathcal{C} \times \mathcal{I}$ such that $V = V_0$ on $(\mathbb{R}^4)\backslash(I^4)$. Let $\mathcal{D}$ be the set of all deterrence systems. Note, for all $(V, I) \in \mathcal{D}$, that $V \in \mathcal{V}(a_I)$.

For any $I \in \mathcal{I}$, we define $\mathcal{C}_I := \{P \in \mathcal{C} \,|\, P = V_0 \text{ on } (\,\overline{3I}\,)^4\}$.

Let $(V, I) \in \mathcal{D}$. For any $(W, J) \in \mathcal{D}$, we say (W, J) is a modification of (V, I) if: both $[\, a_I < a_J \,]$ and $[\, W = V \text{ on } \overline{I}^4 \,]$.

We define

$$\mathcal{M}(V, I) \;:=\; \{\, (W, J) \in \mathcal{D} \;|\; (W, J) \text{ is a modification of } (V, I) \,\}.$$

Let $I \in \mathcal{I}$. We define $\mathcal{D}^\times_I := \{(P, K) \in \mathcal{D} \,|\, 4I \subseteq K \text{ and } P \in \mathcal{C}_I\}$. We denote by $\mathcal{P}_I$ the set of $(P, K) \in \mathcal{D}^\times_I$ such that, for some integer $m > 2a_I$, we have:

- Φ^P_m agrees with Id_4 to all orders at $(0, 0, 0, -a_I)$ and
- for all $\tau \in I^3 \times \{-a_I\}$, for all $t \in (0, m)$,

$$[\, \Phi^P_t(\tau) \in I^4 \,] \qquad \Leftrightarrow \qquad [\, t < 2a_I \,].$$

Lemma 1. *Let $(V, I) \in \mathcal{D}$. Assume V is porous. Let $\sigma' \in \mathcal{U}(V)$. Then, there exists $(V', I') \in \mathcal{M}(V, I)$ such that V' is porous and such that (V', σ') is periodic to all orders.*

Proof. See Lemma 16.1 of [6]. □

5. A Result about Cyclic Groups

Lemma 2. *If G is a connected real Lie group whose center $Z(G)$ is noncompact, then $Z(G)$ has an infinite cyclic closed subgroup.*

Proof. This is Lemma 17.3 of [6]. □

6. The Counterexamples

Note that Theorem 1 below applies when G is discrete and isomorphic to the additive group $\mathbb{Z}$. Lemma 2 shows that Theorem 1 also applies when G is a connected real Lie group whose center $Z(G)$ is noncompact. In addition, note by (ii) below that, if

$$\text{either} \qquad G \text{ is connected} \qquad \text{or} \qquad G = Z,$$

then M is connected.

For any C^∞ manifold M, for any integer $k \geq 0$, let $\pi^M_k : F_k M \to M$ denote the kth order frame bundle of M. Let $\Pi : \mathbb{R}^4 \to \mathbb{R}$ be projection onto the 4th coordinate, defined by $\Pi(x_1, x_2, x_3, x_4) = x_j$.

Theorem 1. *Let G be a real Lie group. Assume the center $Z(G)$ of G admits an infinite cyclic closed subgroup Z. Then, there is a C^∞ manifold M and a fixpoint rare C^∞ action of G on M such that:*

(i) for any integer $k \geq 0$, there is a dense subset D of $F_k M$ such that, for all $\delta \in D$, the stabilizer $\mathrm{Stab}_Z(\delta)$ is infinite and
(ii) the number of connected components of M and G/Z are equal.

Proof. We have $(V_0, I_0) \in \mathcal{D}$. In addition, $\mathcal{U}(V_0) = \mathbb{R}^4$, so V_0 is porous.

Let $|\bullet| : \mathbb{R}^4 \to [0,\infty)$ be a norm on $\mathbb{R}^4$. Let $\{\omega_1,\omega_2,\omega_3,\ldots\}$ be a countable dense subset of $\mathbb{R}^4$. Choose $\sigma_1 \in \mathbb{R}^4$ such that $|\sigma_1 - \omega_1| < 1$. Then, $\sigma_1 \in \mathbb{R}^4 = \mathcal{U}(V_0)$.

By Lemma 1, choose $(V_1, I_1) \in \mathcal{M}(V_0, I_0)$ such that V_1 is porous and such that (V_1,σ_1) is periodic to all orders. Because V_1 is porous, $\mathcal{U}(V_1)$ is dense in $\mathbb{R}^4$, so fix $\sigma_2 \in \mathcal{U}(V_1)$ such that $|\sigma_2 - \omega_2| < 1/2$.

By Lemma 1, choose $(V_2, I_2) \in \mathcal{M}(V_1, I_1)$ such that V_2 is porous and such that (V_2,σ_2) is periodic to all orders. Because V_2 is porous, $\mathcal{U}(V_2)$ is dense in $\mathbb{R}^4$, so fix $\sigma_3 \in \mathcal{U}(V_2)$ such that $|\sigma_3 - \omega_3| < 1/3$.

By Lemma 1, choose $(V_3, I_3) \in \mathcal{M}(V_2, I_2)$ such that V_3 is porous and such that (V_3,σ_3) is periodic to all orders. Because V_3 is porous, $\mathcal{U}(V_3)$ is dense in $\mathbb{R}^4$, so fix $\sigma_4 \in \mathcal{U}(V_3)$ such that $|\sigma_4 - \omega_4| < 1/4$.

Continuing yields a countable dense subset $\{\sigma_1,\sigma_2,\sigma_3,\ldots\}$ of $\mathbb{R}^4$, and a sequence $(V_1,I_1),(V_2,I_2),(V_3,I_3),\ldots$ in $\mathcal{D}$. For each integer $j \geq 1$,

- $a_{I_{j+1}} > a_{I_j}$,
- $V_{j+1} = V_j$ on I_j^4,
- V_j is porous, and
- (V_j,σ_j) is periodic to all orders.

We have $a_{I_1} < a_{I_2} < a_{I_3} < \cdots$ and $a_{I_1}, a_{I_2}, a_{I_3}, \ldots \in \mathbb{N}$. It follows both that $I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots$ and that $I_1 \cup I_2 \cup I_3 \cup \cdots = \mathbb{R}$. Then, $I_1^4 \subseteq I_2^4 \subseteq I_3^4 \subseteq \cdots$ and $I_1^4 \cup I_2^4 \cup I_3^4 \cup \cdots = \mathbb{R}^4$. Define $V_\infty \in \mathcal{C}$ by the rule: For all integers $j \geq 1$, $V_\infty = V_j$ on I_j^4. Let $X := \bigcap_{k=1}^{\infty} (\mathcal{U}(V_k))$.

Claim 1: For all $t \in \mathbb{R}\backslash\{0\}$, for all $\sigma \in X$, we have $\Phi_t^{V_\infty}(\sigma) \neq \sigma$.

Proof of Claim 1. Let $t \in \mathbb{R}\backslash\{0\}$ and let $\sigma \in X$. Assume, for a contradiction, that we have $\Phi_t^{V_\infty}(\sigma) = \sigma$.

Since $\Phi_{\mathbb{R}}^{V_\infty}(\sigma) = \Phi_{[0,t]}^{V_\infty}(\sigma)$, we see that $\Phi_{\mathbb{R}}^{V_\infty}(\sigma)$ is compact. Then, since $I_1^4 \subseteq I_2^4 \subseteq I_3^4 \subseteq \cdots$ and $I_1^4 \cup I_2^4 \cup I_3^4 \cup \cdots = \mathbb{R}^4$, fix an integer $j \geq 1$ such that $\Phi_{\mathbb{R}}^{V_\infty}(\sigma) \subseteq I_j^4$. We have $V_j = V_\infty$ on I_j^4. Then, by Lemma 6.1 of [6], $\Phi_{\mathbb{R}}^{V_j}(\sigma) = \Phi_{\mathbb{R}}^{V_\infty}(\sigma)$.

Then, $\Pi(\Phi_{\mathbb{R}}^{V_j}(\sigma)) = \Pi(\Phi_{\mathbb{R}}^{V_\infty}(\sigma)) \subseteq \Pi(I_j^4) = I_j$. However because $\sigma \in X \subseteq \mathcal{U}(V_j)$, it follows that $\Pi(\Phi_{\mathbb{R}}^{V_j}(\sigma)) = \mathbb{R}$. Thus, we have $\mathbb{R} \subseteq I_j$, contradiction. □

Claim 2: For all integers $j \geq 1$, (V_∞,σ_j) is periodic to all orders.

Proof of Claim 2. Fix an integer $j \geq 1$. We wish to show that (V_∞,σ_j) is periodic to all orders.

Because $(V_j, I_j) \in \mathcal{D}$ and (V_j,σ_j) is periodic, we conclude, from Lemma 8.13 of [6], that $\Phi_{\mathbb{R}}^{V_j}(\sigma_j) \subseteq I_j^4$. For all $\tau \in I_j^4$, because V_∞ and V_j agree on I_j^4, which is an open neighborhood of τ, it follows that they agree to all orders at τ. Thus, for all $t \in \mathbb{R}$, V_∞ and V_j agree to all orders at $\Phi_t^{V_j}(\sigma_j)$. Then, by Lemma 6.3 of [6], for all $t \in \mathbb{R}$ $\Phi_t^{V_\infty} : \mathbb{R}^4 \to \mathbb{R}^4$ and $\Phi_t^{V_j} : \mathbb{R}^4 \to \mathbb{R}^4$ agree to all orders at σ_j. Thus, as (V_j,σ_j) is periodic to all orders, it follows (V_∞,σ_j) is periodic to all orders as well. □

Because Z is infinite cyclic, it follows that Z is isomorphic to the additive discrete group $\mathbb{Z}$. Let $f : Z \to \mathbb{Z}$ be an isomorphism. Define a Z-action on $\mathbb{R}^4$ by: for all $z \in Z$, for all $\sigma \in \mathbb{R}^4$, $z\sigma = \Phi_{f(z)}^{V_\infty}(\sigma)$. Let $M := G \times_Z \mathbb{R}^4$. Because the Z-action on $\mathbb{R}^4$ is C^∞, it follows that the G-action on M is C^∞ as well. By construction, M is (the total space of) a fiber bundle over G/Z with fiber $\mathbb{R}^4$, so, because $\mathbb{R}^4$ is connected, M has the same number of connected components as does G/Z.

By Corollary 8.4(i) of [6], $\mathcal{U}(V)$ is an open subset of $\mathbb{R}^4$. For all integers $k \geq 1$, V_k is porous. Then, $\mathcal{U}(V_k)$ is a dense open subset of $\mathbb{R}^4$. Then, because $X = \bigcap_{k=1}^{\infty} (\mathcal{U}(V_k))$, we see, by the Baire Category Theorem, that X is dense in $\mathbb{R}^4$. By Claim 1, for all $z \in Z\backslash\{1_Z\}$, for all $\sigma \in X$, we have $\Phi_{f(z)}^{V_\infty}(\sigma) \neq \sigma$, i.e., we have $z\sigma \neq \sigma$. Thus, the Z-action on $\mathbb{R}^4$ is fixpoint rare. Then, the G-action on M is also fixpoint rare.

Let $p : G \times \mathbb{R}^4 \to G \times_Z \mathbb{R}^4 = M$ be the canonical map. Define an injection $\iota : \mathbb{R}^4 \to M$ by $\iota(\sigma) = p(1_G, \sigma)$. Let $\Sigma := \iota(\{\sigma_1, \sigma_2, \sigma_3, \ldots\})$. Because $\{\sigma_1, \sigma_2, \sigma_3, \ldots\}$ is dense in $\mathbb{R}^4$, $A := G\Sigma$ is dense in M.

Fix an integer $k \geq 0$, and let $\pi := \pi_k^M : F_k M \to M$ be the kth order frame bundle of M. Let $D := \pi^{-1}(A)$. Since A is dense in M, and since $\pi : F_k M \to M$ is open, it follows that D is dense in $F_k M$. Fix $\delta \in D$. We wish to show that $\mathrm{Stab}_Z(\delta)$ is infinite.

Let $S := \mathrm{Stab}_G(\delta)$. Then, $\mathrm{Stab}_Z(\delta) = S \cap Z$. We therefore wish to show that $S \cap Z$ is infinite.

Since $\pi(\delta) \in A = G\Sigma$, fix $g_0 \in G$, $\tau_0 \in \Sigma$ such that $\pi(\delta) = g_0\tau_0$. Let $\delta_0 := g_0^{-1}\delta$. Then, $\pi(\delta_0) = \tau_0$, i.e., $\delta_0 \in \pi^{-1}(\tau_0)$. Let $S_0 := \mathrm{Stab}_G(\delta_0)$. Then, $S = g_0 S_0 g_0^{-1}$. As $Z \subseteq Z(G)$, we have $Z = g_0 Z g_0^{-1}$. Then, $S \cap Z = g_0(S_0 \cap Z)g_0^{-1}$, so it suffices to show that $S_0 \cap Z$ is infinite.

Recall that $\mathrm{Id}_4 : \mathbb{R}^4 \to \mathbb{R}^4$ is the identity map defined by $\mathrm{Id}_4(\sigma) = \sigma$. Let $\mathrm{Id} : M \to M$ be the identity map defined by $\mathrm{Id}(\rho) = \rho$.

Since $\tau_0 \in \Sigma = \iota(\{\sigma_1, \sigma_2, \sigma_3, \ldots\})$, fix an integer $j \geq 1$ such that $\tau_0 = \iota(\sigma_j)$. By Claim 2, fix $n_0 \in \mathbb{Z}\backslash\{0\}$ such that $\Phi_{n_0}^{V_\infty} : \mathbb{R}^4 \to \mathbb{R}^4$ agrees with the identity $\mathrm{Id}_4 : \mathbb{R}^4 \to \mathbb{R}^4$ to all orders at σ_j.

Let $z_0 := f^{-1}(n_0)$. Then, $z_0 \in Z\backslash\{1_Z\}$, and, for all $\sigma \in \mathbb{R}^4$, we have $z_0\sigma = \Phi_{n_0}^{V_\infty}(\sigma)$. Then, the map $\sigma \mapsto z_0\sigma : \mathbb{R}^4 \to \mathbb{R}^4$ is equal to $\Phi_{n_0}^{V_\infty} : \mathbb{R}^4 \to \mathbb{R}^4$, and therefore agrees with $\mathrm{Id}_4 : \mathbb{R}^4 \to \mathbb{R}^4$ to all orders at σ_j. Then, since $Z \subseteq Z(G)$ and since $\iota(\sigma_j) = \tau_0$, it follows that the map $\rho \mapsto z_0\rho : M \to M$ agrees with $\mathrm{Id} : M \to M$ at τ_0 to all orders. In particular, $\rho \mapsto z_0\rho : M \to M$ agrees with $\mathrm{Id} : M \to M$ at τ_0 to order k. Then, for all $\rho \in \pi^{-1}(\tau_0)$, we have $z_0\rho = \rho$. Thus, since $\delta_0 \in \pi^{-1}(\tau_0)$, we get $z_0\delta_0 = \delta_0$. That is, $z_0 \in \mathrm{Stab}_Z(\delta_0) = S_0 \cap Z$. Let C_0 be the cyclic subgroup of $S_0 \cap Z$ generated by z_0. Every nontrivial subgroup of an infinite cyclic group is infinite, so C_0 is infinite. Thus, because $C_0 \subseteq S_0 \cap Z$, it follows that $S_0 \cap Z$ is infinite, as desired. $\square$

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