Article

On the Unitary Representations of the Braid Group $B_6$

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Abstract: We consider a non-abelian leakage-free qudit system that consists of two qubits each composed of three anyons. For this system, we need to have a non-abelian four dimensional unitary representation of the braid group $B_6$ to obtain a totally leakage-free braiding. The obtained representation is denoted by $\rho$. We first prove that $\rho$ is irreducible. Next, we find the points $y \in \mathbb{C}^*$ at which the representation $\rho$ is equivalent to the tensor product of a one dimensional representation $\chi(y)$ and $\hat{\mu}_6(\pm i)$, an irreducible four dimensional representation of the braid group $B_6$. The representation $\hat{\mu}_6(\pm i)$ was constructed by E. Formanek to classify the irreducible representations of the braid group $B_n$ of low degree. Finally, we prove that the representation $\chi(y) \otimes \hat{\mu}_6(\pm i)$ is a unitary relative to a hermitian positive definite matrix.

Keywords: braid group; unitarity; equivalence; irreducibility

1. Introduction

Due to Artin, the braid group $B_n$ is represented in the group $\text{Aut}(F_n)$ of automorphisms of the free group $F_n$ generated by $x_1, \ldots, x_n$. The matrix representation of $B_n$ was published by W. Burau in 1936. This representation was known as a Burau representation. Since then, other matrix representations of $B_n$ have been constructed. For more details, see [1].

Braid group unitary representations have been essential in topological quantum computations. To understand the $d$-dimensional systems in which anyons are exchanged, a lot of work has been made. The exchange of $n$ anyons inside the qudit system, the $d$-dimensional analogues of qubits, has been governed by the braid group $B_n$ which has $n-1$ generators $\tau_1, \ldots, \tau_{n-1}$. Here, $\tau_i$ exchanges the particle $i$ with its neighbor, particle $i+1$.

When the topological charge of the qudits changes due to the braiding of the anyons from different qudits, a leakage of some of the information will occur in the computational Hilbert space, the fusion space of the anyons.

The leakage-free braiding of anyons has been under investigation for a while. To perform universal quantum computation without any leakage, the requirement would be to consider two-qubit gates. This would be very restrictive and this property can only be realized for two-qubit systems related to the Ising-like anyons model [2].

R. Ainsworth and J.K. Slingerland showed that a non-abelian, leakage-free qudit of dimension $d$ involving $n$ anyons is equivalent to a non-abelian $d$-dimensional representation of the braid group $B_n$. Here, $n$ is the sum of the number of anyons $n_1$ inside the first qudit and the number of anyons $n_2$ inside the second qudit. As for the dimension $d$ of the representation of $B_n$, it is the product of the dimensions $d_1$ and $d_2$ of the Hilbert spaces of the individual qudits.

Moreover, it was proved in [2] that the number of anyons per qubit is either 3 or 4. Thus, there are mainly 3 different types of two-qubit systems and a 4-dimensional representation of the corresponding braid group is constructed for each. Taking into account E. Formanek’s result that there
is no \(d\)-dimensional representation of \(B_n\) with \(d < n - 2\), it was verified in [2] that the only possible type of two-qubit system is having 2 qubits of which each is composed of 3 anyons.

This system is a non-abelian leakage-free qudit system of dimension 4 involving 6 anyons. It is equivalent to a non-abelian 4-dimensional representation of the braid group \(B_6\). This representation is denoted by \(\rho\). Since the number of anyons is 6, there are 5 elementary exchanges \(\tau_1, \ldots, \tau_5\). The exchanges \(\tau_1, \tau_2, \tau_4, \) and \(\tau_5\) satisfy the following relations:

\[
\rho(\tau_i) = \rho_1(\tau_i) \otimes I_{d_2} \quad (1 \leq i \leq n_1 - 1)
\]

and

\[
\rho(\tau_i) = I_{d_1} \otimes \rho_2(\tau_i) \quad (n_1 + 1 \leq i \leq n - 1),
\]

where \(\rho_1\) and \(\rho_2\) are the \(d_1\) and \(d_2\)-dimensional representations of \(B_{n_1}\) and \(B_{n_2}\) on the Hilbert spaces of the first and second qudit respectively. \(I_{d_1}\) and \(I_{d_2}\) are the \(d_1\) and \(d_2\)-dimensional identity matrices respectively. Here, \(n_1 = n_2 = 3\) and \(d_1 = d_2 = 2\).

The matrix \(\rho(\tau_3)\) is constructed by imposing braid group relations. For more details, see [2].

In our work, we consider the unitary representation \(\rho\) and the irreducible representation \(\hat{\mu}_6(\pm i)\) which is defined by E. Formanek in [3]. Both representations are 4-dimensional representations of the braid group \(B_6\).

First, we prove that the unitary representation \(\rho : B_6 \rightarrow GL_4(\mathbb{C})\) is irreducible.

As the representation \(\rho\) is proved to be irreducible, it follows that it is equivalent to the tensor product of a one-dimensional representation \(\chi(y)\) and the irreducible 4-dimensional representation \(\hat{\mu}_6(\pm i)\), where \(y \in \mathbb{C}^{+}\). For more details, see [3].

We then determine the points \(y \in \mathbb{C}^{+}\) at which the two representations \(\rho\) and \(\chi(y) \otimes \hat{\mu}_6(\pm i)\) are equivalent.

Finally, we show that the representation \(\chi(y) \otimes \hat{\mu}_6(\pm i)\) is a unitary relative to a hermitian positive definite matrix.

2. Preliminaries

Definition 1 (See [4]). The braid group on \(n\) strings, \(B_n\), is the abstract group with presentation

\(B_n = \{\sigma_1, \ldots, \sigma_{n-1} | \sigma_i \sigma_j = \sigma_j \sigma_i \text{ if } |i - j| \geq 2, \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \text{ for } i = 1, \ldots, n - 2\}\).

The Hecke algebra representation of \(B_6\) was constructed by V.F.R. Jones in [5]. E. Formanek obtained a low-degree representation of \(B_6\) by conjugating the representation constructed by V.F.R. Jones by a certain permutation matrix. For more details, see [3].

Definition 2 (See [3]). The representation \(\mu_6 : B_6 \rightarrow GL_5(\mathbb{Z}[t^{\pm 1}])\) is given by:

\[
\mu_6(\sigma_1) = \begin{pmatrix}
1 & 0 & 0 & 0 & -t \\
0 & -t & 0 & 0 & 0 \\
0 & -1 & 1 & 0 & 0 \\
0 & -1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & -t
\end{pmatrix},
\]

\(\mu_6(\sigma_2) = \begin{pmatrix}
-t & 0 & 0 & 0 & 0 \\
0 & 1 & -t & 0 & 0 \\
0 & 0 & -t & 0 & 0 \\
-1 & 0 & 0 & 1 & 0 \\
-1 & 0 & 0 & 0 & 1
\end{pmatrix},
\]

\[
\mu_6(\sigma_3) = \begin{pmatrix}
1 & 0 & 0 & -t & 0 \\
0 & -t & 0 & 0 & 0 \\
0 & -1 & 1 & 0 & 0 \\
0 & 0 & 0 & -t & 0 \\
0 & -1 & 0 & 0 & 1
\end{pmatrix},
\]

\(\mu_6(\sigma_4) = \begin{pmatrix}
-t & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & -t \\
-1 & 0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & -t
\end{pmatrix},
\]

and
Theorem 2 (See [3]). A representation \( \rho : B_5 \to GL_5(\mathbb{C}) \) is of Burau type if \( r \geq 2 \) and it is equivalent to an irreducible representation \( \chi(y) \otimes \beta_5(z) : B_n \to GL_{n-1}(\mathbb{C}) \) or \( \chi(y) \otimes \beta_6(z) : B_n \to GL_{n-2}(\mathbb{C}) \) where \( \chi(y) \) is a one-dimensional representation, \( \beta_5(z) \) is the reduced Burau representation, and \( \beta_6(z) \) is the composition factor of the reduced Burau representation.

Definition 3 (See [3]). Let \( y \in \mathbb{C}^* \). The representation \( \chi(y) : B_n \to C^* \) is the one-dimensional representation defined by \( \chi(y)(\sigma_i) = y \).

Theorem 1 (See [3]). 1. For \( z \in \mathbb{C}^* \), \( \mu_5(z) \) and \( \mu_6(z) \) are irreducible unless \( z \) is a root of \((t^2 + t + 1)(t^2 + 1)\);
2. If \( z \) is a root of \( t^2 + t + 1 \), then the composition factors of \( \mu_6(z) \) are \( \chi(-z) \) and \( \beta_6(z) \) and the composition factors of \( \mu_5(z) \) are \( \chi(z) \) and \( \beta_5(z) \);
3. If \( z \) is a root of \( t^2 + 1 \), then the composition factors of \( \mu_6(z) \) are \( \chi(1) \) and an irreducible representation \( \mu_6(z) : B_6 \to GL_4(\mathbb{C}) \), where \( \mu_6(z)(\sigma_1) \) has eigen values \( 1, 1, -z, -z \). The composition factors of \( \mu_5(z) \) are \( \chi(1) \) and an irreducible representation \( \mu_5(z) \), which is the restriction of \( \mu_6(z) \) to \( B_5 \).

Theorem 2 (See [3]). Let \( \rho : B_6 \to GL_r(\mathbb{C}) \) be an irreducible representation, where \( 2 \leq r \leq 5 \). Then one of the following is true.

1. The representation \( \rho \) is of a Burau type;
2. For some \( y \in \mathbb{C}^* \), \( \rho \) is equivalent to \( \chi(y) \otimes \mu_6(\pm i) : B_6 \to GL_4(\mathbb{C}) \). Distinct pairs \( (y, \pm i) \) give inequivalent representations;
3. For some \( y, z \in \mathbb{C}^* \), \( \rho \) is equivalent to \( \chi(y) \otimes \mu_6(z) : B_6 \to GL_5(\mathbb{C}) \), where \( z \) is not a root of \((t^2 + t + 1)(t^2 + 1)\).

3. Irreducibility of \( \rho : B_6 \to GL_4(\mathbb{C}) \)

The construction of a two-qubit system with a minimum amount of leakage has been of great interest. The only two-qubit system that can be realized without leakage is the system of two 3-anyon qubits. This system is equivalent to a 4-dimensional representation of the braid group \( B_6 \). This representation which was constructed in [2] is denoted by \( \rho \).

In this section, we prove that \( \rho : B_6 \to GL_4(\mathbb{C}) \) is irreducible. We denote \( \tau_i \), the exchange of the \( i^{th} \) and \( (i + 1)^{th} \) anyon, by \( \sigma_i \), where \( 1 \leq i \leq 5 \).

Definition 5 (See [2]). The representation \( \rho : B_6 \to GL_4(\mathbb{C}) \) is defined as follows:

\[
\rho(\sigma_1) = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & a \end{pmatrix}, \quad \rho(\sigma_2) = \begin{pmatrix} \frac{1}{\bar{a}-\bar{a}^2} & 0 & c & 0 \\ 0 & \frac{1}{\bar{a}-\bar{a}^2} & \bar{a} & 0 \\ \bar{a} & 0 & \bar{a} & 0 \\ 0 & \bar{a} & 0 & \frac{1}{\bar{a}-\bar{a}^2} \end{pmatrix},
\]

The restriction of \( \mu_6 \) to \( B_5 \) is denoted by \( \mu_5 : B_5 \to GL_5(\mathbb{C}) \). For \( z \in \mathbb{C}^* \), the complex specializations \( \mu_6(z) \) and \( \mu_5(z) \) are obtained from \( \mu_6 \) and \( \mu_5 \) respectively by letting \( t = z \).
As the system considered in our work is leakage-free, the eigenvalues of the generators \( \sigma_i \). That is,

\[
\rho(\sigma_3) = \begin{pmatrix} x & 0 & 0 & 0 \\
0 & \bar{x} & 0 & 0 \\
0 & 0 & \bar{x} & 0 \\
0 & 0 & 0 & x \end{pmatrix}, \quad \rho(\sigma_4) = \begin{pmatrix} \frac{1}{\sqrt{f-1}} & e & 0 & 0 \\
e & \frac{1}{\sqrt{f-1}} & 0 & 0 \\
0 & 0 & \frac{1}{\sqrt{f-1}} & e \\
0 & 0 & -e & \frac{1}{\sqrt{f-1}} \end{pmatrix},
\]

and

\[
\rho(\sigma_5) = \begin{pmatrix} f & 0 & 0 & 0 \\
0 & f & 0 & 0 \\
0 & 0 & f & 0 \\
0 & 0 & 0 & f \end{pmatrix}.
\]

In [2], it was shown that the multiplicities among the eigenvalues of the generators of the braid group result in the formation of topological charges during the fusion of the anyons in the system. As the system considered in our work is leakage-free, the eigenvalues of the generators \( \sigma_i \) of the representation \( \rho \) should be the same. Thus, \( f = a \) or \( f = \bar{a} \) and \( e = c = \sqrt{1 - \frac{1}{2^{-a^2-\bar{a}^2}}} \).

By simple computations, the relation \( \rho(\sigma_2)\rho(\sigma_3)\rho(\sigma_2) = \rho(\sigma_3)\rho(\sigma_2)\rho(\sigma_3) \) yields the equation \( a^2 = -\bar{a}^2 \). But, \( a = e^{i\theta} \). Therefore, \( e^{2i\theta} = -e^{-2i\theta} \). This implies that \( e^{i\theta} = -1 \). Consequently, \( e^{2i\theta} = 1 \).

That is, \( a \) must be a primitive eighth root of unity. Furthermore, \( c = \sqrt{1 - \frac{1}{2^{-a^2-\bar{a}^2}}} = \sqrt{1 - \frac{1}{2}} = \frac{1}{\sqrt{2}} \).

Note that since \( a \) is a primitive eighth root of unity, \( a^8 = 1 \) and \( a^2 \neq 1 \). Then, \( a^3 \neq a \). Consequently, \( a^3 \neq a, f \neq f^3, \) and \( f \neq f^3 \). This emphasizes that the defined matrices \( \rho(\sigma_i), 1 \leq i \leq 5 \), are well-defined.

Now we study the irreducibility of \( \rho \). For simplicity, we denote \( \rho(\sigma_i) \) by \( \rho_i \) for \( 1 \leq i \leq 5 \).

**Lemma 1.** The representation \( \rho : B_6 \to \text{GL}_4(\mathbb{C}) \) has no non trivial proper invariant subspaces of dimension 1.

**Proof.** Let \( S \) be a proper invariant subspace of dimension 1. We consider all the possible cases.

**Case 1:** \( S = \langle e_i \rangle, i = 1, 2, 3, 4 \).

For simplicity, we take \( i = 1 \). Since \( S \) is invariant, it follows that \( \sigma_2(e_1) = \begin{pmatrix} \bar{a} \& 0 \\
0 & -c \\
0 & 0 \end{pmatrix} \in S. \)

This implies that \( c = 0 \), a contradiction.

**Case 2:** \( S = \langle e_i + \mu e_{i+1} \rangle, i = 1, 3, \mu \in \mathbb{C}^* \).

For simplicity, we take \( i = 1 \). Since \( S \) is invariant, it follows that \( \sigma_2(e_1 + \mu e_2) = \begin{pmatrix} \frac{1}{\bar{a}^2} \\
\bar{a}^2 \mu & -\bar{c} \\
0 & -\bar{c} \end{pmatrix} \in S. \)

This implies that \( c = 0 \), a contradiction.

Thus, there are no non trivial proper invariant subspaces of dimension 1.

**Lemma 2.** The representation \( \rho : B_6 \to \text{GL}_4(\mathbb{C}) \) has no non trivial proper invariant subspace of dimension 2.

**Proof.** Let \( S \) be a proper invariant subspaces of dimension 2. We consider all the possible cases.

**Case 1:** \( S = \langle e_i, e_{i+1} \rangle, i = 1, 3 \).
For simplicity, we take $i = 1$. Since $S$ is invariant, it follows that $\sigma_2(e_1) = \begin{pmatrix} \frac{1}{x^2 - y^2} \\ 0 \\ -c \\ 0 \end{pmatrix} \in S$.

This implies that $c = 0$, a contradiction.

**Case 2:** $S = \langle e_i, e_j \rangle\, i = 1, 2 \, j = 3, 4$.

For simplicity, we take $i = 1$. Since $S$ is invariant, it follows that $\sigma_4(e_1) = \begin{pmatrix} \frac{1}{x^2 - y^2} \\ -e \\ 0 \\ 0 \end{pmatrix} \in S$.

This implies that $e = 0$. But, $e = c$. Thus, $c = 0$, a contradiction.

**Case 3:** $S = \langle e_i, e_1 + u e_2 \rangle\, i = 3, 4 \, u \in \mathbb{C}^*$.

For simplicity, we take $i = 3$. Since $S$ is invariant, it follows that $\sigma_4(e_3) = \begin{pmatrix} 0 \\ 0 \\ -e \\ 0 \end{pmatrix} \in S$.

This implies that $e = 0$. But, $e = c$. Thus, $c = 0$, a contradiction.

**Case 4:** $S = \langle e_i, e_3 + u e_4 \rangle\, i = 1, 2 \, u \in \mathbb{C}^*$.

For simplicity, we take $i = 1$. Since $S$ is invariant, it follows that $\sigma_4(e_1) = \begin{pmatrix} \frac{1}{x^2 - y^2} \\ -e \\ 0 \\ 0 \end{pmatrix} \in S$.

This implies that $e = 0$. But, $e = c$. Thus, $c = 0$, a contradiction.

**Case 5:** $S = \langle e_1 + u e_2, e_3 + v e_4 \rangle\, u, v \in \mathbb{C}^*$.

Since $S$ is invariant, it follows that $\sigma_3(e_1 + u e_2) = \begin{pmatrix} x \\ y u \\ 0 \\ 0 \end{pmatrix} \in S$. This implies that $a = \bar{a}$. That is, $a^2 - \bar{a}^2 = 0$.

But, from the equation $\sigma_2 \sigma_3 \sigma_2 = \sigma_3 \sigma_2 \sigma_3$, we have $a^2 - \bar{a}^2 = 0$. Thus, $2a^2 = 0$ which gives a contradiction.

Thus, there are no non trivial proper invariant subspaces of dimension 2. □

Now, we state the theorem of irreducibility.

**Theorem 3.** The representation $\rho : B_6 \to GL_4(\mathbb{C})$ is irreducible.

**Proof.** By Lemma 1 and Lemma 2, there are no proper invariant subspaces of dimensions 1 and 2. Clearly, the representation $\rho$ is unitary, that is $\sigma_i \sigma_i^* = I_4$ for $1 \leq i \leq 5$.

We note that if the representation is unitary, then the orthogonal complement of a proper invariant subspace is again a proper invariant subspace. As there is no proper invariant subspace of dimension 1, there is no proper invariant subspace of dimension 3.

As a result, all the proper possible subspaces are not invariant. Consequently, $\rho$ is irreducible. □

**4. The Representations $\rho$ and $\chi(y) \otimes \hat{\mu}_6(\pm i)$ Are Equivalent**

By Theorem 3, the representation $\rho$ is irreducible. The eigen values of $\rho(\sigma_i)$ for $1 \leq i \leq 5$ are different from those of $\hat{\mu}_4(z)$, the composition factor of the reduced Burau representation. Therefore,
Theorem 4. The images of the generators of $B$ are not equivalent to the tensor product of a one dimensional representation $\chi(y)$ and $\hat{\beta}_4(z)$. That is, $\rho$ is not of a Burau type.

Moreover, $\rho$ is a 4-dimensional representation. Consequently, Theorem 2 implies that the representation $\rho$ is equivalent to the representation $\chi(y) \otimes \hat{\mu}_6(\pm i)$ for some $y \in \mathbb{C}^*$.

Note that, by Theorem 1, the representation $\hat{\mu}_6(z)$ is irreducible for $z = \pm i$ since the roots of the polynomial $t^2 + 1$ are clearly $\pm i$.

In this section, we determine the points $y \in \mathbb{C}^*$ at which the representations $\rho$ and $\chi(y) \otimes \hat{\mu}_6(\pm i)$ are equivalent.

Since the two representations are equivalent, the determinants of the matrices $(\chi(y) \otimes \hat{\mu}_6(\pm i))(\sigma_i)$ and $\rho(\sigma_i)$ are equal for $1 \leq i \leq 5$.

As a result, the two considered representations are equivalent at the following points: $y_1 = \sqrt{i}, y_2 = \sqrt{-1}, y_3 = -\sqrt{i}$, and $y_4 = -\sqrt{-1}$, where $i$ is the complex number such that $i^2 = -1$.

5. Unitarity of $\chi(y) \otimes \hat{\mu}_6(\pm i) : B_6 \rightarrow GL_4(\mathbb{C})$

As the representation $\rho$ is proved to be unitary and equivalent to the representation $\chi(y) \otimes \hat{\mu}_6(\pm i)$ for some $y \in \mathbb{C}^*$, the representation $\chi(y) \otimes \hat{\mu}_6(\pm i)$ is a unitary relative to a matrix $M$.

In this section, we find the matrix $M$ and we prove that $M$ is a hermitian and positive definite.

\textbf{Definition 6} (See [3]). The representation $\hat{\mu}_6(\pm i) : B_6 \rightarrow GL_4(\mathbb{C})$ is defined as follows:

\[ \sigma_1 \mapsto \begin{pmatrix} \pm i & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -i & 0 & \pm i & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}, \quad \sigma_2 \mapsto \begin{pmatrix} 1 & \pm i & 0 & 0 \\ 0 & \pm i & 0 & 0 \\ 0 & 0 & 1 & \pm i \\ 0 & 0 & 0 & \pm i \end{pmatrix}, \]

\[ \sigma_3 \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & \pm i & 0 & 0 \\ 0 & \pm i & 0 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}, \quad \sigma_4 \mapsto (1/2) \begin{pmatrix} 1 \pm i & 0 & -1 \pm i & 0 \\ -1 \pm i & 0 & 1 \pm i & 0 \\ 0 & 1 \pm i & 0 & -1 \pm i \\ 0 & -1 \pm i & 0 & 1 \pm i \end{pmatrix}, \]

and \[ \sigma_5 \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \pm i & 0 \\ 0 & 0 & 0 & \pm i \end{pmatrix}. \]

Now, we state the following theorem:

\textbf{Theorem 4}. The images of the generators of $B_6$ under $\hat{\mu}_6(\pm i)$ are unitary relative to a hermitian positive definite matrix $M$.

\textbf{Proof}. Let,

\[ M = \begin{pmatrix} 2 & 1 - i & 0 & 0 \\ 1 + i & 2 & 0 & 0 \\ 0 & 0 & 2 & 1 - i \\ 0 & 0 & 1 + i & 2 \end{pmatrix}. \]

Here, $i$ is the complex number such that $i^2 = -1$ and $M$ is an invertible matrix whose determinant equals 4.

For simplicity, we denote $(\hat{\mu}_6(\pm i))(\sigma_i)$ by $\sigma_i$ for $1 \leq i \leq 5$. 

By direct computations, \( \sigma_1 M \sigma_1^* = \sigma_2 M \sigma_2^* = \sigma_3 M \sigma_3^* = \sigma_4 M \sigma_4^* = \sigma_5 M \sigma_5^* = M \), where \( \sigma_i^* \) is the complex conjugate transpose of \( \sigma_i \) for \( 1 \leq i \leq 5 \). Therefore, the representation \( \hat{\mu}_6(\pm i) \) is a unitary relative to the matrix \( M \).

Let \( M^* \) be the complex conjugate transpose of \( M \). Clearly, \( M^* = M \). This implies that the invertible matrix \( M \) is hermitian.

By computations, the eigen values of the matrix \( M \) are \( 2 + \sqrt{2} \) and \( 2 - \sqrt{2} \). Clearly, both values are positive. Consequently, \( M \) is a positive definite matrix.

As a result, the representation \( \hat{\mu}_6(\pm i) \) is a unitary relative to an invertible hermitian positive definite matrix \( M \). \( \square \)

Note that the unitarity of the representation \( \hat{\mu}_6(\pm i) \) relative to the matrix \( M \) clearly implies that the representation \( \chi(\gamma) \otimes \hat{\mu}_6(\pm i) \) is also a unitary relative to the same matrix \( M \).

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