On Jacobi-Type Vector Fields on Riemannian Manifolds

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Abstract: In this article, we study Jacobi-type vector fields on Riemannian manifolds. A Killing vector field is a Jacobi-type vector field while the converse is not true, leading to a natural question of finding conditions under which a Jacobi-type vector field is Killing. In this article, we first prove that every Jacobi-type vector field on a compact Riemannian manifold is Killing. Then, we find several necessary and sufficient conditions for a Jacobi-type vector field to be a Killing vector field on non-compact Riemannian manifolds. Further, we derive some characterizations of Euclidean spaces in terms of Jacobi-type vector fields. Finally, we provide examples of Jacobi-type vector fields on non-compact Riemannian manifolds, which are non-Killing.

Keywords: Jacobi-type vector fields; Killing vector fields; conformal vector fields; Euclidean space

MSC: 53C20; 53B21

1. Introduction

Throughout this article, we assume that manifolds are connected and differentiable. There are several important types of smooth vector fields on an $n$-dimensional Riemannian manifold $(M, g)$, whose existence influences the geometry of the Riemannian manifold $M$. A smooth vector field $\xi$ on $M$ is called a Killing vector field if its local flow consists of local isometries of the Riemannian manifold $M$. The geometry of Riemannian manifolds with Killing vector fields has been studied quite extensively (cf., e.g., [1–6]). The presence of a non-zero Killing vector field on a compact Riemannian manifold constrains its geometry, as well as topology; for instance, it does not allow the Riemannian manifold to have negative Ricci curvature, and on a Riemannian manifold of positive curvature, its fundamental group contains a cyclic subgroup with a constant index depending only on $n$ (cf. [1,2]).

In Riemannian geometry, Jacobi vector fields are vector fields along a geodesic defined by the Jacobi equation that arise naturally in the study of the exponential map. More precisely, a vector field $J$ along a geodesic $\gamma$ in a Riemannian manifold $M$ is called a Jacobi vector field if it satisfies the Jacobi equation (cf. [7]):

$$\frac{d^2}{dt^2} J(t) + R(J(t), \dot{\gamma}(t)) \dot{\gamma}(t) = 0,$$

where $D$ denotes the covariant derivative with respect to the Levi–Civita connection $\nabla$, $R$ is the Riemann curvature tensor of $M$, $\dot{\gamma}(t)$ is the tangent vector field, and $t$ is the parameter of the geodesic. Clearly, the Jacobi equation is a linear, second order ordinary differential equation; in particular, the values of $J$ and $\frac{d}{dt} J(t)$ at one point of $\gamma$ uniquely determine the Jacobi vector field. Further, a Killing
vector field $\xi$ on a Riemannian manifold $(M, g)$ is a Jacobi vector field along each geodesic, since it satisfies the differential equation: $\dot{\gamma} + R(\xi, \dot{\gamma})\dot{\gamma} = 0$. Furthermore, it follows from the Jacobi equation that Jacobi vector fields on a Euclidean space are simply those vector fields that are linear in $t$.

As a natural extension of Jacobi vector fields, one of the authors introduced in [8] the notion of Jacobi-type vector fields as follows. A vector field $\eta$ on a Riemannian manifold $M$ is called a Jacobi-type vector field if it satisfies the following Jacobi-type equation:

$$\nabla_X \nabla_Y \eta - \nabla_{\nabla_Y X} \eta + R(\eta, X)X = 0, \quad X, Y \in \mathfrak{X}(M),$$

where $\mathfrak{X}(M)$ denotes the Lie algebra of smooth vector fields on $M$. Obviously, every Jacobi-type vector field is a Jacobi vector field along each geodesic on $M$.

Since each Killing vector field is a Jacobi-type vector field (see [8]), a natural question is the following:

Question 1: “For a given Riemannian manifold $M$, under which topological or geometric conditions is every Jacobi-type vector on $M$ Killing?”

One objective of this article is to study this question. In Section 3, we prove that if a Riemannian manifold $M$ is compact, then every Jacobi-type vector field on $M$ is Killing. In contrast, not every Jacobi-type vector field on a non-compact Riemannian manifold is Killing (see the examples in Section 6). Therefore, the second interesting question is

Question 2: “Under what conditions is a Jacobi-type vector field on a non-compact Riemannian manifold a Killing vector field?”

In Section 4, we obtain three necessary and sufficient conditions for a Jacobi-type vector field on a non-compact Riemannian manifold to be Killing (see Theorems 2–4). In Section 5, we prove two characterizations of Euclidean spaces using Jacobi-type vector fields (see Theorems 6 and 7). In the last section, we provide some explicit examples of non-Killing Jacobi-type vector fields.

2. Preliminaries

First, we recall the following result from [8].

**Proposition 1.** A Killing vector field on a Riemannian manifold is a Jacobi-type vector field.

Although each Killing vector field on a Riemannian manifold is a Jacobi-type vector field, there do exist Jacobi-type vector fields that are non-Killing. For instance, let us consider the Euclidean space $(\mathbb{R}^n, g)$ with the canonical Euclidean metric $g = \sum_{i=1}^n dx^i \otimes dx^i$. Then, it is easy to verify that the position vector field $\psi$ of $\mathbb{R}^n$:

$$\psi = \sum x^i \frac{\partial}{\partial x^i}$$

is of the Jacobi type and it satisfies $(\mathcal{L}_\psi g)(X, Y) = 2g(X, Y)$, where $\mathcal{L}$ denotes the Lie derivative. Hence, $\psi$ is a non-Killing vector field.

We need the following lemma from [8].

**Lemma 1.** If $\eta$ is a Jacobi-type vector field on a Riemannian manifold $M$, then we have the following equation:

$$\nabla_X \nabla_Y \eta - \nabla_{\nabla_Y X} \eta + R(\eta, X)X = 0, \quad X, Y \in \mathfrak{X}(M).$$
For a given Jacobi-type vector field $\eta$ on a Riemannian manifold $M$, let us denote by $\omega$ the one-form dual to $\eta$. Furthermore, we define a symmetric tensor field $B$ of type $(1,1)$ and a skew-symmetric tensor field $\phi$ of type $(1,1)$ respectively by:

$$(L_\eta g)(X,Y) = 2g(BX,Y) \quad \text{and} \quad d\omega(X,Y) = 2g(\phi X,Y)$$

for $X, Y \in \mathfrak{X}(M)$. By applying Koszul’s formula, we find:

$$\nabla_X \eta = BX + \phi X, \quad X \in \mathfrak{X}(M). \quad (1)$$

Combining this with Lemma 1 yields:

$$(\nabla_X B)Y + (\nabla_X \phi)Y + R(\eta, X)Y = 0, \quad (2)$$

where $(\nabla_X A)Y = \nabla_X(AY) - A\nabla_X Y$ for a tensor field $A$ of type $(1,1)$. If we define a smooth function $h$ on $M$ by $h = \text{Tr} B$, then for a local orthonormal frame $\{e_1, \ldots, e_n\}$ on $M$, by choosing $Y = e_i$ in Equation (2) and by taking the inner product with $e_i$, we find:

$$\sum_{i=1}^n g((\nabla_X B)e_i, e_i) = 0,$$

where we have used the skew-symmetry of the tensor $\phi$. Hence, the above equation gives $Xh = 0$ for any $X \in \mathfrak{X}(M)$. Thus, $h$ is a constant function. Consequently, we have the following.

**Lemma 2.** Let $\eta$ be a Jacobi-type vector field on a Riemannian manifold $(M, g)$. If $B$ is the symmetric operator associated with $\eta$ defined by $(L_\eta g)(X,Y) = 2g(BY, X)$, then $\text{Tr} B$ is a constant.

3. Jacobi-Type Vector Fields on Compact Riemannian Manifolds

For Question 1, we prove the following.

**Theorem 1.** Every Jacobi-type vector field on a compact Riemannian manifold is a Killing vector field.

**Proof.** Let $\eta$ be a Jacobi-type vector field on an $n$-dimensional compact Riemannian manifold $(M, g)$. Consider the Ricci operator $Q$ defined by:

$$g(QX, Y) = \text{Ric}(X, Y), \quad X, Y \in \mathfrak{X}(M),$$

where $\text{Ric}$ is the Ricci tensor. Then, for a local orthonormal frame $\{e_1, \ldots, e_n\}$ on $M$, we have:

$$QX = \sum_{i=1}^n R(X, e_i)e_i, \quad X \in \mathfrak{X}(M)$$

and consequently, Equation (2) gives:

$$\sum_{i=1}^n (\nabla_{e_i} B)e_i + \sum_{i=1}^n (\nabla_{e_i} \phi)e_i + Q(\xi) = 0. \quad (3)$$

Furthermore, using Equation (1), we get:

$$R(X, Y)\eta = (\nabla_X B)Y + (\nabla_X \phi)Y - (\nabla_Y B)X - (\nabla_Y \phi)X,$$
which yields:

\[ \text{Ric}(Y, \eta) = g \left( Y, \sum_{i=1}^{n} (\nabla_{e_{i}} B) e_{i} \right) - g \left( Y, \sum_{i=1}^{n} (\nabla_{e_{i}} \varphi) e_{i} \right), \]

where we have applied Lemma 2 and the facts that \( B \) is symmetric and \( \varphi \) is skew-symmetric. The above equation implies:

\[ Q(\eta) = \sum_{i=1}^{n} (\nabla_{e_{i}} B) e_{i} - \sum_{i=1}^{n} (\nabla_{e_{i}} \varphi) e_{i}, \]

which together with Equation (3) gives:

\[ \sum_{i=1}^{n} (\nabla_{e_{i}} B) e_{i} = 0 \quad \text{and} \quad \sum_{i=1}^{n} (\nabla_{e_{i}} \varphi) e_{i} + Q(\eta) = 0. \quad (4) \]

Since \( B \) is a symmetric operator, we can choose a local orthonormal frame \( \{e_{1}, \ldots, e_{n}\} \) on \( M \) that diagonalizes \( B \), and as \( \varphi \) is skew-symmetric, we have:

\[ \sum_{i=1}^{n} g(B e_{i}, \varphi e_{i}) = 0. \quad (5) \]

Recall that the divergence of a vector field \( X \) on \( M \) is given by (see, e.g., [9]):

\[ \text{div} \ X = \sum_{i=1}^{n} (\nabla_{e_{i}} X, e_{i}). \quad (6) \]

Now, by using Equations (1), (5), and (6), we see that the divergence of the vector field \( B \eta \) satisfies:

\[ \text{div}(B \eta) = \|B\|^2 , \]

where \( \|B\|^2 \) denotes the squared norm of \( B \). Thus, after integrating the above equation over the compact \( M \), we get \( B = 0 \). Consequently, Equation (1) confirms that \( \eta \) is a Killing vector field. \( \square \)

**Remark 1.** Let \( M \) be a compact real hypersurface of a Kähler manifold with a unit normal vector field \( N \). In view of Theorem 1, we observe that the assumption “the characteristic vector field \( \xi = -JN \) is of the Jacobi type” in the results of [10,11] is redundant.

### 4. Jacobi-Type Vector Fields on Non-Compact Riemannian Manifolds

On a compact Riemannian manifold, the notions of Jacobi-type vector fields and Killing vector fields are equivalent according to Theorem 1, yet on non-compact Riemannian manifolds, they are different in general (see the examples in Section 6). Therefore, it is an interesting question to seek some conditions under which a Jacobi-type vector field is a Killing vector field on a non-compact Riemannian manifold.

Note that if \( \eta \) is a Killing vector field on an \( n \)-dimensional Riemannian manifold \( M \), then \( B = 0 \) in Equation (1). Thus, we have \( \varphi \eta = \nabla_{\eta} \eta \). Hence, we obtain:

\[ \text{div}(\varphi \eta) = -\|\varphi\|^2 - g \left( \eta, \sum_{i=1}^{n} (\nabla_{e_{i}} \varphi) e_{i} \right). \]

Using Equation (4) in the above equation shows that, for a Killing vector field \( \eta \), we have:

\[ \text{div}(\varphi \eta) = \text{Ric}(\eta, \eta) - \|\varphi\|^2. \]
Moreover, if we define a smooth function \( f \) on \( M \) by \( f = \frac{1}{2} \| \eta \|^2 \), we get \( \nabla f = -\varphi \eta \), and thus, for a Killing vector field \( \eta \), the Laplacian \( \Delta f \) is given by:
\[
\Delta f = \| \varphi \|^2 - Ric(\eta, \eta) .
\] (7)

A natural question is the following:

Question 3: “Does the function \( f = \frac{1}{2} \| \eta \|^2 \) for a Jacobi-type vector field \( \eta \) on a Riemannian manifold satisfying (7) make \( \eta \) a Killing vector field?”

The next theorem provides an answer to this question.

**Theorem 2.** Let \( \eta \) be a Jacobi-type vector field on a Riemannian manifold \( M \). Then, \( \eta \) is a Killing vector field if and only if the function \( f = \frac{1}{2} \| \eta \|^2 \) satisfies:
\[
\Delta f \leq \| \varphi \|^2 - Ric(\eta, \eta) .
\]

**Proof.** Let \( \eta \) be a Jacobi-type vector field on an \( n \)-dimensional Riemannian manifold \( M \). Then, using Equation (1), the gradient \( \nabla f \) of \( f = \frac{1}{2} \| \eta \|^2 \) is given by:
\[
\nabla f = B \eta - \varphi \eta .
\] (8)

Now, using Equations (1) and (4), we compute:
\[
\text{div}(B \eta) = \| B \|^2 \quad \text{and} \quad \text{div}(\varphi \eta) = -\| \varphi \|^2 - g \left( \eta, \sum_{i=1}^{n} (\nabla e_i \varphi) e_i \right).
\] (9)

Thus, by using Equation (8), we conclude:
\[
\Delta f = \| B \|^2 + \| \varphi \|^2 + g \left( \eta, \sum_{i=1}^{n} (\nabla e_i \varphi) e_i \right).
\] (10)

Applying Equation (2) and Lemma 2, we find:
\[
Ric(\eta, \eta) = -g \left( \eta, \sum_{i=1}^{n} (\nabla e_i \varphi) e_i \right).
\] (11)

which together with Equation (10) yields:
\[
\Delta f = \| B \|^2 + \| \varphi \|^2 - Ric(\eta, \eta).
\]

Hence, if the inequality \( \Delta f \leq \| \varphi \|^2 - Ric(\eta, \eta) \) holds, then the above equation implies \( B = 0 \), that is \( \eta \) is a Killing vector field.

The converse is trivial as a Killing vector field is a Jacobi vector field and the function \( f \) satisfies Equation (7).

Recall that the flow \( \{ \psi_t \} \) of a vector field \( X \in \mathfrak{X}(M) \) on a Riemannian manifold \( M \) is called a geodesic flow, if for each point \( p \in M \), the curve \( \sigma(t) = \psi_t(p) \) is a geodesic on \( M \) passing through the point \( p \). As the local flow of a Killing vector field on a Riemannian manifold \( M \) consists of isometries of \( M \), it follows that a local flow of a Killing vector field is a geodesic flow, but the converse is not true. For example, the Reeb vector field \( \zeta \) of a proper trans-Sasakian manifold has as the local flow a geodesic flow, yet \( \zeta \) is not a Killing vector field (cf. [12]).

In the next theorem, we provide a very simple characterization for a Killing vector field to have constant length via a Jacobi-type vector field on a Riemannian manifold.
Theorem 3. Let $\eta$ be a Jacobi-type vector field on a Riemannian manifold $M$ with the flow of $\eta$ a geodesic flow. Then, $\eta$ is a Killing vector field of constant length if and only if the Ricci curvature $\text{Ric} (\eta, \eta)$ satisfies:

$$\text{Ric} (\eta, \eta) \geq \|\phi\|^2.$$ 

Proof. Let $\eta$ be a Jacobi-type vector field on an $n$-dimensional Riemannian manifold $M$. Since the local flow of $\eta$ is a geodesic flow, Equation (1) implies:

$$B\eta + \phi\eta = 0. \quad (12)$$

Now, using Equation (9), we conclude:

$$\|B\|^2 - \|\phi\|^2 - g(\eta, \sum_{i=1}^n (\nabla e_i \phi) e_i) = 0,$$

which upon using Equation (11) gives:

$$\text{Ric} (\eta, \eta) = \|\phi\|^2 - \|B\|^2.$$

Using the inequality $\text{Ric} (\eta, \eta) \geq \|\phi\|^2$ in the above equation, we get $B = 0$, that is $\eta$ is a Killing vector field. Moreover, Equation (12) gives $\phi\eta = 0$, and consequently, Equation (8) implies $\nabla f = 0$, that is $\eta$ has constant length.

Conversely, if $\eta$ is a Killing vector field of constant length, then using $B = 0$ and Equation (1) in $X(\|\eta\|^2) = 0$ gives $g(X, \phi\eta) = 0, X \in \mathfrak{X}(M)$. This gives $\phi\eta = 0$, which together with Equation (1) confirms $\nabla \eta = 0$, that is the local flow of $\eta$ is a geodesic flow. As $f$ is a constant, Equation (7) implies the equality $\text{Ric} (\eta, \eta) = \|\phi\|^2$. \hfill \Box

Recall that a smooth function $f$ on a Riemannian manifold $M$ is said to be harmonic if $\Delta f = 0$ and superharmonic if $\Delta f \leq 0$. The Hessian operator $A_f$ of a smooth function $f$ is a symmetric operator defined by:

$$A_f X = \nabla X \nabla f, \quad X \in \mathfrak{X}(M),$$

and the Hessian of $f$, denoted by $\text{Hess}(f)$, is given by:

$$\text{Hess}(f)(X, Y) = g(A_f X, Y), \quad X, Y \in \mathfrak{X}(M).$$

Now, we prove the following characterization of a Killing vector field using a Jacobi-type vector field on a Riemannian manifold.

Theorem 4. A Jacobi-type vector field $\eta$ on a Riemannian manifold $M$ is a Killing vector field of constant length if and only if the function $f = \frac{1}{2} \|\eta\|^2$ is superharmonic and the Hessian of $f$ satisfies $\text{Hess}(f)(\eta, \eta) \leq 0$.

Proof. Let $\eta$ be a Jacobi-type vector field on a Riemannian manifold $M$. Suppose the function $f = \frac{1}{2} \|\eta\|^2$ satisfies:

$$\text{Hess}(f)(\eta, \eta) \leq 0 \quad \text{and} \quad \Delta f \leq 0. \quad (13)$$

Using Equation (1), we have:

$$\nabla_\eta \eta = B\eta + \phi\eta. \quad (14)$$

After combining (14) with Equation (8), we get:

$$2B\eta = \nabla f + \nabla_\eta \eta, \quad 2\phi\eta = \nabla_\eta \eta - \nabla f. \quad (15)$$
Now, by taking the inner product in Equation (8) with \( \eta \), we get \( \eta(f) = g(B\eta, \eta) \), which gives:
\[
\eta(f) = g\left( (\nabla_\eta B) \eta, \eta \right) + 2g(\eta, \nabla_\eta \eta).
\]
(16)

Furthermore, the first equation in Equation (15) implies:
\[
2g(B\eta, \nabla_\eta \eta) = \nabla_\eta \eta(f) + \|\nabla_\eta \eta\|^2.
\]

Using the above equation in Equation (16) gives:
\[
\text{Hess}(f)(\eta, \eta) = g\left( (\nabla_\eta B) \eta, \eta \right) + \|\nabla_\eta \eta\|^2.
\]
(17)

Note that Equation (2) implies \( (\nabla_\eta B) \eta = - (\nabla_\eta \varphi) \eta \), and as \( \varphi \) is skew-symmetric, we obtain \( g\left( (\nabla_\eta \varphi) \eta, \eta \right) = 0 \). Consequently, the above equation implies \( g\left( (\nabla_\eta B) \eta, \eta \right) = 0 \). Thus, Equation (17) reduces to:
\[
\text{Hess}(f)(\eta, \eta) = \|\nabla_\eta \eta\|^2
\]
and using the condition in Equation (13) forces the above equation to yield \( \nabla_\eta \eta = 0 \). Consequently, the first equation in Equation (15) gives \( \nabla f = 2B\eta \), and on account of Equation (9), we conclude that \( \Delta f = 2\|B\|^2 \).

Now, using the fact that the function \( f \) is superharmonic, we conclude \( B = 0 \). Hence, \( \eta \) is a Killing vector field. Moreover, using \( \nabla_\eta \eta = 0 \) and \( B = 0 \) in Equation (15), we find \( \nabla f = 0 \) on the connected \( M \), which proves that \( f \) is a constant. Thus, \( \eta \) is a Killing vector field of constant length.

Conversely, if \( \eta \) is a Killing vector field of constant length, then obviously, \( \eta \) is a Jacobi-type vector field that satisfies \( \text{Hess}(f)(\eta, \eta) = 0 \) and \( \Delta f = 0 \).

**5. Jacobi-Type Vector Fields on Euclidean Spaces**

A vector field \( X \) on a Riemannian manifold \((M, g)\) is called conformal if it satisfies (cf. e.g., [7,13]):
\[
\mathcal{L}_X g = 2\rho g
\]
(18)
for some smooth function \( \rho : M \to \mathbb{R} \). The conformal vector field \( X \) is called non-trivial if the function \( \rho \) in (18) is a non-zero function. Further, a conformal vector field \( X \) is called a gradient conformal vector field if \( X \) is the gradient of some smooth function. Non-Killing conformal vector fields have been used, e.g., in [2,3,5,14–18] to characterize spheres among compact Riemannian manifolds.

We already known from Section 2 that the position vector field \( \xi \) of the Euclidean \( n \)-space \( \mathbb{R}^n \) is a Jacobi-type vector field satisfying \( \mathcal{L}_\xi g = 2g \). Hence, \( \xi \) is conformal. In fact, it is also a gradient conformal vector field with \( \xi = \nabla f \) with \( f = \frac{1}{2} \|\xi\|^2 \). Furthermore, it is known that if \( \psi \) denotes the position vector field on the complex Euclidean \( n \)-space \( \mathbb{C}^n \), then \( \xi = \psi + J\psi \) is of the Jacobi type, which is a non-gradient conformal vector field on \( \mathbb{C}^n \), where \( J \) denotes the complex structure on \( \mathbb{C}^n \).

From these properties of the vector fields \( \xi \), we ask the next question.

**Question 4:** “Is a Jacobi-type vector field on a complete Riemannian manifold that is also a conformal vector field characterized as a Euclidean space?”

The main purpose of this section is to study this question. First, we show that a complete Riemannian manifold admits a Jacobi-type vector field that is also a non-trivial gradient conformal vector field if and only if it is isometric to the Euclidean space \( \mathbb{R}^n \). Then, we prove that a complete Riemannian manifold admits a Jacobi-type vector field that is also a conformal vector field (not necessarily a gradient conformal vector field) that annihilates the operator \( \varphi \) if and only if it is isometric to the Euclidean space \( \mathbb{R}^n \).

To prove these results mentioned above, we need the following result from [19] (cf. Theorem 1).
Theorem 5. Let $M$ be a complete Riemannian manifold. If there exists a smooth function $f : M \to \mathbb{R}$ satisfying $\text{Hess}(f) = cg$ for some non-zero constant $c$, then $M$ is isometric to $\mathbb{R}^n$.

Now, we prove the following result, which is an easy application of Theorem 5.

Theorem 6. Let $M$ be a complete Riemannian manifold. Then, $M$ admits a Jacobi-type vector field that is also a non-trivial gradient conformal vector field if and only if $M$ is isometric to a Euclidean space.

Proof. Clearly, if $M$ is isometric to the Euclidean $n$-space $\mathbb{R}^n$, then its position vector field $\xi$ is a Jacobi-type vector field, which is also a non-trivial gradient conformal vector field.

Conversely, suppose that the complete Riemannian manifold $M$ admits a Jacobi-type vector field $\eta$ that is also a non-trivial gradient conformal vector field. Then, as $\eta$ is closed, we have that $\varphi = 0$ and $B = \rho I$ in Equation (1) and that the smooth function $\rho$ is a constant by Lemma 2. Moreover, $\xi$ being a gradient conformal vector field, there is a smooth function $f : M \to \mathbb{R}$ that satisfies $\eta = \nabla f$, and consequently, Equation (1) takes the form:

$$\nabla_X \nabla f = \rho X,$$

where the constant $\rho \neq 0$ is guaranteed by the fact that $\eta$ is a non-trivial gradient conformal vector field. The above equation implies that $\text{Hess}(f) = \rho g$ with non-zero constant $\rho$. Consequently, by Theorem 5, we conclude that $M$ is isometric to a Euclidean space.  

Finally, we prove the following.

Theorem 7. Let $M$ be a complete Riemannian manifold. Then, $M$ admits a Jacobi-type vector field $\eta$, which is also a non-trivial conformal vector field that annihilates the operator $\varphi$ (associated with $\eta$) if and only if $M$ is isometric to a Euclidean space.

Proof. Clearly, if $M$ is isometric to the Euclidean $n$-space $\mathbb{R}^n$, then its position vector field $\xi$ is a Jacobi-type vector field with $\varphi = 0$, which is also a non-trivial conformal vector field.

Conversely, if the complete Riemannian manifold $(M, g)$ admits a Jacobi-type vector field $\eta$ that is also a non-trivial conformal vector field with $\varphi(\eta) = 0$, then as $\eta$ is a conformal vector field, we have $B = \rho I$ in Equation (1), which thus takes the form:

$$\nabla_X \eta = \rho X + \varphi X, \quad X \in \mathfrak{X}(M),$$

and the smooth function $\rho$ is a constant by Lemma 2.

Define a smooth function $f : M \to \mathbb{R}$ by $f = \frac{1}{2} ||\xi||^2$, whose gradient is easily found using Equation (19), as:

$$\nabla f = \rho \eta - \varphi(\eta) = \rho \eta.$$

Then, after taking the covariant derivative in the above equation with respect to $X \in \mathfrak{X}(M)$ and using Equation (19), we conclude that:

$$\nabla_X \nabla f = \rho^2 X + \rho \varphi X.$$

Thus, we get:

$$\text{Hess}(f)(X, X) = \rho^2 g(X, X).$$

Now, using polarization in the above equation, we get $\text{Hess}(f) = \rho^2 g$. Note that the constant $\rho$ has to be non-zero as the vector field $\eta$ is a non-trivial conformal vector field. Hence, by Theorem 5, we conclude that $M$ is isometric to a Euclidean space.  


Remark 2. It was proven in [20] that a complete Kähler n-manifold \((M, J, g)\) is isometric to a complex Euclidean n-space \(\mathbb{C}^n\) if and only if \((M, J, g)\) admits a “special kind” of non-trivial Jacobi-type vector field.

6. Examples of Non-Killing Jacobi-Type Vector Fields

In this section, we provide some examples of Jacobi-type vector fields that are non-trivial conformal vector fields.

Example 1. Let \(x^1, \ldots, x^n\) be Euclidean coordinates of the Euclidean n-space \((\mathbb{R}^n, \langle \cdot, \cdot \rangle)\). Consider the vector field:

\[
\xi = \psi - \left( \psi \frac{\partial}{\partial x^i} \right) \frac{\partial}{\partial x^i} + \left( \psi \frac{\partial}{\partial x^j} \right) \frac{\partial}{\partial x^j},
\]

where \(\psi\) is the position vector field of \(\mathbb{R}^n\) and \(i, j\) are two fixed indices with \(i \neq j\). If we denote by \(\nabla\) the covariant derivative operator with respect to the Euclidean connection on \(\mathbb{R}^n\), then it is easy to verify that:

\[
\nabla_X \xi = X + \varphi(X), \quad X \in \mathfrak{X}(\mathbb{R}^n),
\]

where:

\[
\varphi(X) = -\langle x^i \partial / \partial x^i, X \rangle + \langle x^j \partial / \partial x^j, X \rangle,
\]

is skew-symmetric. Hence:

\[
\mathcal{L}_\xi \langle \cdot, \cdot \rangle = 2 \langle \cdot, \cdot \rangle,
\]

that is, \(\xi\) is a conformal vector field, which is non-closed. Moreover, we have:

\[
\nabla_X \nabla_Y \xi - \nabla_{\nabla X Y} \xi = -\text{Hess}(x^i)(X, X) \frac{\partial}{\partial x^i} + \text{Hess}(x^j)(X, X) \frac{\partial}{\partial x^j},
\]

where \(\text{Hess}(f)\) is the Hessian of \(f\). However, the Hessians \(\text{Hess}(x^i)\) and \(\text{Hess}(x^j)\) of the coordinate functions \(x^i\) and \(x^j\) are zero. Therefore, the above equation confirms that \(\xi\) is a Jacobi-type vector field on \((\mathbb{R}^n, \langle \cdot, \cdot \rangle)\). Therefore, \(\xi\) is a Jacobi-type vector field, which is a non-trivial conformal vector field. Hence, \(\xi\) is a non-Killing vector field on \(\mathbb{R}^n\).

Example 2. Let \(M'(q', q'', q', g')\) be a \((2n + 1)\)-dimensional Sasakian manifold (cf. [21]). Then:

\[
\nabla'_{q'} q'' = -q' X, \quad (\nabla' q')(X, Y) = g'(X, Y)q' - \eta'(Y)X, \quad X, Y \in \mathfrak{X}(M'),
\]

where \(\nabla'\) denotes the covariant derivative operator with respect to the Riemannian connection on \(M'\). Using the above equation, we conclude:

\[
R'(X, Y)q'' = \eta'(Y)X - \eta'(X)Y, \quad X, Y \in \mathfrak{X}(M'),
\]

which upon taking the inner product with \(Z \in \mathfrak{X}(M')\) gives:

\[
R''(X, Y, q''; Z) = \eta'(Y)g'(X, Z) - \eta'(X)g'(Y, Z),
\]

that is,

\[
R'(q'', Z)X = g'(X, Z)q' - \eta'(X)Z, \quad X, Z \in \mathfrak{X}(M').
\]

Now, let \(M = (0, \infty) \times R\) be the warped product with the warping function the coordinate function \(t\) on the open interval \((0, \infty)\) and with the warped product metric \(g = dt^2 + t^2 q'\). We shall show that the vector field \(\xi \in \mathfrak{X}(M)\) defined by:

\[
\xi = t \frac{\partial}{\partial t} - q'
\]
is a Jacobi-type vector field, as well as a non-trivial conformal vector field, which is non-Killing on \( M \).

We denote by \( \nabla \) the covariant derivative operator with respect to the Riemannian connection on the Riemannian manifold \( (M, g) \), and let \( E = h \frac{\partial}{\partial t} + V \), where \( V \in \mathfrak{X}(M) \) is a vector field on \( M \) and \( h : (0, \infty) \to \mathbb{R} \) is a smooth function. Then, using Proposition 35 in [22], an easy computation gives:

\[
\nabla_E \xi = \left( h \frac{\partial}{\partial t} + V \right) + \phi'(V) + \eta' \left( \frac{\partial}{\partial t} \right) - \frac{h}{t} \xi = E + \phi(E), \tag{22}
\]

where \( \phi \) is a \((1,1)\)-tensor field on \( M \) defined by:

\[
\phi(E) = \phi'(V) + \eta' \left( \frac{\partial}{\partial t} \right) - \frac{h}{t} \xi.
\]

It is easy to verify that \( \phi \) is a skew-symmetric tensor field. Furthermore, we may compute that:

\[
\nabla_E E = \left( hh' - t g'(V, V) \right) \frac{\partial}{\partial t} + \nabla'_V V + \frac{2h}{t} V. \tag{23}
\]

Now, using Equation (22), we conclude:

\[
\nabla_E \nabla_E \xi = \nabla_E E + \frac{2h}{t} \phi'(V) + \nabla'_V \phi'(V) + \eta' \left( \frac{\partial}{\partial t} \right) - \frac{h}{t} \xi = \nabla'_V V + \frac{2h}{t} V \tag{24}
\]

which upon using Equations (22) and (23), gives:

\[
\nabla \nabla_E \xi = \nabla_E E + \frac{2h}{t} \phi'(V) + \phi' \left( \frac{\partial}{\partial t} \right) - \frac{h}{t} \xi + g'(V, V) \xi + \left( 2h \eta'(V) + tV \right) \frac{\partial}{\partial t}. \tag{25}
\]

Using Proposition 40 in [22] (note the difference in sign convention for the curvature tensor in our work and [22]), first we get:

\[
R(\xi, E)E = -R(\xi', V) V,
\]

where \( R \) is the curvature tensor field for the Riemannian manifold \( (M, g) \), and then, using (5) of Proposition 40 in [22] or by a direct calculation, we find:

\[
R(\xi, E)E = -R'(\xi', V) V + g'(V, V) \xi' - \eta'(V) V. \tag{26}
\]

Hence, from Equations (20), (21), and (24)–(26), we may conclude:

\[
\nabla_E \nabla_E \xi - \nabla \nabla_E E + R(\xi, E)E = 0.
\]

Hence, \( \xi \) is a Jacobi-type vector field on the Riemannian manifold \( M \). Furthermore, using Equation (22), it is easy to verify that \( \xi \) is a non-trivial conformal vector field, which is non-Killing on the Riemannian manifold \( M \).


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