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Fractional Cauchy Problems for Infinite Interval Case-II

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Abstract: We consider fractional abstract Cauchy problems on infinite intervals. A fractional abstract Cauchy problem for possibly degenerate equations in Banach spaces is considered. This form of degeneration may be strong and some convenient assumptions about the involved operators are required to handle the direct problem. Required conditions on spaces are also given, guaranteeing the existence and uniqueness of solutions. The fractional powers of the involved operator B_X have been investigated in the space which consists of continuous functions u on $[0, \infty)$ without assuming $u(0) = 0$. This enables us to refine some previous results and obtain the required abstract results when the operator B_X is not necessarily densely defined.

Keywords: fractional derivative; abstract Cauchy problem; evolution equations; degenerate equations

MSC: 26A33

1. Introduction

In recent years, many studies were devoted to the problem of recovering the solution u to

$$BMu - Lu = f \tag{1}$$

where B, M, and L are closed linear operators on the complex Banach space E with $D(L) \subseteq D(M)$, $0 \in \rho(L)$, $f \in E$ and u is unknown. The first approach to handle existence and uniqueness of the solution u to Equation (1) was given by Favini-Yagi [1] (see in particular the monograph [2]). By using the real interpolation space $(E, D(B))_{\theta,\infty}$, $0 < \theta < 1$ (see [3,4]), suitable assumptions on the operators B, M, L guarantee that Equation (1) has a unique solution. This result was improved by Favini, Lorenzi, and Tanabe [5] (see also [6–8]).

In all cases, the basic assumptions read as follows:

(H_1) Operator B has a resolvent $(z - B)^{-1}$ for any $z \in \mathbb{C}$, $\operatorname{Re} z < a$, $a > 0$ satisfying

$$\|(z - B)^{-1}\|_{\mathcal{L}(E)} \le \frac{c}{|Re\, z| + 1}, \quad Re\, z < a\,. \tag{2}$$

(H_2) Operators L, M satisfy

$$\|M(zM - L)^{-1}\|_{\mathcal{L}(E)} \le \frac{c}{(|z| + 1)^{\beta}} \tag{3}$$

for any $z \in \Sigma_\alpha := \{z \in \mathbb{C} : Re\, z \ge -c(1 + |Im\, z|)^{\alpha},\ c > 0,\ 0 < \beta \le \alpha \le 1\}$.

(H_3) Let A be the possibly multivalued linear operator $A = LM^{-1}$, $D(A) = M(D(L))$. Then, A and B commute in the resolvent sense:

$$B^{-1}A^{-1} = A^{-1}B^{-1}.$$

Very recently, Al Horani et al. [9], see also [10], generalized the previous results to the interpolation space $(E, D(B))_{\theta,p}$, $1 < p \le \infty$, i.e.,

Lemma 1. *Let B, M, L be three closed linear operators on the complex Banach space E satisfying (H_1)–(H_3), $0 < \beta \le \alpha \le 1$, $\alpha + \beta > 1$. Then, for all $f \in (E, D(B))_{\theta,p}$, $2 - \alpha - \beta < \theta < 1$, $1 < p \le \infty$, Equation (1) admits a unique solution u such that $Lu, BMu \in (E, D(B))_{\theta,p}$.*

There are many choices of the operator B verifying Assumption (H_1). In [9], the authors handled the abstract equation of the form

$$D_t^{\tilde{\alpha}}(Mu(t)) - Lu(t) = f(t), \;\; 0 \le t \le T < \infty \tag{4}$$

in the Banach space X with initial condition $\big(g_{1-\tilde{\alpha}} * My\big)(0) = 0$, where

$$g_\beta(t) = \begin{cases} \frac{1}{\Gamma(\beta)}\, t^{\beta-1} & t > 0, \\ 0 & t \le 0, \end{cases} \quad \beta \ge 0,$$

and $\Gamma(\beta)$ is the Gamma function.

For Riemann–Liouville derivative $D_t^{\tilde{\alpha}}$ of order $\tilde{\alpha}$, we address the monograph [11] (see also [12,13]). Very recent applications concerning Caputo fractional derivative operator are also discussed in [14] by the same authors using a completely different method than Sviridyuk's group (see [15,16]). Some related topics can be found in [17–19].

In [20], the authors extended the results of direct and inverse problems, given in [9], to degenerate differential equations on the half line $[0, \infty)$. Precisely, let X be a complex Banach space and

$$E = \{u \in C([0,\infty); X); u \text{ is uniformly continuous and bounded in } [0,\infty), u(0) = 0\}$$

endowed with the sup norm. If B_X is the operator defined by

$$D(B_X) = \{u \in C^1([0,\infty); X); u \text{ and } u' \text{ are uniformly continuous and bounded in } [0,\infty), u(0) = 0 = u'(0)\},$$
$$B_X u = u' \;\text{ for }\; u \in D(B_X),$$

and M, L are two closed linear operators in the complex Banach space E satisfying

$$\|M(\lambda M - L)^{-1}\| \le \frac{\tilde{c}}{(|\lambda| + 1)^\beta} \quad \forall\, \lambda \in \Sigma_\alpha = \{\lambda; \mathrm{Re}\lambda \ge -c(1 + |\mathrm{Im}\lambda|)^\alpha\}, \;\; c > 0, \;\; \tilde{c} > 0,$$

$0 < \beta \le \alpha \le 1$, $0 < \tilde{\alpha} < 1$, then for all $f \in (E, D(B_X^{\tilde{\alpha}}))_{\theta,p}$, $2 - \alpha - \beta < \theta < 1$, $1 < p \le \infty$, equation

$$B_X^{\tilde{\alpha}} Mu - Lu = f$$

admits a unique solution u. Moreover, $Lu, B_X^{\tilde{\alpha}} Mu \in (E, D(B_X^{\tilde{\alpha}}))_{\omega,p}$, $\omega = \theta + \alpha + \beta - 2$.

In this paper, we refine our results in [20] by investigating the fractional power of the operator B_X in the space of continuous functions u defined on $[0, \infty)$ without assuming $u(0) = 0$, i.e.,

$$E = \{u \in C([0,\infty);X); u \text{ is uniformly continuous and bounded in } [0,\infty)\},$$
$$\|u\|_E = \sup_{0\le t<\infty} \|u(t)\|_X,$$

where X is a complex Banach space. In this case, B_X is not densely defined. In such a case, it is not known whether $B_X^{\alpha} B_X^{\beta} = B_X^{\alpha+\beta}$ is true or not, since in the proof of Lemma A2 of T. Kato [21] it seems it is essentially used that A is densely defined. To obtain our results on such a new space E, we should investigate the previous fractional power problem in case $A = B_X$.

The interpolation space $(E, D(B_X^{\tilde{\alpha}}))_{\theta,p}$, $0 < \theta < 1$, $p \in (1,\infty]$ could be characterized by using the famous results of P. Grisvard. Since the operator B_X is of type $(\pi/2, 1)$ and $\tilde{\alpha} \in (0,1)$, $-B_X^{\tilde{\alpha}}$ is the infinitesimal generator of an analytic semigroup $\{e^{-tB_X^{\tilde{\alpha}}}\}_{t>0}$ (see [2], Proposition 0.9, p. 19), the interpolation space $(E, D(B_X^{\tilde{\alpha}}))_{\theta,p}$ could be characterized by

$$\begin{aligned}(E, D(B_X^{\tilde{\alpha}}))_{\theta,p} &= \left\{u \in E;\ \|t^{1-\theta} B_X^{\tilde{\alpha}} e^{-tB_X^{\tilde{\alpha}}} u\|_{L_p^*(E)} + \|u\|_E < \infty\right\} \\ &= \left\{u \in E;\ \|\xi^{\theta} B_X^{\tilde{\alpha}} (\xi + B_X^{\tilde{\alpha}})^{-1} u\|_{L_p^*(E)} < \infty\right\},\end{aligned}$$

where $L_p^*(E)$ denotes the space of all strongly measurable $E-$valued functions f on $(0,\infty)$ such that

$$\int_0^\infty \|f(t)\|_E^p \frac{dt}{t} < \infty,\ \ 1 \le p < \infty,$$
$$\|f(t)\|_{L_\infty^*(E)} = \sup_{0<t<\infty} \|f(t)\|_E,\ \ p = \infty.$$

The following lemma is also needed:

Lemma 2. $\|f * g\|_{L_*^p} \le \|f\|_{L_*^p} \|g\|_{L_*^1}$ $\ \forall f \in L_*^p(\mathbb{R}_+),\ \forall g \in L_*^1(\mathbb{R}_+)$

Section 2 is devoted to our main results. In Section 3, we present our conclusions and remarks.

2. Main Results

Let X be a complex Banach space and

$$E = \{u \in C([0,\infty);X); u \text{ is uniformly continuous and bounded in } [0,\infty)\}, \tag{5}$$
$$\|u\|_E = \sup_{0\le t<\infty} \|u(t)\|_X. \tag{6}$$

Let B_X be an operator defined by

$$\begin{aligned} D(B_X) &= \{u \in C^1([0,\infty);X); u,\ u' \in E,\ u(0) = 0\} \\ &= \{u \in C^1([0,\infty);X); u \text{ and } u' \text{ are uniformly continuous and bounded in } [0,\infty) \\ &\text{and u(0)=0}\} \qquad (7) \\ &= \{u \in C^1([0,\infty);X); u \text{ and } u' \text{ are bounded and } u' \text{ is uniformly continuous in } [0,\infty) \\ &\text{and } u(0) = 0\}, \end{aligned}$$
$$B_X u = u' \ \text{ for }\ u \in D(B_X). \tag{8}$$

Let $f \in E$, $\mathrm{Re}\lambda > 0$. Consider the problem

$$\begin{aligned} &\frac{d}{dt}u(t) + \lambda u(t) = f(t), \quad 0 < t < \infty, \\ &u(0) = 0. \end{aligned} \tag{9}$$

The solution is

$$u(t) = \int_0^t e^{-\lambda(t-s)} f(s) ds, \tag{10}$$

and

$$\|u(t)\| = \left\| \int_0^t e^{-\lambda(t-s)} f(s) ds \right\| \leq \int_0^t e^{-\mathrm{Re}\lambda(t-s)} ds \|f\|_E = \frac{1 - e^{-\mathrm{Re}\lambda t}}{\mathrm{Re}\lambda} \|f\|_E \leq \frac{1}{\mathrm{Re}\lambda} \|f\|_E.$$

Hence, u is bounded in $[0, \infty)$, and so is $u' = f - \lambda u$. This implies that u is uniformly continuous in $[0, \infty)$. Furthermore, $u' = f - \lambda u$ is uniformly continuous. Therefore, $u \in D(B_X)$ and $(B_X + \lambda)u = f$. Since $f \in E$ is arbitrary, one concludes that $R(B_X + \lambda) = E$ and

$$\big((B_X + \lambda)^{-1} f\big)(t) = \int_0^t e^{-\lambda(t-s)} f(s) ds, \tag{11}$$

$$\|(B_X + \lambda)^{-1}\|_{\mathcal{L}(E)} \leq \frac{1}{\mathrm{Re}\lambda} \quad \forall \lambda : \mathrm{Re}\lambda > 0. \tag{12}$$

Here, we make some preparations. Suppose that A is a not necessarily densely defined closed linear operator in a Banach space X satisfying

(i) $\rho(-A) \supset \{\lambda; |\arg\lambda| < \pi - \omega\}$, $0 < \omega < \pi$;
(ii) $\lambda(\lambda + A)^{-1}$ is uniformly bounded in each smaller sector $\{\lambda; |\arg\lambda| < \pi - \omega - \epsilon\}$, $0 < \epsilon < \pi - \omega$; and
(iii) $\|\lambda(\lambda + A)^{-1}\|_{\mathcal{L}(X)} \leq M$ for $\lambda > 0$ with some $M > 0$.

The first Assumption (i) is equivalent to $\rho(A) \supset \{\lambda; \omega < |\arg\lambda| \leq \pi\}$.

CASE $0 \in \rho(A)$. Set for $\alpha > 0$

$$R_\alpha(\lambda) = -\frac{1}{2\pi i} \int_C (\lambda + z^\alpha)^{-1} (z - A)^{-1} dz, \ \ \lambda \geq 0, \tag{13}$$

where C runs in the resolvent set of A from $+\infty e^{-i\theta}$ to $+\infty e^{i\theta}$, $\omega < \theta \leq \pi$, avoiding the negative real axis and 0, where $+\infty\, e^{\pm i\infty} = \lim_{r \to \infty} re^{\pm i\infty}$. Let λ, $\mu \geq 0$. Let C' be another contour which has the same property as C and is located to the right of C without intersecting C. Then,

$$\begin{aligned} R_\alpha(\lambda) R_\alpha(\mu) &= \frac{1}{2\pi i} \int_C (\lambda + z^\alpha)^{-1} (z - A)^{-1} dz \frac{1}{2\pi i} \int_{C'} (\mu + \zeta^\alpha)^{-1} (\zeta - A)^{-1} d\zeta \\ &= \left(\frac{1}{2\pi i}\right)^2 \int_{C'} \int_C (\lambda + z^\alpha)^{-1} (\mu + \zeta^\alpha)^{-1} (z - A)^{-1} (\zeta - A)^{-1} dz d\zeta. \end{aligned}$$

$$R_\alpha(\lambda)R_\alpha(\mu) = \left(\frac{1}{2\pi i}\right)^2 \int_{C'}\int_C (\lambda+z^\alpha)^{-1}(\mu+\zeta^\alpha)^{-1}\frac{(z-A)^{-1}-(\zeta-A)^{-1}}{\zeta-z}dzd\zeta$$
$$= \left(\frac{1}{2\pi i}\right)^2 \int_{C'}\int_C (\lambda+z^\alpha)^{-1}(\mu+\zeta^\alpha)^{-1}\frac{(z-A)^{-1}}{\zeta-z}dzd\zeta$$
$$- \left(\frac{1}{2\pi i}\right)^2 \int_{C'}\int_C (\lambda+z^\alpha)^{-1}(\mu+\zeta^\alpha)^{-1}\frac{(\zeta-A)^{-1}}{\zeta-z}dzd\zeta$$
$$= \frac{1}{2\pi i}\int_C (\lambda+z^\alpha)^{-1}\frac{1}{2\pi i}\int_{C'}\frac{(\mu+\zeta^\alpha)^{-1}}{\zeta-z}d\zeta(z-A)^{-1}dz$$
$$- \frac{1}{2\pi i}\int_{C'}\frac{1}{2\pi i}\int_C \frac{(\lambda+z^\alpha)^{-1}}{\zeta-z}dz(\mu+\zeta^\alpha)^{-1}(\zeta-A)^{-1}d\zeta$$
$$= -\frac{1}{2\pi i}\int_{C'}(\lambda+\zeta^\alpha)^{-1}(\mu+\zeta^\alpha)^{-1}(\zeta-A)^{-1}d\zeta$$
$$= -\frac{1}{2\pi i}\int_{C'}\frac{\left[(\lambda+\zeta^\alpha)^{-1}-(\mu+\zeta^\alpha)^{-1}\right]}{\mu-\lambda}(\zeta-A)^{-1}d\zeta.$$

This yields

$$(\mu-\lambda)R_\alpha(\lambda)R_\alpha(\mu) = -\frac{1}{2\pi i}\int_{C'}(\lambda+\zeta^\alpha)^{-1}(\zeta-A)^{-1}d\zeta + \frac{1}{2\pi i}\int_{C'}(\mu+\zeta^\alpha)^{-1}(\zeta-A)^{-1}d\zeta$$
$$= R_\alpha(\lambda) - R_\alpha(\mu).$$

Hence, $\{R_\alpha(\lambda),\ \lambda \geq 0\}$ is a pseudo resolvent:

$$R_\alpha(\lambda) - R_\alpha(\mu) = (\mu-\lambda)R_\alpha(\lambda)R_\alpha(\mu), \tag{14}$$

and

$$R_\alpha(0) = -\frac{1}{2\pi i}\int_C z^{-\alpha}(z-A)^{-1}dz. \tag{15}$$

For $\alpha > 0$, $\beta > 0$,

$$R_\alpha(0)R_\beta(0) = \left(\frac{1}{2\pi i}\right)^2 \int_{C'}\int_C z^{-\alpha}\zeta^{-\beta}(z-A)^{-1}(\zeta-A)^{-1}dzd\zeta$$
$$= \left(\frac{1}{2\pi i}\right)^2 \int_{C'}\int_C z^{-\alpha}\zeta^{-\beta}\frac{(z-A)^{-1}-(\zeta-A)^{-1}}{\zeta-z}dzd\zeta$$
$$= \left(\frac{1}{2\pi i}\right)^2 \int_{C'}\int_C z^{-\alpha}\zeta^{-\beta}\frac{(z-A)^{-1}}{\zeta-z}dzd\zeta - \left(\frac{1}{2\pi i}\right)^2 \int_{C'}\int_C z^{-\alpha}\zeta^{-\beta}\frac{(\zeta-A)^{-1}}{\zeta-z}dzd\zeta.$$

The first term of the last side vanishes, and the second term is equal to

$$-\frac{1}{2\pi i}\int_{C'}\frac{1}{2\pi i}\int_C \frac{z^{-\alpha}}{\zeta-z}dz\zeta^{-\beta}(\zeta-A)^{-1}d\zeta = -\frac{1}{2\pi i}\int_{C'}\zeta^{-\alpha-\beta}(\zeta-A)^{-1}d\zeta = R_{\alpha+\beta}(0).$$

Therefore, the following formula is obtained:

$$R_\alpha(0)R_\beta(0) = R_{\alpha+\beta}(0),\ \ \alpha > 0,\ \beta > 0. \tag{16}$$

By virtue of Cauchy's representation formula of holomorphic functions, one has

$$R_1(0) = -\frac{1}{2\pi i}\int_C z^{-1}(z-A)^{-1}dz = A^{-1}. \tag{17}$$

Let $0 < \alpha < 1$. Then,

$$R_{1-\alpha}(0)R_\alpha(0) = R_\alpha(0)R_{1-\alpha}(0) = R_1(0) = A^{-1}.$$

Therefore, if $R_\alpha(0)u = 0$, then $A^{-1}u = R_{1-\alpha}(0)R_\alpha(0)u = 0$. This implies $u = 0$. Hence, $R_\alpha(0)$ has an inverse. Since

$$R_\alpha(0)u - R_\alpha(\lambda)u = \lambda R_\alpha(0)R_\alpha(\lambda)u,$$

$R_\alpha(\lambda)u = 0$ implies $R_\alpha(0)u = 0$, and hence $u = 0$, $R_\alpha(\lambda)$ has an inverse $\forall\lambda \geq 0$. Since $\forall u \in X$

$$R_\alpha(\lambda)u - R_\alpha(\mu)u = (\mu - \lambda)R_\alpha(\lambda)R_\alpha(\mu)u,$$

one observes $R_\alpha(\mu)u \in R(R_\alpha(\lambda)) = D(R_\alpha(\lambda)^{-1})$, and

$$u - R_\alpha(\lambda)^{-1}R_\alpha(\mu)u = (\mu - \lambda)R_\alpha(\mu)u.$$

Let $v \in D(R_\alpha(\mu)^{-1}) = R(R_\alpha(\mu))$ and $v = R_\alpha(\mu)u$. Then, $u = R_\alpha(\mu)^{-1}v$, $v \in D(R_\alpha(\lambda)^{-1})$ and

$$R_\alpha(\mu)^{-1}v - R_\alpha(\lambda)^{-1}v = (\mu - \lambda)v.$$

This yields

$$R_\alpha(\mu)^{-1}v - \mu v = R_\alpha(\lambda)^{-1}v - \lambda v, \;\; \forall v \in D(R_\alpha(\lambda)^{-1}) = D(R_\alpha(\mu)^{-1}).$$

Set $A^\alpha = R_\alpha(\lambda)^{-1} - \lambda$ for some $\lambda \geq 0$.
Then, $A^\alpha = R_\alpha(\lambda)^{-1} - \lambda \;\; \forall\lambda \geq 0$, and $R_\alpha(\lambda)^{-1} = A^\alpha + \lambda$.
This implies

$$R_\alpha(\lambda) = (\lambda + A^\alpha)^{-1}.$$

Furthermore, $A^\alpha = R_\alpha(0)^{-1}$ and

$$(A^\alpha)^{-1} = R_\alpha(0) = -\frac{1}{2\pi i}\int_C z^{-\alpha}(z - A)^{-1}dz.$$

Letting $\alpha = 1$, one gets $(A^1)^{-1} = R_1(0) = A^{-1}$. Hence, $A^1 = A$. Thus, writing $(A^\alpha)^{-1} = A^{-\alpha}$

$$\begin{aligned} A^{-\alpha} &= R_\alpha(0) = -\frac{1}{2\pi i}\int_C z^{-\alpha}(z - A)^{-1}dz, \;\; \alpha > 0, \\ A^{-\alpha}A^{-\beta} &= R_\alpha(0)R_\beta(0) = R_{\alpha+\beta}(0) = A^{-\alpha-\beta}, \;\; \alpha > 0, \;\; \beta > 0. \end{aligned}$$

It is not difficult to show that the following relation holds if $0 < \alpha < 1$:

$$(\lambda + A^\alpha)^{-1} = \frac{\sin \pi\alpha}{\pi}\int_0^\infty \frac{\mu^\alpha}{\lambda^2 + 2\lambda\mu^\alpha \cos \pi\alpha + \mu^{2\alpha}}(\mu + A)^{-1}d\mu, \;\; \lambda \geq 0. \tag{18}$$

Proposition 1. *Let A be a not necessarily densely defined closed linear operator in a Banach space X satisfying* (i)–(iii) *and $0 \in \rho(A)$. Then, the fractional power A^α of A is defined for $\alpha > 0$, and the followings hold:*

$$A^\alpha A^\beta = A^{\alpha+\beta}, \alpha > 0, \beta > 0, \quad A^1 = A, \quad A^{-\alpha} = (A^\alpha)^{-1} \;\; \text{is bounded}$$

and Equation (10) *holds.*

GENERAL CASE In what follows, we assume (i)–(iii). Let $0 < \alpha < 1$. Set for $\lambda > 0$

$$R_\alpha(\lambda) = \frac{\sin \pi\alpha}{\pi}\int_0^\infty \frac{\mu^\alpha}{\lambda^2 + 2\lambda\mu^\alpha \cos \pi\alpha + \mu^{2\alpha}}(\mu + A)^{-1}d\mu. \tag{19}$$

If $\epsilon > 0$, $A_\epsilon = A + \epsilon$ satisfies (i)–(iii), and $0 \in \rho(A_\epsilon)$. Hence, A_ϵ^α is defined, and

$$(\lambda + A_\epsilon^\alpha)^{-1} = \frac{\sin \pi\alpha}{\pi} \int_0^\infty \frac{\mu^\alpha}{\lambda^2 + 2\lambda\mu^\alpha \cos \pi\alpha + \mu^{2\alpha}} (\mu + A_\epsilon)^{-1} d\mu, \tag{20}$$

$$(\lambda + A_\epsilon^\alpha)^{-1} - (\mu + A_\epsilon^\alpha)^{-1} = (\mu - \lambda)(\lambda + A_\epsilon^\alpha)^{-1}(\mu + A_\epsilon^\alpha)^{-1},\ \lambda > 0,\ \mu > 0. \tag{21}$$

In view of (iii)

$$\|(\mu + A_\epsilon)^{-1} - (\mu + A)^{-1}\|_X = \| - \epsilon(\mu + \epsilon + A)^{-1}(\mu + A)^{-1}\|_X \leq \epsilon \frac{M}{\mu + \epsilon} \frac{M}{\mu}.$$

Therefore, with the aid of the dominated convergence theorem one obtains from Equations (17) and (18)

$$(\lambda + A_\epsilon^\alpha)^{-1} \to R_\alpha(\lambda),\ \ \lambda > 0\ \text{ as } \epsilon \to 0. \tag{22}$$

Thus, we have obtained:

Proposition 2. *Let A be a not necessarily densely defined closed linear operator in a Banach space X satisfying* (i)–(iii). *Then, the bounded operator valued function $\{R_\alpha(\lambda),\ \lambda > 0\}$ defined by Equation (19) is a pseudo resolvent:*

$$R_\alpha(\lambda) - R_\alpha(\mu) = (\mu - \lambda)R_\alpha(\lambda)R_\alpha(\mu),\ \lambda > 0,\ \mu > 0. \tag{23}$$

CASE $A = B_X$. Let $\epsilon > 0$. Then, $B_X + \epsilon > 0$ has a bounded inverse and

$$\|(B_X + \epsilon + \lambda)^{-1}\|_{\mathcal{L}(E)} \leq \frac{1}{\text{Re}\lambda + \epsilon} < \frac{1}{\text{Re}\lambda} \quad \forall \lambda : \text{Re}\lambda \geq 0. \tag{24}$$

By virtue of Proposition 1, the fractional power $(B_X + \epsilon)^\alpha$ of $B_X + \epsilon$ is defined for $\alpha > 0$, and the followings hold:

$$\begin{aligned} &(B_X + \epsilon)^\alpha (B_X + \epsilon)^\beta = (B_X + \epsilon)^{\alpha+\beta},\ \alpha > 0,\ \beta > 0, \quad (B_X + \epsilon)^1 = B_X + \epsilon, \\ &(B_X + \epsilon)^{-\alpha} = ((B_X + \epsilon)^\alpha)^{-1} \ \text{ is bounded,} \end{aligned} \tag{25}$$

and for $0 < \alpha < 1$

$$(\lambda + (B_X + \epsilon)^\alpha)^{-1} = \frac{\sin \pi\alpha}{\pi} \int_0^\infty \frac{\mu^\alpha}{\lambda^2 + 2\lambda\mu^\alpha \cos \pi\alpha + \mu^{2\alpha}} (\mu + B_X + \epsilon)^{-1} d\mu,\ \ \lambda \geq 0. \tag{26}$$

Especially,

$$(B_X + \epsilon)^{-\alpha} = \frac{\sin \pi\alpha}{\pi} \int_0^\infty \mu^{-\alpha} (\mu + B_X + \epsilon)^{-1} d\mu. \tag{27}$$

Therefore,

$$\begin{aligned} &\big((\lambda + (B_X + \epsilon)^\alpha)^{-1} f\big)(t) \\ &= \frac{\sin \pi\alpha}{\pi} \int_0^\infty \frac{\mu^\alpha}{\lambda^2 + 2\lambda\mu^\alpha \cos \pi\alpha + \mu^{2\alpha}} \big((\mu + B_X + \epsilon)^{-1} f\big)(t) d\mu \\ &= \frac{\sin \pi\alpha}{\pi} \int_0^\infty \frac{\mu^\alpha}{\lambda^2 + 2\lambda\mu^\alpha \cos \pi\alpha + \mu^{2\alpha}} \int_0^t e^{-(\mu+\epsilon)(t-s)} f(s) ds d\mu \\ &= \frac{\sin \pi\alpha}{\pi} \int_0^t \int_0^\infty \frac{\mu^\alpha e^{-(\mu+\epsilon)(t-s)}}{\lambda^2 + 2\lambda\mu^\alpha \cos \pi\alpha + \mu^{2\alpha}} d\mu f(s) ds \end{aligned} \tag{28}$$

and

$$\big((B_X + \epsilon)^{-\alpha} f\big)(t) = \frac{\sin \pi\alpha}{\pi} \int_0^t \int_0^\infty \mu^{-\alpha} e^{-(\mu+\epsilon)(t-s)} d\mu f(s) ds. \tag{29}$$

By the change of the independent variable $\mu(t-s)=\tau$,

$$\int_0^\infty \mu^{-\alpha}e^{-(\mu+\epsilon)(t-s)}d\mu = e^{-\epsilon(t-s)}\int_0^\infty \mu^{-\alpha}e^{-\mu(t-s)}d\mu$$
$$= e^{-\epsilon(t-s)}\int_0^\infty (t-s)^\alpha\tau^{-\alpha}e^{-\tau}(t-s)^{-1}d\tau = (t-s)^{\alpha-1}e^{-\epsilon(t-s)}\int_0^\infty \tau^{-\alpha}e^{-\tau}d\tau$$
$$= (t-s)^{\alpha-1}e^{-\epsilon(t-s)}\Gamma(1-\alpha).$$

Hence, using $\Gamma(\alpha)\Gamma(1-\alpha)=\pi/\sin\pi\alpha$, one observes

$$\big((B_X+\epsilon)^{-\alpha}f\big)(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}e^{-\epsilon(t-s)}f(s)ds. \tag{30}$$

Set for $\lambda>0$,

$$R(\lambda) = \frac{\sin\pi\alpha}{\pi}\int_0^\infty \frac{\mu^\alpha}{\lambda^2+2\lambda\mu^\alpha\cos\pi\alpha+\mu^{2\alpha}}(\mu+B_X)^{-1}d\mu. \tag{31}$$

Then, in view of Equation (23),

$$R(\lambda)-R(\mu) = (\mu-\lambda)R(\lambda)R(\mu),\ \ \lambda>0,\ \mu>0. \tag{32}$$

From Equations (12) and (31), it follows that

$$\|R(\lambda)\|_{\mathcal{L}(E)} \le \frac{\sin\pi\alpha}{\pi}\int_0^\infty \frac{\mu^{\alpha-1}}{(\lambda+\mu^\alpha\cos\pi\alpha)^2+(\mu\sin\pi\alpha)^2}d\mu. \tag{33}$$

For $f\in E$, in view of Equations (11) and (31),

$$\big(R(\lambda)f\big)(t) = \frac{\sin\pi\alpha}{\pi}\int_0^\infty \frac{\mu^\alpha}{\lambda^2+2\lambda\mu^\alpha\cos\pi\alpha+\mu^{2\alpha}}\big((\mu+B_X)^{-1}f\big)(t)d\mu$$
$$= \frac{\sin\pi\alpha}{\pi}\int_0^\infty \frac{\mu^\alpha}{\lambda^2+2\lambda\mu^\alpha\cos\pi\alpha+\mu^{2\alpha}}\int_0^t e^{-\mu(t-s)}f(s)dsd\mu$$
$$= \frac{\sin\pi\alpha}{\pi}\int_0^t\int_0^\infty \frac{\mu^\alpha e^{-\mu(t-s)}}{\lambda^2+2\lambda\mu^\alpha\cos\pi\alpha+\mu^{2\alpha}}d\mu f(s)ds. \tag{34}$$

Using

$$\Gamma(\alpha)\Gamma(1-\alpha) = \frac{\pi}{\sin\pi\alpha},\quad \int_0^\infty \mu^{-\alpha}e^{-(t-s)\mu}d\mu = \Gamma(1-\alpha)(t-s)^{\alpha-1}, \tag{35}$$

one deduces from Equation (34)

$$\big(R(\lambda)f\big)(t) - \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s)ds$$
$$= \frac{\sin\pi\alpha}{\pi}\int_0^t\int_0^\infty \frac{\mu^\alpha e^{-\mu(t-s)}}{\lambda^2+2\lambda\mu^\alpha\cos\pi\alpha+\mu^{2\alpha}}d\mu f(s)ds - \frac{\sin\pi\alpha}{\pi}\int_0^t\int_0^\infty \mu^{-\alpha}e^{-\mu(t-s)}d\mu f(s)ds$$
$$= \frac{\sin\pi\alpha}{\pi}\int_0^t\int_0^\infty \left(\frac{\mu^\alpha}{\lambda^2+2\lambda\mu^\alpha\cos\pi\alpha+\mu^{2\alpha}} - \mu^{-\alpha}\right)e^{-\mu(t-s)}d\mu f(s)ds.$$

One has

$$\begin{aligned}
&\left\| \int_0^t \int_0^\infty \left(\frac{\mu^\alpha}{\lambda^2 + 2\lambda\mu^\alpha \cos \pi\alpha + \mu^{2\alpha}} - \mu^{-\alpha} \right) e^{-\mu(t-s)} d\mu f(s) ds \right\|_X \\
&\leq \int_0^t \left| \int_0^\infty \left(\frac{\mu^\alpha}{\lambda^2 + 2\lambda\mu^\alpha \cos \pi\alpha + \mu^{2\alpha}} - \mu^{-\alpha} \right) e^{-\mu(t-s)} d\mu \right| \|f(s)\|_X ds \\
&\leq \int_0^t \int_0^\infty \left| \frac{\mu^\alpha}{(\lambda + \mu^\alpha \cos \pi\alpha)^2 + (\mu^\alpha \sin \pi\alpha)^2} - \mu^{-\alpha} \right| e^{-\mu(t-s)} d\mu ds \|f\|_E. \qquad (36)
\end{aligned}$$

Since

$$\begin{aligned}
&\left| \frac{\mu^\alpha}{(\lambda + \mu^\alpha \cos \pi\alpha)^2 + (\mu^\alpha \sin \pi\alpha)^2} - \mu^{-\alpha} \right| e^{-\mu(t-s)} \\
&\leq \left(\frac{\mu^\alpha}{(\lambda + \mu^\alpha \cos \pi\alpha)^2 + (\mu^\alpha \sin \pi\alpha)^2} + \mu^{-\alpha} \right) e^{-\mu(t-s)} \leq \left(\frac{1}{(\sin \pi\alpha)^2} + 1 \right) \mu^{-\alpha} e^{-\mu(t-s)}, \\
&\int_0^t \int_0^\infty \mu^{-\alpha} e^{-\mu(t-s)} d\mu ds = \Gamma(1-\alpha) \int_0^t (t-s)^{\alpha-1} ds = \Gamma(1-\alpha) \frac{t^\alpha}{\alpha} < \infty
\end{aligned}$$

and

$$\left| \frac{\mu^\alpha}{(\lambda + \mu^\alpha \cos \pi\alpha)^2 + (\mu^\alpha \sin \pi\alpha)^2} - \mu^{-\alpha} \right| e^{-\mu(t-s)} \to 0 \text{ as } \lambda \to 0, \ \forall (\mu, s) \in (0,\infty) \times (0,t),$$

the last right hand-side of Inequality (36) tends to 0 as $\lambda \to 0$. Therefore,

$$\big(R(\lambda) f\big)(t) \to \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds \text{ as } \lambda \to 0, \ \forall t \in (0,\infty). \qquad (37)$$

Suppose $R(\lambda)f = 0 \ \exists \lambda > 0$. Then, $R(\lambda)f = 0 \ \forall \lambda > 0$, i.e., $\big(R(\lambda)f\big)(t) \equiv 0 \ \forall \lambda > 0$. Hence, by virtue of Equation (37),

$$\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds \equiv 0.$$

This yields

$$\begin{aligned}
&0 \equiv \int_0^t (t-\tau)^{-\alpha} \int_0^\tau (\tau-s)^{\alpha-1} f(s) ds d\tau = \int_0^t \int_s^t (t-\tau)^{-\alpha} (\tau-s)^{\alpha-1} d\tau f(s) ds \\
&= B(1-\alpha,\alpha) \int_0^t f(s) ds \Longrightarrow f(t) \equiv 0.
\end{aligned}$$

Therefore, $R(\lambda)$ has an inverse $\forall \lambda > 0$. Set $B_X^\alpha = R(\lambda)^{-1} - \lambda$ for some $\lambda > 0$. Then, $B_X^\alpha = R(\lambda)^{-1} - \lambda, \forall \lambda > 0$, and $R(\lambda) = (\lambda + B_X^\alpha)^{-1}, \forall \lambda > 0$. Hence, in view of Equation (31),

$$(\lambda + B_X^\alpha)^{-1} = \frac{\sin \pi\alpha}{\pi} \int_0^\infty \frac{\mu^\alpha}{\lambda^2 + 2\lambda\mu^\alpha \cos \pi\alpha + \mu^{2\alpha}} (\mu + B_X)^{-1} d\mu, \ \lambda > 0. \qquad (38)$$

By virtue of Equations (34) and (38), one observes that, for $\lambda > 0$ and $f \in E$,

$$((\lambda + B_X^\alpha)^{-1} f)(t) = \frac{\sin \pi\alpha}{\pi} \int_0^t \int_0^\infty \frac{\mu^\alpha e^{-\mu(t-s)}}{\lambda^2 + 2\lambda\mu^\alpha \cos \pi\alpha + \mu^{2\alpha}} d\mu f(s) ds, \ 0 < t < \infty. \qquad (39)$$

Since

$$\begin{aligned}
&\left\| \int_0^t \int_0^\infty \frac{\mu^\alpha e^{-(\mu+\epsilon)(t-s)}}{\lambda^2 + 2\lambda\mu^\alpha \cos \pi\alpha + \mu^{2\alpha}} d\mu f(s) ds - \int_0^t \int_0^\infty \frac{\mu^\alpha e^{-\mu(t-s)}}{\lambda^2 + 2\lambda\mu^\alpha \cos \pi\alpha + \mu^{2\alpha}} d\mu f(s) ds \right\|_X \\
&= \left\| \int_0^t \left(e^{-\epsilon(t-s)} - 1\right) \int_0^\infty \frac{\mu^\alpha e^{-\mu(t-s)}}{\lambda^2 + 2\lambda\mu^\alpha \cos \pi\alpha + \mu^{2\alpha}} d\mu f(s) ds \right\|_X \\
&\le \int_0^t \left(1 - e^{-\epsilon(t-s)}\right) \int_0^\infty \frac{\mu^\alpha e^{-\mu(t-s)}}{\lambda^2 + 2\lambda\mu^\alpha \cos \pi\alpha + \mu^{2\alpha}} d\mu \|f(s)\|_X ds \\
&\le \int_0^t \left(1 - e^{-\epsilon(t-s)}\right) \int_0^\infty \frac{\mu^\alpha e^{-\mu(t-s)}}{(\mu^\alpha \sin \pi\alpha)^2} d\mu ds \|f\|_E \\
&= \frac{1}{(\sin \pi\alpha)^2} \int_0^t \left(1 - e^{-\epsilon(t-s)}\right) \int_0^\infty \mu^{-\alpha} e^{-\mu(t-s)} d\mu ds \|f\|_E \\
&= \frac{\Gamma(1-\alpha)}{(\sin \pi\alpha)^2} \int_0^t \left(1 - e^{-\epsilon(t-s)}\right) (t-s)^{\alpha-1} ds \|f\|_E \\
&= \frac{\Gamma(1-\alpha)}{(\sin \pi\alpha)^2} \int_0^t (1 - e^{-\epsilon s}) s^{\alpha-1} ds \|f\|_E \to 0 \text{ uniformly in } [0,T],\ 0 < T < \infty, \text{ as } \epsilon \to 0,
\end{aligned}$$

noting Equations (28) and (39), one observes that, if $\lambda > 0$,

$$\left((\lambda + (B_X + \epsilon)^\alpha)^{-1} f\right)(t) \to \left((\lambda + B_X^\alpha)^{-1} f\right)(t)$$

uniformly in $[0, T]$, $0 < T < \infty$ as $\epsilon \to 0$. By virtue of Equations (12) and (38), one deduces

$$\begin{aligned}
\|(\lambda + B_X^\alpha)^{-1}\|_{\mathcal{L}(E)} &\le \frac{\sin \pi\alpha}{\pi} \int_0^\infty \frac{\mu^{\alpha-1}}{\lambda^2 + 2\lambda\mu^\alpha \cos \pi\alpha + \mu^{2\alpha}} d\mu \\
&= \frac{\sin \pi\alpha}{\pi} \int_0^\infty \frac{r^{\alpha-1}}{1 + 2r^\alpha \cos \pi\alpha + r^{2\alpha}} dr \frac{1}{\lambda} = \frac{1}{\lambda},\ \ \lambda > 0.
\end{aligned}$$

For an arbitrary λ with $\mathrm{Re}\lambda > 0$, let μ be so large that $\mu > |\lambda|^2/(2\,Re\,\lambda)$. Then, $\mu > |\mu - \lambda|$. One has

$$\begin{aligned}
&(\lambda + B_X^\alpha)^{-1} = (\mu + B_X^\alpha + \lambda - \mu)^{-1} = \left(\left(I + (\lambda-\mu)(\mu + B_X^\alpha)^{-1}\right)(\mu + B_X^\alpha)\right)^{-1} \\
&= (\mu + B_X^\alpha)^{-1}\left(I + (\lambda-\mu)(\mu + B_X^\alpha)^{-1}\right)^{-1} = (\mu + B_X^\alpha)^{-1} \sum_{n=0}^\infty (-1)^n (\lambda-\mu)^n (\mu + B_X^\alpha)^{-n}.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\left\|(\lambda + B_X^\alpha)^{-1}\right\|_{\mathcal{L}(E)} \le \left\|(\mu + B_X^\alpha)^{-1}\right\|_{\mathcal{L}(E)} \sum_{n=0}^\infty |\lambda-\mu|^n \left\|(\mu + B_X^\alpha)^{-1}\right\|_{\mathcal{L}(E)}^n \\
&= \frac{1}{\mu} \sum_{n=0}^\infty \left(\frac{|\lambda-\mu|}{\mu}\right)^n = \frac{1}{\mu} \frac{1}{1 - |\lambda-\mu|/\mu} = \frac{1}{\mu - |\lambda-\mu|}.
\end{aligned}$$

Since

$$\begin{aligned}
&\frac{1}{\mu - |\lambda-\mu|} = \frac{\mu + |\lambda-\mu|}{\mu^2 - |\lambda-\mu|^2} = \frac{\mu + |\lambda-\mu|}{\mu^2 - (\mu^2 - 2\mu Re\,\lambda + |\lambda|^2)} = \frac{\mu + |\lambda-\mu|}{2\mu Re\,\lambda - |\lambda|^2} \\
&= \frac{1 + |\lambda/\mu - 1|}{2Re\,\lambda - |\lambda|^2/\mu} \longrightarrow \frac{1}{Re\,\lambda} \ \text{ as } \ \mu \longrightarrow \infty,
\end{aligned}$$

one concludes

$$\rho(B^{\alpha}_{X}) \supset \{\lambda; \operatorname{Re}\lambda < 0\} \text{ and } \|(\lambda + B^{\alpha}_{X})^{-1}\|_{\mathcal{L}(E)} \leq \frac{1}{\operatorname{Re}\lambda}, \ \operatorname{Re}\lambda > 0. \tag{40}$$

One has $\forall \lambda > 0, \forall f \in D(B^{\alpha}_{X})$

$$R(\lambda)^{-1} f = \lambda f + B^{\alpha}_{X} f \Longrightarrow \lambda R(\lambda) f + R(\lambda) B^{\alpha}_{X} f = f.$$

Hence,

$$\big(R(\lambda)B^{\alpha}_{X} f\big)(t) = f(t) - \lambda\big(R(\lambda)f\big)(t), \ t > 0.$$

Since Equation (37) holds for any $f \in E$, one has

$$\begin{aligned} &\big(R(\lambda)B^{\alpha}_{X} f\big)(t) \longrightarrow \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \big(B^{\alpha}_{X} f\big)(s)ds\,, \ \forall\, t \in (0,\infty)\,, \\ &\lambda\big(R(\lambda)f\big)(t) \longrightarrow 0\,, \ \forall\, t \in (0,\infty)\,. \end{aligned}$$

Therefore, one obtains

$$\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \big(B^{\alpha}_{X} f\big)(s)ds = f(t).$$

This implies

$$\frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{-\alpha} \int_0^\tau (\tau-s)^{\alpha-1} \big(B^{\alpha}_{X} f\big)(s)dsd\tau = \int_0^t (t-\tau)^{-\alpha} f(\tau)d\tau.$$

The left hand side is equal to

$$\frac{1}{\Gamma(\alpha)} \int_0^t \int_s^t (t-\tau)^{-\alpha} (\tau-s)^{\alpha-1} d\tau \big(B^{\alpha}_{X} f\big)(s)ds = \Gamma(1-\alpha) \int_0^t \big(B^{\alpha}_{X} f\big)(s)ds.$$

Hence,

$$\int_0^t \big(B^{\alpha}_{X} f\big)(s)ds = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} f(s)ds.$$

From this, it follows that

$$\int_0^t (t-\tau)^{\alpha-1} \int_0^\tau \big(B^{\alpha}_{X} f\big)(s)dsd\tau = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{\alpha-1} \int_0^\tau (\tau-s)^{-\alpha} f(s)dsd\tau.$$

By the change of the order of the integration

$$\frac{1}{\alpha} \int_0^t (t-s)^{\alpha} \big(B^{\alpha}_{X} f\big)(s)ds = \Gamma(\alpha) \int_0^t f(s)ds.$$

By the differentiation of both sides

$$\int_0^t (t-s)^{\alpha-1} \big(B^{\alpha}_{X} f\big)(s)ds = \Gamma(\alpha) f(t).$$

Therefore, B^{α}_{X} has an inverse $B^{-\alpha}_{X}$, and for $f \in D(B^{-\alpha}_{X})$

$$\big(B^{-\alpha}_{X} f\big)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)ds, \quad 0 < t < \infty. \tag{41}$$

Consequently, the following proposition is established:

Proposition 3. *Let B_X be the operator defined by Equations (7) and (8).*

Then, B_X satisfies $\rho(B_X) \supset \{\lambda; \mathrm{Re}\lambda < 0\}$ and Equations (11) and (12) hold. The fractional power B_X^α, $0 < \alpha < 1$, of B_X is defined implicitly by Equation (38) or Equation (39). B_X^α has an inverse $B_X^{-\alpha}$ and for $f \in D(B_X^{-\alpha})$ Equation (41) holds.

Especially if $f \in D(B_X^{-\alpha})$, then the function $\int_0^t (t-s)^{\alpha-1} f(s)ds$ belongs to E. The converse is given in the next proposition.

Proposition 4. *Suppose that both functions f and $\frac{1}{\Gamma(\alpha)} \int_0^{\cdot} (\cdot - s)^{\alpha-1} f(s)ds$, $0 < \alpha < 1$, belong to E. Then, $f \in D(B_X^{-\alpha})$ and Equation (41) holds.*

Proof. As a preparation, we first consider the case of a finite interval. Let $0 < T < \infty$. Let

$$D(B_X) = \{u \in C^1([0,T]; X); u(0) = 0\}, \quad B_X u = u'.$$

Then, $\forall \lambda \in \mathbb{C}$

$$\big((\lambda + B_X)^{-1} f\big)(t) = \int_0^t e^{-\lambda(t-s)} f(s)ds, \text{ especially } \big(B_X^{-1} f\big)(t) = \int_0^t f(s)ds, \ 0 \le t \le T. \tag{42}$$

Therefore, B_X satisfies the assumptions of Proposition 1 with $\omega = \pi/2$, and hence its fractional power B_X^α is defined for $0 < \alpha < 1$, and we have:

$$(\lambda + B_X^\alpha)^{-1} = \frac{\sin \pi\alpha}{\pi} \int_0^\infty \frac{\mu^\alpha}{\lambda^2 + 2\lambda\mu^\alpha \cos \pi\alpha + \mu^{2\alpha}} (\mu + B_X)^{-1} d\mu, \ \lambda \ge 0. \tag{43}$$

Analogously to Equation (40), the following statement is established:

$$\rho(B_X^\alpha) \supset \{\lambda; \mathrm{Re}\,\lambda < 0\} \text{ and } \|(\lambda + B_X^\alpha)^{-1}\|_{\mathcal{L}(C([0,T];X))} \le \frac{1}{\mathrm{Re}\,\lambda}, \ \mathrm{Re}\,\lambda > 0. \tag{44}$$

It follows from Equations (42) and (43) that for $f \in C([0,T]; X)$, $\lambda \ge 0$

$$\big((\lambda + B_X^\alpha)^{-1} f\big)(t) = \frac{\sin \pi\alpha}{\pi} \int_0^t \int_0^\infty \frac{\mu^\alpha e^{-\mu(t-s)}}{\lambda^2 + 2\lambda\mu^\alpha \cos \pi\alpha + \mu^{2\alpha}} d\mu f(s) ds, \quad 0 \le t \le T. \tag{45}$$

Especially if $\lambda = 0$,

$$\left(B_X^{-\alpha} f\right)(t) = \frac{\sin \pi\alpha}{\pi} \int_0^t \int_0^\infty \mu^{-\alpha} e^{-\mu(t-s)} d\mu f(s) ds = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds, \ 0 \le t \le T. \tag{46}$$

For $f \in C([0,T]; X)$ and $\lambda > 0$,

$$(\lambda + B_X^\alpha)^{-1}(\lambda B_X^{-\alpha} + 1) f = (\lambda + B_X^\alpha)^{-1}(\lambda + B_X^\alpha) B_X^{-\alpha} f = B_X^{-\alpha} f. \tag{47}$$

From Equation (45) with f replaced by $(\lambda B_X^{-\alpha}+1)f$ and Equation (46), it follows that

$$\begin{aligned}
&\left((\lambda + B_X^{\alpha})^{-1}(\lambda B_X^{-\alpha}+1)f\right)(t)\\
&=\frac{\sin\pi\alpha}{\pi}\int_0^t\int_0^\infty \frac{\mu^\alpha e^{-\mu(t-s)}}{\lambda^2+2\lambda\mu^\alpha\cos\pi\alpha+\mu^{2\alpha}}d\mu\left((\lambda B_X^{-\alpha}+1)f\right)(s)ds\\
&=\frac{\sin\pi\alpha}{\pi}\int_0^t\int_0^\infty \frac{\mu^\alpha e^{-\mu(t-s)}}{\lambda^2+2\lambda\mu^\alpha\cos\pi\alpha+\mu^{2\alpha}}d\mu\left(\frac{\lambda}{\Gamma(\alpha)}\int_0^s(s-\sigma)^{\alpha-1}f(\sigma)d\sigma+f(s)\right)ds\\
&=\frac{\sin\pi\alpha}{\pi}\frac{\lambda}{\Gamma(\alpha)}\int_0^t\left(\int_0^\infty \frac{\mu^\alpha e^{-\mu(t-s)}}{\lambda^2+2\lambda\mu^\alpha\cos\pi\alpha+\mu^{2\alpha}}d\mu\right)\int_0^s(s-\sigma)^{\alpha-1}f(\sigma)d\sigma ds\\
&+\frac{\sin\pi\alpha}{\pi}\int_0^t\int_0^\infty \frac{\mu^\alpha e^{-\mu(t-s)}}{\lambda^2+2\lambda\mu^\alpha\cos\pi\alpha+\mu^{2\alpha}}d\mu f(s)ds. \qquad (48)
\end{aligned}$$

By the changes of the order of integration,

$$\begin{aligned}
&\int_0^t\left(\int_0^\infty \frac{\mu^\alpha e^{-\mu(t-s)}}{\lambda^2+2\lambda\mu^\alpha\cos\pi\alpha+\mu^{2\alpha}}d\mu\right)\int_0^s(s-\sigma)^{\alpha-1}f(\sigma)d\sigma ds\\
&=\int_0^t\int_\sigma^t\left(\int_0^\infty \frac{\mu^\alpha e^{-\mu(t-s)}(s-\sigma)^{\alpha-1}}{\lambda^2+2\lambda\mu^\alpha\cos\pi\alpha+\mu^{2\alpha}}d\mu\right)ds f(\sigma)d\sigma\\
&=\int_0^t\int_0^\infty \frac{\mu^\alpha}{\lambda^2+2\lambda\mu^\alpha\cos\pi\alpha+\mu^{2\alpha}}\int_\sigma^t e^{-\mu(t-s)}(s-\sigma)^{\alpha-1}ds d\mu f(\sigma)d\sigma. \qquad (49)
\end{aligned}$$

Substituting Equation (49) (with s and σ interchanged) into Equation (48), one deduces

$$\begin{aligned}
&\left((\lambda + B_X^{\alpha})^{-1}(\lambda B_X^{-\alpha}+1)f\right)(t)\\
&=\frac{\sin\pi\alpha}{\pi}\frac{\lambda}{\Gamma(\alpha)}\int_0^t\int_0^\infty \frac{\mu^\alpha}{\lambda^2+2\lambda\mu^\alpha\cos\pi\alpha+\mu^{2\alpha}}\int_s^t e^{-\mu(t-\sigma)}(\sigma-s)^{\alpha-1}d\sigma d\mu f(s)ds *\\
&+\frac{\sin\pi\alpha}{\pi}\int_0^t\int_0^\infty \frac{\mu^\alpha e^{-\mu(t-s)}}{\lambda^2+2\lambda\mu^\alpha\cos\pi\alpha+\mu^{2\alpha}}d\mu f(s)ds\\
&=\frac{\sin\pi\alpha}{\pi}\int_0^t\int_0^\infty \frac{\mu^\alpha}{\lambda^2+2\lambda\mu^\alpha\cos\pi\alpha+\mu^{2\alpha}}\\
&\times\left[\frac{\lambda}{\Gamma(\alpha)}\int_s^t e^{-\mu(t-\sigma)}(\sigma-s)^{\alpha-1}d\sigma+e^{-\mu(t-s)}\right]d\mu f(s)ds. \qquad (50)
\end{aligned}$$

From Equations (46), (47), and (50), it follows that the following equality holds $\forall f\in C([0,T];X)$:

$$\begin{aligned}
&\frac{\sin\pi\alpha}{\pi}\int_0^t\int_0^\infty \frac{\mu^\alpha}{\lambda^2+2\lambda\mu^\alpha\cos\pi\alpha+\mu^{2\alpha}}\left[\frac{\lambda}{\Gamma(\alpha)}\int_s^t e^{-\mu(t-\sigma)}(\sigma-s)^{\alpha-1}d\sigma+e^{-\mu(t-s)}\right]d\mu f(s)ds\\
&=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f(s)ds,\ \ 0\le t\le T.
\end{aligned}$$

This yields that, for $< t\le T$:

$$\frac{\sin\pi\alpha}{\pi}\int_0^\infty \frac{\mu^\alpha}{\lambda^2+2\lambda\mu^\alpha\cos\pi\alpha+\mu^{2\alpha}}\left[\frac{\lambda}{\Gamma(\alpha)}\int_0^t e^{-\mu(t-\sigma)}\sigma^{\alpha-1}d\sigma+e^{-\mu t}\right]d\mu=\frac{t^{\alpha-1}}{\Gamma(\alpha)}. \qquad (51)$$

Since $T>0$ is arbitrary, one concludes that Equation (51) holds for $0<t<\infty$.

We return to the case of the infinite interval $(0,\infty)$. Suppose that both functions f and $u(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s)ds$ belong to E. One has by virtue of Equation (39) with $\lambda = 1$

$$
\begin{aligned}
\left((1+B_X^\alpha)^{-1}u\right)(t) &= \frac{\sin\pi\alpha}{\pi}\int_0^t\int_0^\infty \frac{\mu^\alpha e^{-\mu(t-s)}}{1+2\mu^\alpha\cos\pi\alpha+\mu^{2\alpha}}d\mu u(s)ds \\
&= \frac{\sin\pi\alpha}{\pi}\int_0^t\left(\int_0^\infty \frac{\mu^\alpha e^{-\mu(t-s)}}{1+2\mu^\alpha\cos\pi\alpha+\mu^{2\alpha}}d\mu\right)\frac{1}{\Gamma(\alpha)}\int_0^s (s-\sigma)^{\alpha-1}f(\sigma)d\sigma ds \\
&= \frac{\sin\pi\alpha}{\pi\Gamma(\alpha)}\int_0^t\int_\sigma^t\left(\int_0^\infty \frac{\mu^\alpha e^{-\mu(t-s)}}{1+2\mu^\alpha\cos\pi\alpha+\mu^{2\alpha}}d\mu\right)(s-\sigma)^{\alpha-1}ds f(\sigma)d\sigma \\
&= \frac{\sin\pi\alpha}{\pi}\int_0^t\int_0^\infty \frac{\mu^\alpha}{1+2\mu^\alpha\cos\pi\alpha+\mu^{2\alpha}}\frac{1}{\Gamma(\alpha)}\int_\sigma^t e^{-\mu(t-s)}(s-\sigma)^{\alpha-1}dsd\mu f(\sigma)d\sigma, \qquad (52)
\end{aligned}
$$

and

$$
\left((1+B_X^\alpha)^{-1}f\right)(t) = \frac{\sin\pi\alpha}{\pi}\int_0^t\int_0^\infty \frac{\mu^\alpha e^{-\mu(t-s)}}{1+2\mu^\alpha\cos\pi\alpha+\mu^{2\alpha}}d\mu f(s)ds. \qquad (53)
$$

Adding Equations (52) and (53), and using Equation (51) with $\lambda = 1$, one observes

$$
\begin{aligned}
&\left((1+B_X^\alpha)^{-1}(u+f)\right)(t) \\
&= \frac{\sin\pi\alpha}{\pi}\int_0^t\int_0^\infty \frac{\mu^\alpha}{1+2\mu^\alpha\cos\pi\alpha+\mu^{2\alpha}}\left[\frac{1}{\Gamma(\alpha)}\int_s^t e^{-\mu(t-\sigma)}(\sigma-s)^{\alpha-1}d\sigma + e^{-\mu(t-s)}\right]d\mu f(s)ds \\
&= \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s)ds = u(t).
\end{aligned}
$$

This yields that $u \in D(B_X^\alpha)$ and $u + f = (I + B_X^\alpha)u = u + B_X^\alpha u$. Consequently, $f \in D(B_X^{-\alpha})$ and $B_X^{-\alpha} f = u$. □

In view of Propositions 3 and 4, the following statement is obtained:

Corollary 1. *Let $f \in E$. Then, $f \in D(B_X^{-\alpha})$ if and only if $\int_0^\cdot (\cdot - s)^{\alpha-1} f(s)ds \in E$. For $f \in D(B_X^{-\alpha})$, Equation (41) holds.*

For $f \in E$, $\alpha > 0$, $\beta > 0$,

$$
\begin{aligned}
&\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\frac{1}{\Gamma(\beta)}\int_0^s (s-\sigma)^{\beta-1}f(\sigma)d\sigma ds \\
&= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^t\int_\sigma^t (t-s)^{\alpha-1}(s-\sigma)^{\beta-1}ds f(\sigma)d\sigma = \frac{1}{\Gamma(\alpha+\beta)}\int_0^t (t-\sigma)^{\alpha+\beta-1}f(\sigma)d\sigma.
\end{aligned}
$$

Suppose $f \in D(B_X^{-\beta})$, $0 < \beta < 1$. Then, in view of Corollary 1 , $\int_0^\cdot (\cdot - s)^{\beta-1} f(s)ds \in E$. Hence,

$$
\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}(B_X^{-\beta}f)(s)d\sigma ds = \frac{1}{\Gamma(\alpha+\beta)}\int_0^t (t-\sigma)^{\alpha+\beta-1}f(\sigma)d\sigma.
$$

Therefore, under the assumption $f \in D(B_X^{-\beta})$ $B_X^{-\beta} f \in D(B_X^{-\alpha})$ if and only if $f \in D(B_X^{-\alpha-\beta})$, and in this case $B_X^{-\alpha}B_X^{-\beta} f = B_X^{-\alpha-\beta} f$ holds. In particular, it is obtained that

$$
B_X^{-\alpha}B_X^{-\beta} \subset B_X^{-\alpha-\beta}. \qquad (54)
$$

PROBLEM $\beta > \alpha \Longrightarrow D(B_X^{-\beta}) \subset D(B_X^{-\alpha})$?

Let $0 < \tilde{\alpha} < 1, 0 < \beta \leq \alpha \leq 1$. Let L and M be densely defined closed linear operators in X such that $0 \in \rho(L)$, $D(L) \subset D(M)$ and

$$\text{(H)} \quad \begin{aligned} &\|M(\lambda M - L)^{-1}\|_{\mathcal{L}(X)} \leq \frac{\tilde{c}}{(|\lambda|+1)^{\beta}}, \ \tilde{c} > 0, \\ &\forall \lambda \in \Sigma_{\alpha} = \{\lambda; \mathrm{Re}\lambda \geq -c(1+|\mathrm{Im}\lambda|)^{\alpha}\}, \ c > 0. \end{aligned} \tag{55}$$

Consider the equation

$$B_X^{\tilde{\alpha}} Mu - Lu = f. \tag{56}$$

Let a_0 and a be such that

$$0 < a_0 < \min\{c, 1\}, \quad c < a < c + a_0. \tag{57}$$

Equation (56) is equivalent to

$$(B_X^{\tilde{\alpha}} + a_0)Mu - (L + a_0 M)u = f. \tag{58}$$

Since $-a_0 > -c$, in view of (H) $M(L + a_0 M)^{-1} \in \mathcal{L}(X)$ and if $\mathrm{Re}(\lambda - a_0) \geq -c(1 + |\mathrm{Im}(\lambda - a_0)|)^{\alpha}$,

$$\|M((\lambda - a_0)M - L)^{-1}\|_{\mathcal{L}(X)} \leq \frac{\tilde{c}}{(|\lambda - a_0| + 1)^{\beta}},$$

i.e., if $\mathrm{Re}\lambda \geq a_0 - c(1 + |\mathrm{Im}\lambda|)^{\alpha}$,

$$\|M(\lambda M - (L + a_0 M))^{-1}\|_{\mathcal{L}(X)} \leq \frac{\tilde{c}}{(|\lambda - a_0| + 1)^{\beta}}. \tag{59}$$

Since $|\lambda - a_0| + 1 \geq |\lambda| - a_0 + 1 = |\lambda| + (1 - a_0) \geq (1 - a_0)(|\lambda| + 1)$,

$$\frac{\tilde{c}}{(|\lambda - a_0| + 1)^{\beta}} \leq \frac{\tilde{c}}{(1 - a_0)^{\beta}(|\lambda| + 1)^{\beta}} = \frac{c_1}{(|\lambda| + 1)^{\beta}}, \ \ c_1 = \tilde{c}(1 - a_0)^{-\beta}.$$

Therefore,

$$\|M(\lambda M - (L + a_0 M))^{-1}\|_{\mathcal{L}(X)} \leq \frac{c_1}{(|\lambda| + 1)^{\beta}} \ \ \forall \lambda : \mathrm{Re}\lambda \geq a_0 - c(1 + |\mathrm{Im}\lambda|)^{\alpha}. \tag{60}$$

The inequality in Equation (40) implies

$$\|(B_X^{\tilde{\alpha}} + a_0 - \lambda)^{-1}\|_{\mathcal{L}(E)} \leq \frac{1}{a_0 - \mathrm{Re}\lambda} \ \ \forall \lambda : \mathrm{Re}\lambda < a_0. \tag{61}$$

By virtue of Equation (38),

$$(B_X^{\tilde{\alpha}} + a_0)^{-1} = \frac{\sin \pi\tilde{\alpha}}{\pi} \int_0^{\infty} \frac{\mu^{\tilde{\alpha}}}{a_0^2 + 2a_0\mu^{\tilde{\alpha}} \cos \pi\tilde{\alpha} + \mu^{2\tilde{\alpha}}} (B_X + \mu)^{-1} d\mu. \tag{62}$$

Let $f \in L^p(0, \infty; X)$ and $T \in \mathcal{L}(X)$. Since

$$\begin{aligned} &(T(B_X + \mu)^{-1} f)(t) = T \cdot ((B_X + \mu)^{-1} f)(t) = T \int_0^t e^{-\mu(t-s)} f(s) ds = \int_0^t e^{-\mu(t-s)} T f(s) ds \\ &= \int_0^t e^{-\mu(t-s)} (Tf)(s) ds = ((B_X + \mu)^{-1} T f)(t), \end{aligned}$$

one observes $T(B_X+\mu)^{-1}f=(B_X+\mu)^{-1}Tf$, i.e., $T(B_X+\mu)^{-1}=(B_X+\mu)^{-1}T$. Therefore, with the aid of Equation (62), one obtains

$$T(B_X^{\tilde{\alpha}}+a_0)^{-1}=(B_X^{\tilde{\alpha}}+a_0)^{-1}T. \tag{63}$$

Applying this to $T=M(L+a_0M)^{-1}$, one obtains

$$(B_X^{\tilde{\alpha}}+a_0)^{-1}M(L+a_0M)^{-1}=M(L+a_0M)^{-1}(B_X^{\tilde{\alpha}}+a_0)^{-1}. \tag{64}$$

Let Γ be the curve

$$\Gamma=\{z=a-c(1+|y|)^{\alpha}+iy,\ y\in\mathbb{R}\}.$$

In view of Equation (57), one has $a_0<c<a$, $a-c<a_0$. Hence, if $\lambda=a-c(1+|y|)^{\alpha}+iy\in\Gamma$ with $y\in\mathbb{R}$, one has

$$a_0-c(1+|\mathrm{Im}\lambda|)^{\alpha}=a_0-c(1+|y|)^{\alpha}<a-c(1+|y|)^{\alpha}=\mathrm{Re}\lambda\le a-c<a_0.$$

Therefore, Equations (60) and (61) hold on Γ.

We verify

$$\|(B_X^{\tilde{\alpha}}+a_0-\lambda)^{-1}\|_{\mathcal{L}(E)}\le\frac{c_2}{1+|\mathrm{Re}\lambda|}\quad\forall\lambda\in\Gamma,\quad c_2=\frac{a-c+1}{a_0-a+c}. \tag{65}$$

We show that, if $\lambda\in\Gamma$, the following inequality holds:

$$a_0-\mathrm{Re}\lambda\ge\frac{a_0-a+c}{a-c+1}(1+|\mathrm{Re}\lambda|). \tag{66}$$

Note here $a_0-a+c>0$ and $a-c+1>a-c>0$ in view of Equation (57). Hence,

$$a_0-\frac{a_0-a+c}{a-c+1}=\frac{a_0a-a_0c+a-c}{a-c+1}=\frac{(a_0+1)(a-c)}{a-c+1}>0. \tag{67}$$

This yields that

$$\begin{aligned}&a_0-\mathrm{Re}\lambda-\frac{a_0-a+c}{a-c+1}(1+\mathrm{Re}\lambda)=a_0-\mathrm{Re}\lambda-\frac{a_0-a+c}{a-c+1}-\frac{a_0-a+c}{a-c+1}\mathrm{Re}\lambda\\&=\frac{(a_0+1)(a-c)}{a-c+1}-\frac{a_0+1}{a-c+1}\mathrm{Re}\lambda=\frac{a_0+1}{a-c+1}(a-c-\mathrm{Re}\lambda)\ge0\end{aligned}$$

if $\lambda\in\Gamma$. Therefore, Equation (66) holds if $\lambda\in\Gamma$ and $\mathrm{Re}\lambda\ge0$. Recalling that $a_0<1$ we see if $\mathrm{Re}\lambda\le0$,

$$a_0-\mathrm{Re}\lambda=a_0+|\mathrm{Re}\lambda|\ge a_0(1+|\mathrm{Re}\lambda|).$$

Therefore, recalling Equation (67) one observes that Equation (66) holds also in the case $\mathrm{Re}\lambda\le0$. Since $\mathrm{Re}\lambda\le a-c<a_0$ if $\lambda\in\Gamma$, it follows from Equations (61) and (66) that Equation (65) holds. Set $B=B_X^{\tilde{\alpha}}+a_0$, $T=M(L+a_0M)^{-1}$. Then, Equations (61) and (64) are expressed as

$$B^{-1}T=TB^{-1}, \tag{68}$$

$$\|(B-\lambda)^{-1}\|_{\mathcal{L}(E)}\le\frac{1}{a_0-\mathrm{Re}\lambda}\quad\forall\lambda:\mathrm{Re}\lambda<a_0\,, \tag{69}$$

respectively. Let $v=(L+a_0M)u$ be the new unknown variable. Then, $Mu=M(L+a_0M)^{-1}v=Tv$ and Equation (58) is expressed as

$$BTv-v=f. \tag{70}$$

Our candidate of the solution to Equation (70) is

$$v = (2\pi i)^{-1} \int_\Gamma z^{-1}(zT-1)^{-1}B(B-z)^{-1} f dz. \tag{71}$$

We have

$$zT - 1 = zM(L+a_0M)^{-1} - 1 = [zM - (L+a_0M)](L+a_0M)^{-1}.$$

If $z = a - c(1+|y|)^\alpha + iy \in \Gamma$, then $\mathrm{Re}\, z = a - c(1+|\mathrm{Im}\, z|)^\alpha > a_0 - c(1+|\mathrm{Im}\, z|)^\alpha$. Hence, in view of Equation (60)

$$\|M(zM-(L+a_0M))^{-1}\|_{\mathcal{L}(X)} \le \frac{c_1}{(|z|+1)^\beta}.$$

Therefore, if $z \in \Gamma$,

$$\begin{aligned}
&\|(zT-1)^{-1}\|_{\mathcal{L}(E)} = \|(zT-1)^{-1}\|_{\mathcal{L}(X)} = \|(L+a_0M)(zM-(L+a_0M))^{-1}\|_{\mathcal{L}(X)} \\
&= \|\{zM - (zM-(L+a_0M))\}(zM-(L+a_0M))^{-1}\|_{\mathcal{L}(X)} \\
&= \|zM(zM-(L+a_0M))^{-1} - I\|_{\mathcal{L}(X)} \le \|zM(zM-(L+a_0M))^{-1}\|_{\mathcal{L}(X)} + 1 \\
&\le \frac{c_1|z|}{(|z|+1)^\beta} + 1 \le c_1(|z|+1)^{1-\beta} + 1 \le c_2(|z|+1)^{1-\beta}, \ \ c_2 = c_1 + 1.
\end{aligned} \tag{72}$$

This yields

$$\|v\|_E \le \frac{c_2}{2\pi} \int_\Gamma |z|^{-(1+\theta)}(1+|z|)^{1-\beta}|z|^\theta \|B(B-z)^{-1}f\|_E |dz|. \tag{73}$$

Let $z = a - c(1+|y|)^\alpha + iy \in \Gamma$. From

$$\begin{aligned}
&B(B-z)^{-1} = B(B+1+|y|)^{-1} + (1+|y|+z)(B-z)^{-1}B(B+1+|y|)^{-1} \\
&= \left[1 + (1+|y|+z)(B-z)^{-1}\right]B(B+1+|y|)^{-1},
\end{aligned}$$

it follows that

$$\begin{aligned}
&\|B(B-z)^{-1}f\|_E = \|\left[1+(1+|y|+z)(B-z)^{-1}\right]B(B+1+|y|)^{-1}f\|_E \\
&\le \|1+(1+|y|+z)(B-z)^{-1}\|_{\mathcal{L}(E)}\|B(B+1+|y|)^{-1}f\|_E \\
&\le \left(1 + \frac{1+|y|+|z|}{a_0 - \mathrm{Re} z}\right)\|B(B+1+|y|)^{-1}f\|_E.
\end{aligned} \tag{74}$$

Since

$$\begin{aligned}
&1+|y|+|z| = 1+|y|+|a-c(1+|y|)^\alpha + iy| \le 1+|y|+a+c(1+|y|)^\alpha + |y| \\
&= 1+2|y|+a+c(1+|y|)^\alpha,
\end{aligned}$$

one has

$$1 + \frac{1+|y|+|z|}{a_0 - \mathrm{Re} z} \le 1 + \frac{1+2|y|+a+c(1+|y|)^\alpha}{a_0 - a + c(1+|y|)^\alpha} = \frac{a_0+1+2|y|+2c(1+|y|)^\alpha}{a_0-a+c(1+|y|)^\alpha},$$

and

$$\begin{aligned}
&a_0+1+2|y|+2c(1+|y|)^\alpha \le (a_0+1)(1+|y|) + 2(1+|y|) + 2c(1+|y|) \\
&= (a_0+3+2c)(1+|y|).
\end{aligned}$$

Since $a_0 < a$,

$$a_0 - a + c(1+|y|)^\alpha \ge c(1+|y|)^\alpha - (a-a_0)(1+|y|)^\alpha = (c-a+a_0)(1+|y|)^\alpha.$$

Note here that $c-a+a_0>0$ (cf. Equation (57)). Hence,

$$1+\frac{1+|y|+|z|}{a_0-\mathrm{Re}z}\le\frac{(a_0+3+2c)(1+|y|)}{(c-a+a_0)(1+|y|)^\alpha}=c_3(1+|y|)^{1-\alpha},\ c_3=\frac{a_0+3+2c}{c-a+a_0}. \tag{75}$$

From Equations (74) and (75), it follows that

$$\|B(B-z)^{-1}f\|_E\le c_3(1+|y|)^{1-\alpha}\|B(B+1+|y|)^{-1}f\|_E. \tag{76}$$

For $z=a-c(1+|y|)^\alpha+iy\in\Gamma,\ \ y\in\mathbb{R}$, one has

$$\begin{aligned}&|z|=|a-c(1+|y|)^\alpha+iy|\le a+c(1+|y|)^\alpha+|y|\le a+c(1+|y|)+|y|\\&=a+c+(c+1)|y|\le c_4(1+|y|),\ \ c_4=\max\{a,1\}+c,\end{aligned} \tag{77}$$

$$\begin{aligned}&(1+|z|)^{1-\beta}\le(1+c_4(1+|y|))^{1-\beta}\le(1+|y|+c_4(1+|y|))^{1-\beta}=c_5(1+|y|)^{1-\beta},\\&c_5=(1+c_4)^{1-\beta},\end{aligned} \tag{78}$$

$$\begin{aligned}&|1+|y|+z|\le1+|y|+|z|\le1+|y|+c_4(1+|y|)=(1+c_4)(1+|y|)=c_6(1+|y|),\\&c_6=1+c_4=1+\max\{a,1\}+c.\end{aligned} \tag{79}$$

Here, we show that $\exists c_7$ such that

$$|z|\ge c_7(1+|y|)\ \ \forall z\in\Gamma:z=a-c(1+|y|)^\alpha+iy,\ y\in\mathbb{R}. \tag{80}$$

Proof. Let $0<b<a-c(\Longleftrightarrow c-b<c<a-b)$.
(i) Case $|a-c(1+|y|)^\alpha|\le b$. In this case,

$$\begin{aligned}&|a-c(1+|y|)^\alpha|\le b\Longleftrightarrow -b\le a-c(1+|y|)^\alpha\le b\Longrightarrow a-b\le c(1+|y|)^\alpha\\&\Longleftrightarrow\frac{a-b}{c}\le(1+|y|)^\alpha\Longleftrightarrow\left(\frac{a-b}{c}\right)^{1/\alpha}\le1+|y|\Longrightarrow|y|\ge\left(\frac{a-b}{c}\right)^{1/\alpha}-1\equiv\delta>0.\end{aligned}$$

Hence,

$$\frac{\delta}{1+\delta}(1+|y|)=\frac{\delta}{1+\delta}+\frac{\delta|y|}{1+\delta}\le\frac{|y|}{1+\delta}+\frac{\delta|y|}{1+\delta}=|y|.$$

Therefore,

$$|z|=|a-c(1+|y|)^\alpha+iy|\ge|y|\ge\frac{\delta}{1+\delta}(1+|y|).$$

(ii) Case $a-c(1+|y|)^\alpha>b$. In this case

$$|z|\ge|a-c(1+|y|)^\alpha|=a-c(1+|y|)^\alpha>b,$$

and

$$\begin{aligned}&a-c(1+|y|)^\alpha>b\Longleftrightarrow(1+|y|)^\alpha<\frac{a-b}{c}\Longleftrightarrow1+|y|<\left(\frac{a-b}{c}\right)^{1/\alpha}\\&\Longleftrightarrow\left(\frac{c}{a-b}\right)^{1/\alpha}(1+|y|)<1.\end{aligned}$$

Therefore,

$$|z|>b>b\left(\frac{c}{a-b}\right)^{1/\alpha}(1+|y|).$$

(iii) Case $c(1+|y|)^\alpha - a > b$. In this case

$$c(1+|y|)^\alpha > a+b \Longleftrightarrow (1+|y|)^\alpha > \frac{a+b}{c} \Longleftrightarrow |y| > \left(\frac{a+b}{c}\right)^{1/\alpha} - 1 \equiv \gamma > 0.$$

Hence,

$$\frac{\gamma}{\gamma+1}(1+|y|) = \frac{\gamma}{\gamma+1} + \frac{\gamma}{\gamma+1}|y| < \frac{|y|}{\gamma+1} + \frac{\gamma}{\gamma+1}|y| = |y| \le |z|.$$

Thus Equation (80) holds with $c_7 = \min\left\{\frac{\delta}{1+\delta}, b\left(\frac{c}{a-b}\right)^{1/\alpha}, \frac{\gamma}{\gamma+1}\right\}$. □

Hence, from Equations (73) and (76)–(80), it follows that

$$\|v\|_E \le c_8 \int_\Gamma (1+|y|)^{1-\alpha-\beta-\theta}(1+|y|)^\theta \|B(B+1+|y|)^{-1}f\|_E |dz|,$$

where $c_8 = c_2 c_7^{-(1+\theta)} c_5 c_4^\theta c_3/2\pi$. For $z = a - c(1+|y|)^\alpha + iy$, $y \ge 0$

$$|dz| = |-c\alpha(1+y)^{\alpha-1}dy + idy| = \{(c\alpha)^2(1+y)^{2(\alpha-1)}+1\}^{1/2}dy \le ((c\alpha)^2+1)^{1/2}dy.$$

Therefore,

$$\begin{aligned}
&\|v\|_E \le 2((c\alpha)^2+1)^{1/2} c_8 \int_0^\infty (1+y)^{1-\alpha-\beta-\theta}(1+y)^\theta \|B(B+1+y)^{-1}f\|_E dy \\
&= 2((c\alpha)^2+1)^{1/2} c_8 \int_1^\infty y^{2-\alpha-\beta-\theta} y^\theta \|B(B+y)^{-1}f\|_E \frac{dy}{y} \\
&\le 2((c\alpha)^2+1)^{1/2} c_8 \left(\frac{p-1}{(\theta+\alpha+\beta-2)p}\right)^{(p-1)/p} \left(\int_1^\infty y^{\theta p}\|B(B+y)^{-1}f\|_E^p \frac{dy}{y}\right)^{1/p} \\
&\le 2((c\alpha)^2+1)^{1/2} c_8 \left(\frac{p-1}{(\theta+\alpha+\beta-2)p}\right)^{(p-1)/p} \|f\|_{(E,D(B))_{\theta,p}}.
\end{aligned}$$

Thus, it has been shown that v is well defined by Equation (71) if $f \in (E, D(B))_{\theta,p}$.

Next, we show that v satisfies Equation (70). We show that the following inequality holds with some constant c_9:

$$1 + |a - c(1+|y|)^\alpha| \ge c_9(1+|y|)^\alpha \quad \forall y \in \mathbb{R}. \tag{81}$$

Proof. (i) Case $|a - c(1+|y|)^\alpha| \le 1/2$. Since

$$\begin{aligned}
&|a - c(1+|y|)^\alpha| \le \frac{1}{2} \Longleftrightarrow -\frac{1}{2} \le a - c(1+|y|)^\alpha \le \frac{1}{2} \Longrightarrow -1 \le 2a - 2c(1+|y|)^\alpha \\
&\Longleftrightarrow 2c(1+|y|)^\alpha \le 2a+1 \Longleftrightarrow \frac{2c}{2a+1}(1+|y|)^\alpha \le 1,
\end{aligned}$$

one observes

$$1 + |a - c(1+|y|)^\alpha| \ge 1 \ge \frac{2c}{2a+1}(1+|y|)^\alpha. \tag{82}$$

(ii) Case $a - c(1+|y|)^\alpha > 1/2$ (this can occur only in case $a - c > 1/2$). Since

$$\begin{aligned}
&a - c(1+|y|)^\alpha > \frac{1}{2} \Longleftrightarrow a - \frac{1}{2} > c(1+|y|)^\alpha \\
&\left(\text{multiplying both sides by } \frac{1+a}{a-1/2}\right) \Longrightarrow 1 + a > \frac{c(1+a)(1+|y|)^\alpha}{a-1/2},
\end{aligned}$$

one has

$$1+a-c(1+|y|)^{\alpha} > \frac{c(1+a)(1+|y|)^{\alpha}}{a-1/2} - c(1+|y|)^{\alpha}$$
$$= \left(\frac{1+a}{a-1/2}-1\right)c(1+|y|)^{\alpha} = \frac{3/2}{a-1/2}c(1+|y|)^{\alpha} = \frac{3c}{2a-1}(1+|y|)^{\alpha}.$$

Therefore,

$$1+|a-c(1+|y|)^{\alpha}| = 1+a-c(1+|y|)^{\alpha} \geq \frac{3c}{2a-1}(1+|y|)^{\alpha}. \tag{83}$$

(iii) Case $c(1+|y|)^{\alpha}-a > 1/2$. In this case,

$$1+|a-c(1+|y|)^{\alpha}| = 1+c(1+|y|)^{\alpha}-a.$$

If $a \leq 1$,

$$1+c(1+|y|)^{\alpha}-a \geq c(1+|y|)^{\alpha}. \tag{84}$$

If $a > 1$,

$$1+c(1+|y|)^{\alpha}-a = c(1+|y|)^{\alpha}-(a-1) \geq c(1+|y|)^{\alpha}-(a-1)(1+|y|)^{\alpha}$$
$$= (c-a+1)(1+|y|)^{\alpha}. \quad (\text{note that } a < c+a_0 < c+1) \tag{85}$$

From Equations (84) and (85), it follows that

$$1+|a-c(1+|y|)^{\alpha}| = 1+c(1+|y|)^{\alpha}-a \geq \min\{c, c-a+1\}(1+|y|)^{\alpha}. \tag{86}$$

Note that $c-a+1 > 0$ since $a < c+a_0 < c+1$.
Consequently, it has been proved that Equation (81) holds with

$$c_9 = \begin{cases} \min\left\{\frac{2c}{2a+1}, \frac{3c}{2a-1}, \min\{c, c-a+1\}\right\} & \text{if } a-c > 1/2, \\ \min\left\{\frac{2c}{2a+1}, \min\{c, c-a+1\}\right\} & \text{if } a-c \leq 1/2. \end{cases}$$

□

Next, we show that v satisfies Equation (70). Since

$$z^{-1}T(zT-1)^{-1}B(B-z)^{-1} = z^{-2}(zT-1+1)(zT-1)^{-1}(B-z+z)(B-z)^{-1}$$
$$= z^{-2}\{1+(zT-1)^{-1}\}\{1+z(B-z)^{-1}\}$$
$$= z^{-2}+z^{-1}(B-z)^{-1}+z^{-2}(zT-1)^{-1}+z^{-1}(zT-1)^{-1}(B-z)^{-1},$$

we have

$$Tv = (2\pi i)^{-1}\int_{\Gamma} z^{-1}T(zT-1)^{-1}B(B-z)^{-1}f dz$$
$$= (2\pi i)^{-1}\int_{\Gamma} z^{-2} f dz + (2\pi i)^{-1}\int_{\Gamma} z^{-1}(B-z)^{-1} f dz$$
$$+(2\pi i)^{-1}\int_{\Gamma} z^{-2}(zT-1)^{-1} f dz + (2\pi i)^{-1}\int_{\Gamma} z^{-1}(zT-1)^{-1}(B-z)^{-1} f dz. \tag{87}$$

Clearly, $(2\pi i)^{-1}\int_{\Gamma} z^{-2} f dz = 0$. By assumption $M\{(z-a_0)M-L\}^{-1}$ is holomorphc in $\mathrm{Re}z \geq a_0 - c(1+|\mathrm{Im}z|)^{\alpha}$. Since

$$zT - 1 = zM(L + a_0M)^{-1} - 1 = \{zM - (L + a_0M)\}(L + a_0M)^{-1}$$
$$= (zM - L - a_0M)(L + a_0M)^{-1},$$
$$(zT - 1)^{-1} = (L + a_0M)(zM - L - a_0M)^{-1} = (L - (z - a_0)M + zM)\{(z - a_0)M - L\}^{-1}$$
$$= \{zM - ((z - a_0)M - L)\}\{(z - a_0)M - L\}^{-1} = zM\{(z - a_0)M - L\}^{-1} - 1,$$

$(zT - 1)^{-1}$ is also holomorphic in $\mathrm{Re}z \geq a_0 - c(1 + |\mathrm{Im}z|)^\alpha$. If $z \in \Gamma$, then

$$\mathrm{Re}z = a - c(1 + |\mathrm{Im}z|)^\alpha > a_0 - c(1 + |\mathrm{Im}z|)^\alpha.$$

Hence, Γ lies in the region where $(zT - 1)^{-1}$ is holomorphc. If

$$\mathrm{Re}z \geq a_0 - c(1 + |\mathrm{Im}z|)^\alpha \big(\iff \mathrm{Re}(z - a_0) \geq -c(1 + |\mathrm{Im}(z - a_0)|)^\alpha \big),$$

then

$$\|M\{(z - a_0)M - L\}^{-1}\|_{\mathcal{L}(X)} \leq \frac{\tilde{c}}{(|z - a_0| + 1)^\beta}.$$

Hence,

$$\|(zT - 1)^{-1}\|_{\mathcal{L}(X)} = \|zM\{(z - a_0)M - L\}^{-1} - 1\|_{\mathcal{L}(X)} \leq |z|\|M\{(z - a_0)M - L\}^{-1}\|_{\mathcal{L}(X)} + 1$$
$$\leq \frac{\tilde{c}|z|}{(|z - a_0| + 1)^\beta} + 1 = O(|z|^{1-\beta}) \text{ as } |z| \to \infty \text{ in } \mathrm{Re}z > a_0 - c(1 + |\mathrm{Im}z|)^\alpha. \tag{88}$$

Therefore,

$$\|z^{-2}(zT - 1)^{-1}\|_{\mathcal{L}(X)} = O(|z|^{-1-\beta}) \text{ as } |z| \to \infty \text{ in } \mathrm{Re}z > a_0 - c(1 + |\mathrm{Im}z|)^\alpha.$$

Hence, one observes

$$\int_\Gamma z^{-2}(zT - 1)^{-1}dz = 0. \tag{89}$$

Let R be a large positive number. The set $\Gamma \cap \{|z| = R\}$ consists of two points z_1, z_2. Let Γ_R be the closed curve which consists of the part of Γ in the disk $|z| \leq R$ and the part of the circle $|z| = R$ in $\mathrm{Re}z \leq \mathrm{Re}z_1 = \mathrm{Re}z_2$. Since $(B - z)^{-1}$ is holomorphic in the region $\mathrm{Re}z < a_0$, which contains the closed set surrounded by Γ_R,

$$(2\pi i)^{-1} \int_{\Gamma_R} z^{-1}(B - z)^{-1}dz = B^{-1}. \tag{90}$$

Since $z_k = a - c(1 + |\mathrm{Im}z_k|)^\alpha + i\mathrm{Im}z_k$ and $|z_k| = R$, $k = 1, 2$, one has

$$\big(a - c(1 + |\mathrm{Im}z_k|)^\alpha\big)^2 + (\mathrm{Im}z_k)^2 = R^2,\ k = 1, 2.$$

This implies $|\mathrm{Im}z_k| = O(R)$ as $R \to \infty$, and hence $|\mathrm{Re}z_k| = O(R^\alpha)$ as $R \to \infty$, $k = 1, 2$. Therefore, by virtue of Equation (69) for $z \in \Gamma_R \cap \{|z| = R\}$

$$\|(B - z)^{-1}\|_{\mathcal{L}(E)} \leq \frac{1}{a_0 - \mathrm{Re}z} = \frac{1}{a_0 + |\mathrm{Re}z|} \leq \frac{1}{a_0 + |\mathrm{Re}z_k|} = O(R^{-\alpha}) \ \ (k = 1, 2),$$

as $R \to \infty$. Letting $R \to \infty$ in Equation (90), one observes

$$(2\pi i)^{-1} \int_\Gamma z^{-1}(B - z)^{-1}dz = B^{-1}. \tag{91}$$

From Equations (87), (89), and (91), one obtains

$$Tv = B^{-1}f + (2\pi i)^{-1}\int_\Gamma z^{-1}(zT-1)^{-1}(B-z)^{-1}f dz,$$

and hence

$$BTv = f + (2\pi i)^{-1}\int_\Gamma z^{-1}(zT-1)^{-1}B(B-z)^{-1}f dz = f + v. \tag{92}$$

Thus, we have established Equation (70).

Our next step is to establish the maximal regularity of solutions to Equation (56) or, equivalently, to Equation (58). By observing the resolvent identity

$$\begin{aligned}&(B+t)^{-1}(B-z)^{-1} = -(t+z)^{-1}\{(B+t)^{-1} - (B-z)^{-1}\}\\&= -(t+z)^{-1}(B+t)^{-1} + (t+z)^{-1}(B-z)^{-1},\end{aligned}$$

we get, for $t > 0$,

$$\begin{aligned}&B(B+t)^{-1}(B-z)^{-1} = [I - t(B+t)^{-1}](B-z)^{-1} = (B-z)^{-1} - t(B+t)^{-1}(B-z)^{-1}\\&= (B-z)^{-1} - t[-(t+z)^{-1}(B+t)^{-1} + (t+z)^{-1}(B-z)^{-1}]\\&= (B-z)^{-1} + t(t+z)^{-1}(B+t)^{-1} - t(t+z)^{-1}(B-z)^{-1}.\end{aligned}$$

Hence,

$$\begin{aligned}&(B+t)^{-1}v = (2\pi i)^{-1}\int_\Gamma z^{-1}(zT-1)^{-1}B(B+t)^{-1}(B-z)^{-1}f dz\\&= (2\pi i)^{-1}\int_\Gamma z^{-1}(zT-1)^{-1}\left[(B-z)^{-1} + t(t+z)^{-1}(B+t)^{-1} - t(t+z)^{-1}(B-z)^{-1}\right]f dz\\&= (2\pi i)^{-1}\int_\Gamma z^{-1}(zT-1)^{-1}(B-z)^{-1}f dz + (2\pi i)^{-1}\int_\Gamma z^{-1}(zT-1)^{-1}t(t+z)^{-1}(B+t)^{-1}f dz\\&-(2\pi i)^{-1}\int_\Gamma z^{-1}(zT-1)^{-1}t(t+z)^{-1}(B-z)^{-1}f dz.\end{aligned}$$

Therefore, we deduce

$$\begin{aligned}B(B+t)^{-1}v &= v + (2\pi i)^{-1}\int_\Gamma z^{-1}(t+z)^{-1}(zT-1)^{-1}dz\, tB(B+t)^{-1}f\\&\quad -(2\pi i)^{-1}\int_\Gamma z^{-1}t(t+z)^{-1}(zT-1)^{-1}B(B-z)^{-1}f dz\\&= v + J_1(f,t) + J_2(f,t),\ t \in \mathbb{R}_+.\end{aligned}$$

One observes that $(zT-1)^{-1} = zM((z-a_0)M-L)^{-1} - I$ is holomorphic in $\{z; \mathrm{Re}z \geq a_0 - c(1+|\mathrm{Im}z|^\alpha)\}$. Hence, the integrand of $J_1(f,t)$ is holomorphic in $\{z; \mathrm{Re}z \geq a - c(1+|\mathrm{Im}z|^\alpha)\}$ and its norm is $O(|z|^{-1-\beta})$ as $|z| \to \infty$ in view of Equation (88). Therefore, $J_1(f,t) = 0$ for any $(t,f) \in (\mathbb{R}_+, E)$.

Moreover, $J_2(f,t)$ satisfies

$$\begin{aligned}&J_2(f,t) = -(2\pi i)^{-1}\int_\Gamma z^{-1}(t+z-z)(t+z)^{-1}(zT-1)^{-1}B(B-z)^{-1}f dz\\&= -(2\pi i)^{-1}\int_\Gamma z^{-1}(zT-1)^{-1}B(B-z)^{-1}f dz + (2\pi i)^{-1}\int_\Gamma (t+z)^{-1}(zT-1)^{-1}B(B-z)^{-1}f dz\\&= -v + (2\pi i)^{-1}\int_\Gamma (t+z)^{-1}(zT-1)^{-1}B(B-z)^{-1}f dz.\end{aligned}$$

Thus, we have obtained

$$B(B+t)^{-1}v = (2\pi i)^{-1}\int_\Gamma (t+z)^{-1}(zT-1)^{-1}B(B-z)^{-1}f dz. \tag{93}$$

Setting $\omega = \theta + \alpha + \beta - 2$, we can estimate $t^{\omega}\|B(B+t)^{-1}v\|$ taking into account the identity

$$\begin{aligned}&B(B-z)^{-1} = B(B+1+|y|)^{-1} + (1+|y|+z)(B-z)^{-1}B(B+1+|y|)^{-1}\\&= \left[1+(1+|y|+z)(B-z)^{-1}\right]B(B+1+|y|)^{-1},\\&z \in \Gamma: \ z = a - c(1+|y|)^{\alpha} + iy, \ y \in \mathbb{R}.\end{aligned}$$

Here, we show that the following inequality holds for $t > 0$, $y \in \mathbb{R}$:

$$|t + a - c(1+|y|)^{\alpha} + iy| \geq c_{10}(t+1+|y|), \ c_{10} = \frac{\min\{1, a-c\}}{\max\{2\sqrt{2}, 2c+1\}}. \tag{94}$$

Proof. (i) Case $|t+a-c(1+|y|)^{\alpha}| < (t+a-c)/2$. Recalling $a > c$, $0 < \alpha \leq 1$, we deduce

$$\begin{aligned}&|t+a-c(1+|y|)^{\alpha}| < (t+a-c)/2 \Longrightarrow t+a-c(1+|y|)^{\alpha} < (t+a-c)/2\\&\Longrightarrow (t+a+c)/2 < c(1+|y|)^{\alpha} \leq c(1+|y|) \Longrightarrow (t+a-c)/2 < c|y| \Longrightarrow t+a-c < 2c|y|.\end{aligned}$$

Hence,

$$\min\{1, a-c\}(t+1) \leq t+a-c < 2c|y|.$$

This implies

$$\min\{1, a-c\}(t+1+|y|) < 2c|y| + |y| = (2c+1)|y|.$$

Therefore,

$$|t+a-c(1+|y|)^{\alpha} + iy| \geq |y| > \frac{\min\{1, a-c\}}{2c+1}(t+1+|y|).$$

(ii) Case $|t+a-c(1+|y|)^{\alpha}| \geq (t+a-c)/2$. From

$$\begin{aligned}&|t+a-c(1+|y|)^{\alpha}+iy|^2 = (t+a-c(1+|y|)^{\alpha})^2 + y^2 \geq (t+a-c)^2/4 + y^2\\&> \frac{1}{8}(2(t+a-c)^2 + 2y^2) \geq \frac{1}{8}(t+a-c+|y|)^2 \geq \frac{1}{8}\big(\min\{1,a-c\}(t+1+|y|)\big)^2\end{aligned}$$

it follows that

$$|t+a-c(1+|y|)^{\alpha} + iy| \geq \frac{\min\{1, a-c\}}{2\sqrt{2}}(t+1+|y|).$$

□

From Equations (72), (76), (78), (93), and (94), it follows that

$$\begin{aligned}&t^{\omega}\|B(B+t)^{-1}v\|_E = t^{\omega}\left\|(2\pi i)^{-1}\int_{\Gamma}(t+z)^{-1}(zT-1)^{-1}B(B-z)^{-1}f dz\right\|_E\\&\leq \frac{t^{\omega}}{2\pi}\int_{\Gamma}\frac{c_2c_5(1+|y|)^{1-\beta}c_3(1+|y|)^{1-\alpha}}{c_{10}(t+1+|y|)}\|B(B+1+|y|)^{-1}f\|_E|dz|\\&= \frac{t^{\omega}}{2\pi}\frac{c_2c_5c_3}{c_{10}}\int_{\Gamma}\frac{(1+|y|)^{2-\alpha-\beta}}{t+1+|y|}\|B(B+1+|y|)^{-1}f\|_E|dz|\\&\leq c_{11}t^{\omega}\int_0^{\infty}\frac{(1+y)^{2-\alpha-\beta}}{t+1+y}\|B(B+1+y)^{-1}f\|_E dy,\end{aligned} \tag{95}$$

where

$$c_{11} = \frac{1}{\pi}\frac{c_2c_5c_3}{c_{10}}((c\alpha)^2+1)^{1/2}.$$

One has

$$t^{\omega}\int_0^{\infty}\frac{(1+y)^{2-\alpha-\beta}}{t+1+y}\|B(B+1+y)^{-1}f\|_E dy = t^{\omega}\int_1^{\infty}\frac{y^{3-\alpha-\beta-\theta}}{t+y}y^{\theta}\|B(B+y)^{-1}f\|_E\frac{dy}{y}$$
$$=\int_1^{\infty}\frac{(ty^{-1})^{\alpha+\beta+\theta-2}}{ty^{-1}+1}y^{\theta}\|B(B+y)^{-1}f\|_E\frac{dy}{y}=\int_1^{\infty}g(ty^{-1})f_1(y)\frac{dy}{y}, \tag{96}$$

where $f_1(y) = y^{\theta}\|B(B+y)^{-1}f\|_E$, $g(y) = \dfrac{y^{\theta+\alpha+\beta-2}}{1+y}$. Applying Lemma 2 and using Equation (96) one obtains

$$\left(\int_0^{\infty}\left(t^{\omega}\int_0^{\infty}\frac{(1+y)^{2-\alpha-\beta}}{t+1+y}\|B(B+1+y)^{-1}f\|_E dy\right)^p\frac{dt}{t}\right)^{1/p}$$
$$\leq\left(\int_0^{\infty}\left(\int_0^{\infty}g(ty^{-1})f_1(y)\frac{dy}{y}\right)^p\frac{dt}{t}\right)^{1/p}\leq\int_0^{\infty}g(t)\frac{dt}{t}\left(\int_0^{\infty}f_1(y)^p\frac{dy}{y}\right)^{1/p}. \tag{97}$$

With the aid of the change of the independent variable $s = (1+t)^{-1}$, one observes

$$\int_0^{\infty}g(t)\frac{dt}{t}=\int_0^{\infty}\frac{t^{\theta+\alpha+\beta-2}}{1+t}\frac{dt}{t}=\int_0^1(1-s)^{\theta+\alpha+\beta-3}s^{-\theta-\alpha-\beta+2}ds$$
$$=\Gamma(\theta+\alpha+\beta-2)\Gamma(3-\theta-\alpha-\beta)=\Gamma(\omega)\Gamma(1-\omega),$$

and

$$\int_0^{\infty}f_1(y)^p\frac{dy}{y}=\int_0^{\infty}y^{\theta p}\|B(B+y)^{-1}f\|_E^p\frac{dy}{y}.$$

Hence, one obtains from Equation (97)

$$\left(\int_0^{\infty}\left(t^{\omega}\int_0^{\infty}\frac{(1+y)^{2-\alpha-\beta}}{t+1+y}\|B(B+1+y)^{-1}f\|_E dy\right)^p\frac{dt}{t}\right)^{1/p}$$
$$\leq\Gamma(\omega)\Gamma(1-\omega)\left(\int_0^{\infty}y^{\theta p}\|B(B+y)^{-1}f\|_E^p\frac{dy}{y}\right)^{1/p}. \tag{98}$$

It follows from Equations (95) and (98) that

$$\left(\int_0^{\infty}\left(t^{\omega}\|B(B+t)^{-1}v\|_E\right)^p\frac{dt}{t}\right)^{1/p}$$
$$\leq\left(\int_0^{\infty}\left(c_{11}t^{\omega}\int_0^{\infty}\frac{(1+y)^{2-\alpha-\beta}}{t+1+y}\|B(B+1+y)^{-1}f\|_E dy\right)^p\frac{dt}{t}\right)^{1/p}$$
$$=c_{11}\left(\int_0^{\infty}\left(t^{\omega}\int_0^{\infty}\frac{(1+y)^{2-\alpha-\beta}}{t+1+y}\|B(B+1+y)^{-1}f\|_E dy\right)^p\frac{dt}{t}\right)^{1/p}$$
$$\leq c_{11}\Gamma(\omega)\Gamma(1-\omega)\left(\int_0^{\infty}y^{\theta p}\|B(B+y)^{-1}f\|_E^p\frac{dy}{y}\right)^{1/p}.$$

Hence, $v = (L+a_0M)u \in (E, D(B))_{\omega,p}$. This implies $BTv = f + v \in (E, D(B))_{\omega,p}$. Since $BTv = BM(L+a_0M)^{-1}v = BMu$, one has $BMu \in (E, D(B))_{\omega,p}$:

$$\int_0^{\infty}t^{\omega p}\|B(B+t)^{-1}BMu\|^p\frac{dt}{t}<\infty.$$

Hence,

$$\int_0^\infty t^{\omega p}\|B(B+t)^{-1}Mu\|^p\frac{dt}{t} = \int_0^\infty t^{\omega p}\|B^{-1}B(B+t)^{-1}BMu\|^p\frac{dt}{t}$$
$$\leq \|B^{-1}\|^p \int_0^\infty t^{\omega p}\|B(B+t)^{-1}BMu\|^p\frac{dt}{t} < \infty,$$

i.e., $Mu \in (E, D(B))_{\omega,p}$. In view of $v = (L + a_0M)u \in (E, D(B))_{\omega,p}$ it follows that $Lu \in (E, D(B))_{\omega,p}$.

Therefore, $B_X^{\tilde{\alpha}}Mu = Lu + f \in (E, D(B))_{\omega,p}$. Thus, the following result is established:

Theorem 1. *Let M, L be two closed linear operators in the complex Banach space E satisfying* (H), *and let $0 < \beta \leq \alpha \leq 1$. Let B_X be the operator defined by Equations* (7) *and* (8) *and $0 < \tilde{\alpha} < 1$. Then, for all $f \in (E, D(B_X^{\tilde{\alpha}}))_{\theta,p}$, $2 - \alpha - \beta < \theta < 1$, $1 < p \leq \infty$ equation $B_X^{\tilde{\alpha}}Mu + Lu = f$ admits a unique solution u. Moreover, $Lu, B_X^{\tilde{\alpha}}Mu \in (E, D(B_X^{\tilde{\alpha}}))_{\omega,p}$, $\omega = \theta + \alpha + \beta - 2$.*

3. Conclusions

The fractional powers of the involved operator B_X are investigated in the space of continuous functions which do not necessarily vanish at the origin. This enables us to prove some previous results in the case where the involved operator B_X is not necessarily densely defined. Precisely, a fractional abstract Cauchy problem for possibly degenerate equations in Banach spaces is considered and refined.

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