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Submanifolds in Normal Complex Contact Manifolds

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Abstract: In the present article we initiate the study of submanifolds in normal complex contact metric manifolds. We define invariant and anti-invariant (CC-totally real) submanifolds in such manifolds and start the study of their basic properties. Also, we establish the Chen first inequality and Chen inequality for the invariant $\delta\,(2,2)$ for CC-totally real submanifolds in a normal complex contact space form and characterize the equality cases. We also prove the minimality of CC-totally real submanifolds of maximum dimension satisfying the equalities.

Keywords: normal complex contact space forms; submanifolds; δ-invariants

MSC: 53C40; 53C25

1. Introduction

Complex and contact geometries represent some of the most studied areas in differential geometry.

The complex contact metric structures are less explored and there is a short list of papers in the mathematical literature on this topic. However, in [1] there is a chapter in which D.E. Blair realizes a comprehensive survey of known results on (normal) complex contact manifolds.

In [2], D.E. Blair and the first author in this work, proved that a locally symmetric normal complex contact metric manifold is locally isometric to the complex projective space $\mathbb{C}P^{2n+1}(4)$ of constant holomorphic curvature 4. They also studied reflections in the integral submanifolds of the vertical subbundle of a normal complex contact manifold and showed that when such reflections are isometries the manifold fibers over a locally symmetric space. If the normal complex contact manifold is Kähler, then the manifold fibers over a quaternionic symmetric space. Also, if the complex contact structure is given by a global holomorphic contact form, then the manifold fibers over a locally symmetric complex symplectic manifold.

In [3], the same authors studied the homogeneity and local symmetry of complex (κ,μ)-spaces. It was proved that for $k<1$, a complex (κ,μ)-space is locally homogeneous and GH-locally symmetric.

On the other hand, in differential geometry, the theory of submanifolds plays a very important role. As we know, articles on submanifolds in normal complex contact manifolds have not been published until now.

In this paper, we define invariant and anti-invariant (CC-totally real) submanifolds of normal complex contact metric manifolds and start the study of their basic properties.

Also, the theory of Chen invariants represents one of the most useful tools for obtaining relationships between extrinsic and intrinsic invariants of a submanifold. In [4], B.-Y. Chen wrote a survey on Chen invariants and Chen inequalities and their applications. We establish the Chen first inequality and Chen inequality for the invariant $\delta\,(2,2)$ for CC-totally real submanifolds in a normal complex contact space form and give the characterizations of the equality cases.

2. Preliminaries

A *complex contact manifold* [1] is a complex manifold M of odd complex dimension $2n+1$ together with an open covering $\{\mathcal{O}\}$ of coordinate neighborhoods such that:

(1) On each $\mathcal{O}$ there is a holomorphic 1-form θ such that $\theta \wedge (d\theta)^n \neq 0$.
(2) On $\mathcal{O} \cap \mathcal{O}' \neq \emptyset$ there is a non-vanishing holomorphic function f such that $\theta' = f\theta$.

The complex contact structure determines a non-integrable subbundle $\mathcal{H}$ by the equation $\theta = 0$; $\mathcal{H}$ is called the *complex contact subbundle* or the *horizontal subbundle*.

A Hermitian manifold M with almost complex structure J, Hermitian metric g, and open covering by coordinate neighborhoods $\{\mathcal{O}\}$ is a *complex almost contact metric manifold* if it satisfies the following two conditions:

(1) In each $\mathcal{O}$ there exist 1-forms u and $v = u \circ J$ with dual vector fields U and $V = -JU$ and (1,1) tensor fields G and $H = GJ$ such that

$$G^2 = H^2 = -I + u \otimes U + v \otimes V,$$

$$GJ = -JG, \quad GU = 0, \quad g(X, GY) = -g(GX, Y).$$

(2) On $\mathcal{O} \cap \mathcal{O}' \neq \emptyset$,

$$u' = Au - Bv, \quad v' = Bu + Av,$$

$$G' = AG - BH, \quad H' = BG + AH,$$

where A and B are functions with $A^2 + B^2 = 1$.

A complex contact manifold admits a complex almost contact metric structure for which the local contact form θ is $u - iv$ to within a non-vanishing complex-valued function multiple. The local tensor fields G and H and du and dv are related by

$$du(X,Y) = \widehat{G}(X,Y) + (\sigma \wedge v)(X,Y),$$

$$dv(X,Y) = \widehat{H}(X,Y) - (\sigma \wedge u)(X,Y)$$

for some 1-form σ and where $\widehat{G}(X,Y) = g(X, GY)$ and $\widehat{H}(X,Y) = g(X, HY)$.

Also, on $\mathcal{O} \cap \mathcal{O}'$ one has $U' \wedge V' = U \wedge V$; it follows that there is a global vertical bundle $\mathcal{V}$ orthogonal to $\mathcal{H}$ (which is generally assumed to be integrable), and, in this case, σ takes the form $\sigma(X) = g(\nabla_X U, V)$, ∇ being the Levi–Civita connection of g. The subbundle $\mathcal{V}$ is the analogue of the characteristic or Reeb vector field of real contact geometry.

A complex contact manifold with a complex almost contact metric structure satisfying these conditions is called a *complex contact metric manifold*.

Example 1. *Examples of complex contact manifolds are given in [1]:*

(i) *Odd-dimensional complex projective space.*
(ii) *Twistor spaces.*
(iii) *The complex Boothby-Wang fibration.*
(iv) $\mathbf{C}^{n+1} \times \mathbf{C}P^n(16)$.

S. Ishihara and M. Konishi [5] introduced a notion of *normality* for complex contact structures. They defined two tensor fields S and T by

$$S(X,Y) = [G,G](X,Y) + 2\widehat{G}(X,Y)U - 2\widehat{H}(X,Y)V + 2(v(Y)HX - v(X)HY) +$$

$$+ \sigma(GY)HX - \sigma(GX)HY + \sigma(X)GHY - \sigma(Y)GHX,$$

$$T(X,Y) = [H,H](X,Y) - 2\widehat{G}(X,Y)U + 2\widehat{H}(X,Y)V + 2(u(Y)GX - u(X)GY) +$$
$$+\sigma(HX)GY - \sigma(HY)GX + \sigma(X)GHY - \sigma(Y)GHX,$$

where $[G,G]$ and $[H,H]$ denote the Nijenhuis tensors of G and H respectively.

A complex contact structure is said to be *normal* if S and T vanish, $S = T = 0$.

This notion is too strong; among its implications is that the underlying Hermitian manifold (M,g) is Kähler. Also one remarks that the canonical examples of a complex contact manifold, the odd-dimensional complex projective space, is normal in this sense, however, the complex Heisenberg group is not.

B. Korkmaz [6] generalized the notion of normality and we will use her definition here. With this notion of normality both odd-dimensional complex projective space and the complex Heisenberg group with their standard complex contact metric structures are normal.

A complex contact metric structure is *normal* [6] if

$$S(X,Y) = T(X,Y) = 0, \text{ for every } X,Y \in \mathcal{H},$$

$$S(U,X) = T(V,X) = 0, \text{ for every } X \in \mathcal{H}.$$

The definition appears to depend on the special nature of U and V, but it respects the change in overlaps, $\mathcal{O} \cap \mathcal{O}'$; then it is a global notion.

The expressions for the covariant derivatives of the structures tensors on a normal complex contact metric manifold are

$$\nabla_X U = -GX + \sigma(X)V, \tag{1}$$

$$\nabla_X V = -HX - \sigma(X)U. \tag{2}$$

Equivalently, a complex contact metric manifold is *normal* if and only if the covariant derivatives of G and H have the following forms:

$$g((\nabla_X G)Y,Z) = \sigma(X)g(HY,Z) + v(X)d\sigma\,(GZ,GY) - \tag{3}$$
$$-2v(X)g(HGY,Z) - u(Y)g(X,Z) - v(Y)g(JX,Z) +$$
$$+u(Z)g(X,Y) + v(Z)g(JX,Y),$$
$$g((\nabla_X H)Y,Z) = -\sigma(X)g(GY,Z) - u(X)d\sigma\,(HZ,HY) - \tag{4}$$
$$-2u(X)g(GHY,Z) + u(Y)g(JX,Z) - v(Y)g(X,Z) +$$
$$+u(Z)g(X,JY) + v(Z)g(X,Y).$$

For the Hermitian structure J we have

$$g((\nabla_X J)Y,Z) = u(X)\big(d\sigma\,(Z,GY) - 2g(HY,Z)\big) + \tag{5}$$

$$+v(X)\big(d\sigma\,(Z,HY) + 2g(GY,Z)\big).$$

B. Korkmaz (see [1,6]) defined the notion of *GH-sectional curvature* for a (normal) complex contact metric manifold $\tilde{M}$.

For $p \in \tilde{M}$ and a unit vector $X \in \mathcal{H}_p$, the plane in $T_p\tilde{M}$ spanned by X and $Y = \lambda\, GX + \mu HX$, $\lambda, \mu \in \mathbb{R}$, $\lambda^2 + \mu^2 = 1$, is called a *GH-plane section*. The sectional curvature of the GH-plane section is denoted by $\tilde{K}(X,Y)$.

For a given tangent vector X, the sectional curvature $\tilde{K}(X,Y)$ is independent of the vector Y in the plane of GX and HX. This is equivalent to $\tilde{K}(X,GX) = \tilde{K}(X,HX)$ and $\tilde{g}(\tilde{R}_{X(GX)}HX,X) = 0$.

Let $\tilde{M}$ be a normal complex contact manifold. If the GH-sectional curvature is independent of the choice of GH-section at each point, it is constant on the manifold $\tilde{M}$ and then $\tilde{M}$ is called a *(normal) complex contact space form*.

According to [6] and the convention used by D.E. Blair, the curvature tensor of $\tilde{M}$ have the expression

$$\begin{aligned}\tilde{R}(X,Y,Z,W) &= g(\tilde{R}_{XY}Z,W) = g(\tilde{R}(X,Y)Z,W) = \\ &= \frac{c+3}{4}\,[g(Y,Z)g(X,W) - g(X,Z)g(Y,W) + \\ &+ g(Z,JY)g(JX,W) - g(Z,JX)g(JY,W) + 2g(X,JY)g(JZ,W)] + \\ &+ \frac{c-1}{4}\,[-\,(u(Y)u(Z)+v(Y)v(Z))\,g(X,W) + (u(X)u(Z)+v(X)v(Z))\,g(Y,W) + \\ &+ 2u\wedge v(Z,Y)g(JX,W) - 2u\wedge v(Z,X)g(JY,W) + 4u\wedge v(X,Y)g(JZ,W) + \\ &+ g(Z,GY)g(GX,W) - g(Z,GX)g(GY,W) + 2g(X,GY)g(GZ,W) + \\ &g(Z,HY)g(HX,W) - g(Z,HX)g(HY,W) + 2g(X,HY)g(HZ,W) + \\ &+ (-u(X)g(Y,Z) + u(Y)g(X,Z) + v(X)g(JY,Z) - v(Y)g(JX,Z) + 2v(Z)g(X,JY))\,g(U,W) + \\ &+ (-v(X)g(Y,Z) + v(Y)g(X,Z) + u(X)g(JY,Z) - u(Y)g(JX,Z) + 2u(Z)g(X,JY))\,g(V,W)] + \\ &+ \frac{4}{3}\,(\Omega(U,V)+c+1)\,[(v(X)u\wedge v(Z,Y) - v(Y)u\wedge v(Z,X) + 2v(Z)u\wedge v(X,Y))\,g(U,W) - \\ &-\,(u(X)u\wedge v(Z,Y) - u(Y)u\wedge v(Z,X) + 2u(Z)u\wedge v(X,Y))\,g(V,W)]\,,\end{aligned} \tag{6}$$

where $\Omega\,(U,V) = g(U,JV)$.

Remark 1 ([1]).

(i) *An odd-dimensional complex projective space with the Fubini–Study metric of constant holomorphic sectional curvature 4 is of constant GH-sectional curvature 1.*

(ii) *The complex Heisenberg group has holomorphic sectional curvature 0 for horizontal and vertical holomorphic sections and constant GH-sectional curvature -3.*

We recall some results.

Theorem 1 ([1])**.** *Let $\tilde{M}$ be a normal complex contact metric manifold. Then $\tilde{M}$ has constant GH-sectional curvature c if and only if for X horizontal, the holomorphic sectional curvature of the plane spanned by X and JX is $c+3$.*

Theorem 2 ([1])**.** *Let $\tilde{M}$ be a normal complex contact metric manifold of constant GH-sectional curvature $+1$ and satisfying $d\sigma\,(V,U) = 2$. Then $\tilde{M}$ has constant holomorphic sectional curvature c.*

If, in addition, $\tilde{M}$ is complete and simply connected, then $\tilde{M}$ is isometric to $\mathbb{CP}^{2n+1}$ with the Fubini–Study metric of constant holomorphic sectional curvature c.

We denote a (normal) complex contact space form by $\tilde{M}(c)$, c being the constant GH-sectional curvature of $\tilde{M}$.

3. Submanifolds

By analogy with the geometry of submanifolds in (real) contact manifolds [7], we shall define certain special classes of submanifolds in normal complex contact manifolds (see also [8]).

Let M be a submanifold of a normal complex contact manifold $\tilde{M}$. Assume that U and V are normal vector fields to M. For $p \in M$ and $X, Y \in T_pM$, we have

$$g(GX, Y) = -g(\tilde{\nabla}_X U, Y) = g(U, \tilde{\nabla}_X Y) = g(U, h(X, Y)).$$

Since the first term of the above equalities is skew-symmetric and the last term is symmetric (in X, Y), then $g(GX, Y) = 0$. Similarly, $g(HX, Y) = 0$.

Then we have the following.

Remark 2. *If the vector fields U and V are normal to M, then for any $p \in M$, $G(T_pM) \subset T_p^{\perp}M$ and $H(T_pM) \subset T_p^{\perp}M$.*

Remark 3. *The above conditions do not imply $J(T_pM) \subset T_p^{\perp}M$, for all $p \in M$.*

Based on the previous two remarks, we define two classes of submanifolds.

Definition 1. *A submanifold M of a normal complex contact manifold $\tilde{M}$ is said to be an* invariant *submanifold if $G(T_pM) \subset T_pM$ and $H(T_pM) \subset T_pM$, for all $p \in M$.*

Remark 4. *It is immediately that, in the case when M is invariant, according to the above definition, T_pM is invariant by J, too.*

We shall prove the minimality of an invariant submanifold in a normal complex complex contact manifold.

First of all, we remark that U and V have to be tangent to the submanifold.

Suppose that U is not tangent to the submanifold and decompose U into its tangential and normal parts, say $U = U^T + U^{\perp}$. Then $0 = GU = GU^T + GU^{\perp}$; since the G-invariance of the tangent space implies the G invariance of the normal space, both GU^T and $GU^{\perp}$ vanish.

Let X be a horizontal tangent vector field; then

$$g(U, [X, GX]) = u([X, GX]) = -2du(X, GX) = -2g(X, G^2X) = 2g(X, X). \quad (7)$$

Now $[X, GX]$ is tangent to the submanifold and therefore

$$g(U, [X, GX]) = g(U^T, [X, GX]) = g(U^T, \tilde{\nabla}_X GX - \tilde{\nabla}_{GX} X) =$$

$$= g(U^T, (\tilde{\nabla}_X G)X + (\tilde{\nabla}_{GX} G)GX).$$

Using the Formula (3) for the covariant derivative of G for a normal complex contact metric manifold, this becomes

$$g(U, [X, GX]) = u(U^T)g(X, X) + u(U^T)g(GX, GX) = 2g(U^T, U^T)g(X, X).$$

Comparing with (7), $g(U^T, U^T) = 1$, i.e., U^T is unit as is U; therefore $U^{\perp} = 0$, contradicting the supposition that U was not tangent.

For this reason, an orthonormal basis of M, $\dim M = 4n + 2$, can be written as

$$\{E_1, ..., E_n, GE_1, ..., GE_n, HE_1, ..., HE_n, JE_1, ..., JE_n, U, V\}.$$

The proof of the minimality uses the formula of the covariant derivative given by B. Foreman in his Ph.D. Thesis [9].

Let $pr : TM \longrightarrow \mathcal{H}$ denote the projection to the horizontal subbundle and $J' = pr \circ J$. We then have

$$2g((\tilde{\nabla}_X G)Y, Z) = g([G, G](Y, Z), GX) -$$

$$- 3v \wedge d\sigma(X, GY, GZ) + 3v \wedge d\sigma(X, Y, Z) -$$

$$- 2\sigma(X)g(Y, HZ) + 4v(X)g(Y, J'Z) - \sigma(Y)g(Z, HX) + \sigma(GY)g(Z, J'X) -$$

$$- 2u(Y)g(X, pZ) - 2v(Y)g(Z, J'X) + \sigma(Z)g(Y, HX) - \sigma(GZ)g(Y, J'X) +$$

$$+ 2u(Z)g(X, pY) + 2v(Z)g(Y, J'X).$$

Now taking X as horizontal and $Y = X$ we make the following computation

$$2g((\tilde{\nabla}_X G)Y, Z) + 2g((\tilde{\nabla}_{GX} G)GY, Z) =$$

$$= g([G, G](X, Z), GX) - 2\sigma(X)g(X, HZ) -$$

$$- \sigma(X)g(Z, HX) + \sigma(GX)g(Z, JX) +$$

$$+ 2u(Z)g(X, X) - g([G, G](GX, Z), X) - 2\sigma(GX)g(GX, HZ) +$$

$$+ \sigma(GX)g(HZ, GX) + \sigma(X)g(Z, HX) + 2u(Z)g(X, X).$$

Expanding the Nijenhuis torsion terms and cancelling as appropriate, we have

$$g((\tilde{\nabla}_X G)Y, Z) + g((\tilde{\nabla}_{GX} G)GY, Z) =$$

$$= \sigma(X)g(HX, Z) + \sigma(GX)g(JX, Z) + 2u(Z)g(X, X).$$

Now suppose that Z is normal to the submanifold. Then the previous formula yields

$$g(h(X, GX) - Gh(X, X), Z) + g(h(GX, -X) - Gh(GX, GX), Z) = 0,$$

giving

$$h(X, X) + h(GX, GX) = 0,$$

where h denotes the second fundamental form of the submanifold M.

If we put JX instead of X, we find $h(JX, JX) + h(HX, HX) = 0$.

Then, we have proven:

Theorem 3. *Any invariant submanifold of a normal complex contact manifold is minimal.*

Example 2. *Example of an invariant submanifold:*

Consider the Segre embedding:

$$\mathbf{C}P^{2m+1} \times \mathbf{C}P^{2n+1} \to \mathbf{C}P^{(2m+1)(2n+1)+2m+2n+2},$$

$$([z^1, ..., z^{2m+2}], [w^1, ..., w^{2n+2}]) \mapsto [z^1 w^1, ..., z^i w^j, ..., z^{2m+2} w^{2n+2}].$$

Its image is an invariant submanifold.

Another important class of submanifolds are the anti-invariant submanifolds.

Definition 2. *A submanifold M of a normal complex contact manifold $\tilde{M}$ is said to be a CC-*totally real (anti-invariant) *submanifold if*

(i) U and V are normal to M;

(ii) *M is a totally real submanifold of $\tilde{M}$ (with respect to J).*

Example 3. *Example of a CC-totally real submanifold: $\mathbb{RP}^n \subset \mathbb{CP}^{2n+1}$.*

For a CC-totally real (anti-invariant) submanifold M of a (normal) complex contact space form $\tilde{M}(c)$ ($\dim_{\mathbb{R}} M = n$; $\dim_{\mathbb{R}} \tilde{M}(c) = m$) of arbitrary codimension, an orthonormal basis of $T_pM,\ p \in M$, can be written as $\{E_1, E_2, \ldots, E_n\}$, $n = \dim_{\mathbb{R}} M$.

Obviously,

$$U, V \perp E_i,\ \forall i = \overline{1, n},$$

$$U, V \perp GE_i,\ U, V \perp HE_i,\ U, V \perp JE_i, \forall i = \overline{1, n}.$$

4. Chen Inequalities

In this section we shall prove the Chen first inequality and Chen inequality for the invariant $\delta\ (2,2)$ for CC-totally real submanifolds in a normal complex contact space form $\tilde{M}(c)$.

Recall the definitions of the Chen first invariant and Chen invariant $\delta\ (2,2)$ (see, for example, [4,10]).

Let (M, g) be a Riemannian manifold of dimension $n \geq 2$ and denote by K and τ the sectional curvature and scalar curvature of M, respectively.

Chen first invariant δ_M is defined by $\delta_M = \tau - \inf K$.

Chen invariant $\delta\ (2,2)$ is given by $\delta\ (2,2) = \tau - \inf[K(\pi_1) + K(\pi_2)]$, where π_1 and π_2 are mutually orthogonal plane sections.

In order to prove Chen first inequality, we state the following lemma.

Lemma 1 ([11])**.** *Let $n \geq 3$ be an integer and let $a_1, ..., a_n$ be n real numbers. Then one has:*

$$\sum_{1 \leq i < j \leq n} a_i a_j - a_1 a_2 \leq \frac{n-2}{2(n-1)} \left(\sum_{i=1}^{n} a_i \right)^2.$$

Moreover, the equality holds if and only if $a_1 + a_2 = a_3 = ... = a_n$.

Let M be an n-dimensional CC-totally real submanifold of a normal complex contact space form of arbitrary codimension m.

The Gauss equation for M in $\tilde{M}(c)$ is given by

$$\tilde{R}(X, Y, Z, W) = R(X, Y, Z, W) - g(h(X, W), h(Y, Z)) + g(h(X, Z), h(Y, W)), \tag{8}$$

for any $X, Y, Z, W \in \Gamma(TM)$, where h is the second fundamental form.

Let $p \in M$, π a plane section in T_pM, $\{E_1, E_2\}$ an orthonormal basis of π, $\{E_1, ..., E_n\}$ an orthonormal basis of T_pM and $\{E_{n+1}, ..., E_m\}$ an orthonormal basis of $T_p^{\perp}M$.

The Gauss equation implies

$$K(\pi) = R(E_1, E_2, E_2, E_1) = \tag{9}$$

$$= \frac{c+3}{4} + g(h(E_1, E_1), h(E_2, E_2)) - g(h(E_1, E_2), h(E_1, E_2)) =$$

$$= \frac{c+3}{4} + \sum_{\alpha=n+1}^{m} [h_{11}^{\alpha}\, h_{22}^{\alpha} - (h_{12}^{\alpha})^2],$$

where $h_{ij}^{\alpha} = g(h(E_i, E_j), E_\alpha), i, j \in \{1, ..., n\}, \alpha \in \{n+1, ..., m\}$.

On the other hand,

$$\tau = \sum_{1\le i<j\le n} R(E_i, E_j, E_j, E_i) = \tag{10}$$

$$= n(n-1)\frac{c+3}{8} + \sum_{1\le i<j\le n} [g(h(E_i,E_i),h(E_j,E_j)) - g(h(E_i,E_j),h(E_i,E_j))] =$$

$$= n(n-1)\frac{c+3}{8} + \sum_{\alpha=n+1}^{m}\sum_{1\le i<j\le n} [h_{ii}^{\alpha}\, h_{jj}^{\alpha} - (h_{ij}^{\alpha})^2].$$

Subtracting the above two equations, one has

$$\tau - K(\pi) = (n-2)(n+1)\frac{c+3}{8} + \sum_{\alpha=n+1}^{m}\left(\sum_{1\le i<j\le n} h_{ii}^{\alpha}\, h_{jj}^{\alpha} - h_{11}^{\alpha}\, h_{22}^{\alpha}\right) -$$

$$- \sum_{\alpha=n+1}^{m}\left[\sum_{1\le i<j\le n} (h_{ij}^{\alpha})^2 - (h_{12}^{\alpha})^2\right] \le$$

$$\le (n-2)(n+1)\frac{c+3}{8} + \sum_{\alpha=n+1}^{m}\left(\sum_{1\le i<j\le n} h_{ii}^{\alpha}\, h_{jj}^{\alpha} - h_{11}^{\alpha}\, h_{22}^{\alpha}\right).$$

By applying Lemma 1, we obtain for all $\alpha \in \{n+1, ..., m\}$:

$$\sum_{1\le i<j\le n} h_{ii}^{\alpha}\, h_{jj}^{\alpha} - h_{11}^{\alpha}\, h_{22}^{\alpha} \le \frac{n-2}{2(n-1)}\left(\sum_{i=1}^{n} h_{ii}^{\alpha}\right)^2 = \frac{n^2(n-2)}{2(n-1)}(H^{\alpha})^2.$$

By summing the above relations, we obtain

$$\tau - K(\pi) \le (n-2)(n+1)\frac{c+3}{8} + \frac{n^2(n-2)}{2(n-1)}\|\vec{H}\|^2,$$

where $\vec{H}$ is the mean curvature vector.

The equality holds at a point $p \in M$ if and only if for any $\alpha \in \{n+1, ..., m\}$,

$$\begin{cases} h_{11}^{\alpha} + h_{22}^{\alpha} = h_{33}^{\alpha} = ... = h_{nn}^{\alpha}, \\ h_{ij}^{\alpha} \ = 0, \forall 1 \le i < j \le n, (i,j) \ne (1,2). \end{cases} \tag{11}$$

If we take E_{n+1} parallel to $\vec{H}(p)$ and E_1, E_2 such that $h_{12}^{n+1} = 0$, the shape operators take the forms given in the following:

Theorem 4. *Let $\tilde{M}(c)$ be a normal complex contact space form and M an n-dimensional $(n \ge 3)$ CC-totally real submanifold. Then we have*

$$\delta_M = \tau - \inf K \le \frac{n-2}{2}\left[\frac{n^2}{n-1}\|\vec{H}\|^2 + (n+1)\frac{c+3}{4}\right].$$

Moreover, the equality case of the inequality holds at a point $p \in M$ if and only if there exist an orthonormal basis $\{E_1, E_2, ..., E_n\}$ of T_pM and an orthonormal basis $\{E_{n+1}, ..., E_m\}$ of $T_p^{\perp}M$ such that the shape operators take the following forms:

$$A_{n+1} = \begin{pmatrix} a & 0 & 0 & \dots & 0 \\ 0 & b & 0 & \dots & 0 \\ 0 & 0 & \mu & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & ... & \mu \end{pmatrix}, \quad a + b = \mu,$$

$$A_r = \begin{pmatrix} h_{11}^r & h_{12}^r & 0 & \dots & 0 \\ h_{12}^r & -h_{11}^r & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & ... & 0 \end{pmatrix}, \quad r \in \{n+2, ..., m\}.$$

If M is of maximum dimension, i.e., $\dim M = n$ and $\dim \tilde{M} = 4n + 2$ (the analogue of a Legendrian submanifold in the real case), we obtain:

Theorem 5. *Let $\tilde{M}(c)$ be a normal complex contact space form of dimension $4n + 2$. Then any n-dimensional CC-totally real submanifold M satisfying the equality case of Chen first inequality, identically, is minimal.*

Proof. Let M be an n-dimensional CC-totally real submanifold in a $(4n + 2)$-dimensional normal complex contact space form $\tilde{M}(c)$ satisfying the equality case of Chen first inequality, identically. Then the Equations (11) can be written as

$$\begin{cases} h(E_1, E_1) + h(E_2, E_2) = h(E_3, E_3) = ... = h(E_n, E_n), \\ h(E_i, E_j) = 0, \forall 1 \leq i < j \leq n, (i, j) \neq (1, 2). \end{cases} \tag{12}$$

Then, by using (1), we have

$$g(h(E_3, E_3), U) = g(\tilde{\nabla}_{E_3} E_3, U) = -g(E_3, \tilde{\nabla}_{E_3} U) = g(E_3, GE_3 - \sigma\,(E_3)V) = 0,$$

and similarly $g(h(E_3, E_3), V) = 0$.

On the other hand, using (3), we find for any $j \neq 3$

$$g(h(E_3, E_3), GE_j) = g(\tilde{\nabla}_{E_3} E_3, GE_j) = -g(E_3, \tilde{\nabla}_{E_3} GE_j) = g(GE_3, h(E_3, E_j)) = 0,$$

and

$$g(h(E_3, E_3), GE_3) = g(h(E_1, E_1) + h(E_2, E_2), GE_3) = g(h(E_1, E_3), GE_1) + g(h(E_2, E_3), GE_2) = 0.$$

Analogously, $g(h(E_3, E_3), HE_j) = g(h(E_3, E_3), JE_j) = 0, \forall j = 1, ..., n$.

Therefore $h(E_3, E_3) = 0$.

It follows by (12) that

$$\vec{H} = \frac{n-1}{n} h(E_3, E_3) = 0,$$

i.e., M is a minimal submanifold. □

Next we shall prove Chen inequality for the invariant $\delta\,(2, 2)$. We shall use the following Lemma.

Lemma 2 ([12])**.** *Let $n \geq 4$ be an integer and let $a_1, ..., a_n$ be n real numbers. Then one has:*

$$\sum_{1 \leq i < j \leq n} a_i a_j - a_1 a_2 - a_3 a_4 \leq \frac{n-3}{2(n-2)} \left(\sum_{i=1}^{n} a_i \right)^2.$$

Moreover, the equality holds if and only if $a_1 + a_2 = a_3 + a_4 = a_5 = ... = a_n$.

Let $p \in M$, π_1 and π_2 mutually orthogonal plane sections in T_pM, $\{E_1, E_2\}$ and $\{E_3, E_4\}$ orthonormal bases of π_1 and π_2, and $\{E_1, ..., E_n\}$ an orthonormal basis of T_pM.

The Gauss equation implies

$$K(\pi_1) = \frac{c+3}{4} + g(h(E_1, E_1), h(E_2, E_2)) - g(h(E_1, E_2), h(E_1, E_2)) =$$

$$= \frac{c+3}{4} + \sum_{\alpha=n+1}^{m} [h_{11}^{\alpha} \, h_{22}^{\alpha} - (h_{12}^{\alpha})^2].$$

$$K(\pi_2) = \frac{c+3}{4} + g(h(E_3, E_3), h(E_4, E_4)) - g(h(E_3, E_4), h(E_3, E_4)) =$$

$$= \frac{c+3}{4} + \sum_{\alpha=n+1}^{m} [h_{33}^{\alpha} \, h_{44}^{\alpha} - (h_{34}^{\alpha})^2].$$

By subtracting the above two equations from (10), we obtain

$$\tau - K(\pi_1) - K(\pi_2) = (n^2 - n - 4)\frac{c+3}{8} + \sum_{\alpha=n+1}^{m} \left(\sum_{1 \leq i < j \leq n} h_{ii}^{\alpha} \, h_{jj}^{\alpha} - h_{11}^{\alpha} \, h_{22}^{\alpha} - h_{33}^{\alpha} \, h_{44}^{\alpha} \right) -$$

$$- \sum_{\alpha=n+1}^{m} \left[\sum_{1 \leq i < j \leq n} (h_{ij}^{\alpha})^2 - (h_{12}^{\alpha})^2 - (h_{34}^{\alpha})^2 \right] \leq$$

$$\leq (n^2 - n - 4)\frac{c+3}{8} + \sum_{\alpha=n+1}^{m} \left(\sum_{1 \leq i < j \leq n} h_{ii}^{\alpha} \, h_{jj}^{\alpha} - h_{11}^{\alpha} \, h_{22}^{\alpha} - h_{33}^{\alpha} \, h_{44}^{\alpha} \right).$$

By applying Lemma 2, we have for all $\alpha \in \{n+1, ..., m\}$:

$$\sum_{1 \leq i < j \leq n} h_{ii}^{\alpha} \, h_{jj}^{\alpha} - h_{11}^{\alpha} \, h_{22}^{\alpha} - h_{33}^{\alpha} h_{44}^{\alpha} \leq \frac{n-3}{2(n-2)} \left(\sum_{i=1}^{n} h_{ii}^{\alpha} \right)^2 = \frac{n^2(n-2)}{2(n-1)} (H^{\alpha})^2.$$

By summing the above relations, we have

$$\tau - K(\pi_1) - K(\pi_2) \leq (n^2 - n - 4)\frac{c+3}{8} + \frac{n^2(n-3)}{2(n-2)} \|\vec{H}\|^2.$$

Then, we found next

Theorem 6. *Let $\tilde{M}(c)$ be an m-dimensional normal complex contact space form and M an n-dimensional ($n \geq 4$) CC-totally real submanifold. Then we have*

$$\delta\,(2,2) \leq \frac{n^2(n-3)}{2(n-2)} \|\vec{H}\|^2 + (n^2 - n - 4)\frac{c+3}{4}.$$

Moreover, the equality holds at a point $p \in M$ if and only if for any $\alpha \in \{n+1, ..., m\}$,

$$\begin{cases} h^{\alpha}_{11} + h^{\alpha}_{22} = h^{\alpha}_{33} + h^{\alpha}_{44} = h^{\alpha}_{55} = ... = h^{\alpha}_{nn}, \\ h^{\alpha}_{ij} \ = 0, \forall 1 \leq i < j \leq n, (i,j) \neq (1,2), (3,4). \end{cases}$$

For a CC-totally real submanifold of maximum dimension, we have

Theorem 7. *Let $\tilde{M}(c)$ be a normal complex contact space form of dimension $4n+2$. Then any n-dimensional CC-totally real submanifold M satisfying the equality case of Chen inequality for $\delta\ (2,2)$, identically, is minimal.*

Its *proof* is similar to the proof of Theorem 5.

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