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Convergence Theorems of Variational Inequality for Asymptotically Nonexpansive Nonself Mapping in $CAT(0)$ Spaces

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Abstract: The aim of this manuscript is to get the strong convergence theorems of the Moudafi's viscosity approximation methods for an asymptotically nonexpansive nonself mapping in $CAT(0)$ spaces.

Keywords: geodesic triangle; $CAT(0)$ space; quasilinearization; retraction; asymptotically nonexpansive nonself mapping; viscosity approximation method; $\triangle$-convergence; variational inequality; fixed point

1. Introduction

Let (X,d) be a metric space. A geodesic path joining $p_1 \in X$ to $p_2 \in X$ (or, a geodesic from p_1 to p_2) is a mapping g from a closed interval $[0,l] \subset \mathbb{R}$ to X such that $g(0) = p_1, g(l) = p_2$, and

$$d(g(t), g(t')) = |t - t'|, \quad \forall\, t, t' \in [0,l].$$

In particular, g is an isometry and $d(p_1, p_2) = l$. The image α of g is said to be a geodesic segment (or metric segment) joining p_1 and p_2. When it is unique, this geodesic segment is denoted by $[p_1, p_2]$. The space (X,d) is called a geodesic space if every two points of X are joined by a geodesic segment, and X is called a uniquely geodesic segment if there is exactly one geodesic segment joining p_1 and p_2 for each $p_1, p_2 \in X$. A subset $Y \subseteq X$ is called convex if Y includes every geodesic segment joining any two of its points.

A geodesic triangle $\triangle(p_1, p_2, p_3)$ is a geodesic metric space (X,d) that consists of three vertices of $\triangle$ (the points $p_1, p_2, p_3 \in X$) and the edges of $\triangle$ (a geodesic segment between each pair of vertices). A comparison triangle for the geodesic triangle $\triangle(p_1, p_2, p_3)$ in (X,d) is a triangle $\bar{\triangle}(p_1, p_2, p_3) = \triangle(\bar{p}_1, \bar{p}_2, \bar{p}_3)$ in $\mathbb{R}^2$ such that

$$d_{\mathbb{R}^2}(\bar{p}_i, \bar{p}_j) = d(p_i, p_j), \quad i, j \in \{1,2,3\}.$$

A comparison triangle for the geodesic triangle always exists (see, [1,2]).

A geodesic metric space is called a $CAT(0)$ space (this term is due to M. Gromov [3] and it is an acronym for E. Cartan, A.D. Aleksandrov and V.A. Toponogov) if all geodesic triangles of appropriate size satisfy the following $CAT(0)$ comparison axiom.

Let $\triangle$ be a geodesic triangle in (X,d) and let $\bar{\triangle} \subset \mathbb{R}^2$ be a comparison triangle for $\triangle$. Then $\triangle$ is said to satisfy the $CAT(0)$ inequality if for all vertices $p_1, p_2 \in \triangle$ and all comparison points $\bar{p}_1, \bar{p}_2 \in \bar{\triangle}$,

$$d(p_1, p_2) \le d_{\mathbb{R}^2}(\bar{p}_1, \bar{p}_2).$$

Let p, p_1, p_2 be points of a $CAT(0)$ space, if p_0 is the midpoint of the segment $[p_1, p_2]$, which we will denote by $\frac{p_1 \oplus p_2}{2}$, then the $CAT(0)$ inequality implies

$$d^2\left(p, \frac{p_1 \oplus p_2}{2}\right) = d^2(p, p_0) \leq \frac{1}{2}d^2(p, p_1) + \frac{1}{2}d^2(p, p_2) - \frac{1}{4}d^2(p_1, p_2).$$

This inequality is called the (CN) inequality ([4]).

Remark 1. *A geodesic metric space (X, d) is a $CAT(0)$ space if and only if it satisfies the (CN) inequality (cf. [1], p. 163).*

The above (CN) inequality has been extended as

$$\begin{aligned} &d^2(p, \alpha p_1 \oplus (1-\alpha)p_2) \\ &\leq \alpha d^2(p, p_1) + (1-\alpha)d^2(p, p_2) - \alpha(1-\alpha)d^2(p_1, p_2), \quad \forall\, p, p_1, p_2 \in X, \end{aligned} \tag{CN*}$$

for all $0 \leq \alpha \leq 1$ [5,6].

In recent years, $CAT(0)$ spaces have attracted many researchers as they treated a very important role in different directions of geometry and mathematics (see [1,7–10]). Complete $CAT(0)$ spaces are often called Hadamard spaces (see [10]).

It is well known that a normed linear space satisfies the (CN) inequality if and only if it satisfies the parallelogram identity, i.e., it is a pre-Hilbert space [1]. Hence it is not so unusual to have an inner product-like notion in Hadamard spaces. In [11], they introduced the concept of quasilinearization as follows

Let us usually denote a pair $(p, q) \in X^2 = X \times X$ by $\overrightarrow{pq}$ and call it a vector. Then quasilinearization is defined as a mapping $\langle \cdot, \cdot \rangle : X^2 \times X^2 \to \mathbb{R}$ by

$$\langle \overrightarrow{xy}, \overrightarrow{uv} \rangle = \frac{1}{2}(d^2(x, v) + d^2(y, u) - d^2(x, u) - d^2(y, v)), \quad \forall\, x, y, u, v \in X.$$

It is easily seen that

$$\langle \overrightarrow{xy}, \overrightarrow{uv} \rangle = \langle \overrightarrow{uv}, \overrightarrow{xy} \rangle, \quad \langle \overrightarrow{xy}, \overrightarrow{uv} \rangle = -\langle \overrightarrow{yx}, \overrightarrow{uv} \rangle$$

and

$$\langle \overrightarrow{xy}, \overrightarrow{uv} \rangle = \langle \overrightarrow{xw}, \overrightarrow{uv} \rangle + \langle \overrightarrow{wy}, \overrightarrow{uv} \rangle$$

for all $x, y, u, v, w \in X$. We say that X satisfies the Cauchy–Schwarz inequality if

$$\langle \overrightarrow{xy}, \overrightarrow{uv} \rangle \leq d(x, y)d(u, v), \quad \forall\, x, y, u, v \in X. \tag{1}$$

Remark 2. *A geodesically connected metric space is a $CAT(0)$ space if and only if it satisfies the Cauchy–Schwarz inequality ([11], Corollary 3).*

In [12], the authors introduced the concept of duality mapping in $CAT(0)$ spaces, by using the concept of quasilinearization, and studied its relation with the subdifferential. Moreover, they proved a characterization of metric projection in $CAT(0)$ spaces as follows.

Theorem 1. ([12], Theorem 2.4) *Let C be a nonempty convex subset of a complete $CAT(0)$ space X. Then*

$$p = P_C x \quad \Leftrightarrow \quad \langle \overrightarrow{yp}, \overrightarrow{px} \rangle \geq 0, \ \forall\, y \in C,$$

for all $x \in X$ and $p \in C$.

In 2015, using the concept of quasilinearization, Wangkeeree et al. [13] proved the strong convergence theorems of the following Moudafi's viscosity iterations for an asymptotically nonexpansive mapping T: For given a contraction mapping f defined on C and $0 < \alpha_n < 1$, let $x_n \in C$ be the unique fixed point of the contraction $x \mapsto \alpha_n f(x) \oplus (1-\alpha_n)T^n x$, i.e.,

$$x_n = \alpha_n f(x_n) \oplus (1-\alpha_n)T^n x_n, \quad \forall n \geq 1 \tag{2}$$

and let $x_1 \in C$ be arbitrarily chosen and

$$x_{n+1} = \alpha_n f(x_n) \oplus (1-\alpha_n)T^n x_n, \quad \forall n \geq 1. \tag{3}$$

They proved the iterative schemes $\{x_n\}$ defined by Equations (2) and (3) strongly converge to the same point $\bar{x} \in F(T)$ with $\bar{x} = P_{F(T)} f(\bar{x})$, which is the unique solution of the variational inequality

$$\langle \overrightarrow{\bar{x}f(\bar{x})}, x\bar{x} \rangle \geq 0, \quad x \in F(T),$$

where $F(T) = \{x : Tx = x\}$.

On the other hand, Shi et al. [14] studied the $\triangle$-convergence of the iteration scheme for asymptotically nonexpansive mappings in $CAT(0)$ spaces.

Let (X,d) be a metric space and C be a nonempty subset of X. A mapping f defined on C is called a contraction with coefficient $0 < \alpha < 1$ if

$$d(f(u), f(v)) \leq \alpha d(u,v)$$

for all $u, v \in C$. A subset C is called a retract of X if there exists a continuous mapping P from X onto C such that $Pu = u$ for all $u \in C$. A mapping $P : X \to C$ is said to be a retraction if $P^2 = P$. Moreover, if a mapping P is a retraction, then $Pv = v$ for all v in the range of P.

Definition 1. *Let C be a nonempty subset of a metric space (X,d). Let $P : X \to C$ be a nonexpansive retraction of X onto C.*

(1) A nonself mapping $T : C \to X$ is said to be nonexpansive (cf. [15]) if

$$d(Tx, Ty) \leq d(x,y),$$

for all $x, y \in C$.

(2) A nonself mapping $T : C \to X$ is said to be asymptotically nonexpansive ([16]) if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$d(T(PT)^{n-1}x, T(PT)^{n-1}y) \leq k_n d(x,y), \quad \forall n \in \mathbb{N},$$

for all $x, y \in C$.

Recently, Kim et al. [17] and Kim [18] presented the existence and $\triangle$-convergence for asymptotically nonexpansive nonself mappings in $CAT(0)$ spaces.

Motivated and inspired by Wangkeeree et al. [13], Shi et al. [14], Kim et al. [17] and Kim [18], the aim of this paper is to obtain the strong convergence theorems of the Moudafi's viscosity approximation methods for an asymptotically nonexpansive nonself mapping in $CAT(0)$ spaces.

Let C be a nonempty closed convex subset of a complete $CAT(0)$ space X. Let $P : X \to C$ be a retraction mapping and $T : C \to X$ be an asymptotically nonexpansive nonself mapping. Given a contraction mapping f defined on C and $0 < \alpha_n < 1$, let $x_n \in C$ be the unique fixed point of the contraction $x \mapsto \alpha_n f(x) \oplus (1-\alpha_n)T(PT)^{n-1}x$, i.e.,

$$x_n = \alpha_n f(x_n) \oplus (1-\alpha_n)T(PT)^{n-1}x_n, \ \ \forall n \geq 1 \tag{4}$$

and let $x_1 \in C$ be arbitrarily chosen and

$$x_{n+1} = \alpha_n f(x_n) \oplus (1-\alpha_n)T(PT)^{n-1}x_n, \ \ \forall n \geq 1. \tag{5}$$

The author proved that the iterative schemes $\{x_n\}$ defined by Equations (4) and (5) strongly converge to the same point $x^* \in \mathcal{F}(T)$ such that $x^* = P_{\mathcal{F}(T)}f(x^*)$ which is the unique solution of the variational inequality

$$\langle \overrightarrow{x^* f(x^*)}, xx^* \rangle \geq 0, \ \ x \in \mathcal{F}(T),$$

where $\mathcal{F}(T) = \{x : Tx = x\}$.

2. Preliminaries

Throughout this paper, $\mathbb{N}$ denotes the set of all positive integers. Let C be a nonempty subset of a metric space (X,d). $\mathcal{F}(T) = \{x : Tx = x\}$ denotes the set of fixed points of T.

We write $(1-t)p_1 \oplus tp_2$ for the unique point p in the geodesic segment joining from p_1 to p_2 such that

$$d(p,p_1) = td(p_1,p_2) \quad \text{and} \quad d(p,p_2) = (1-t)d(p_1,p_2).$$

We also denote by $[p_1,p_2]$ the geodesic segment joining from p_1 to p_2, i.e., $[p_1,p_2] = \{(1-t)p_1 \oplus tp_2 : T \in [0,1]\}$. A subset C of a $CAT(0)$ space is convex if $[p_1,p_2] \subset C$ for all $p_1,p_2 \in C$.

In the sequel we need the following useful lemmas.

Lemma 1. ([1], Proposition 2.2, p. 176) *Let X be a $CAT(0)$ space, then the distance function $d : X \times X \to \mathbb{R}$ is convex, i.e., given any pair of geodesics $g : [0,1] \to X$ and $g' : [0,1] \to X$, parameterized proportional to arc length, the following inequality holds for all $t \in [0,1]$:*

$$d(g(t), g'(t)) \leq (1-t)d(g(0), g'(0)) + td(g(1), g'(1)).$$

Lemma 2. ([6]) *Let X be a $CAT(0)$ space, $p_1, p_2, z \in X$ and $t \in [0,1]$. Then*

(i) $d(tp_1 \oplus (1-t)p_2, z) \leq td(p_1,z) + (1-t)d(p_2,z)$,
(ii) $d^2(tp_1 \oplus (1-t)p_2, z) \leq td^2(p_1,z) + (1-t)d^2(p_2,z) - t(1-t)d^2(p_1,p_2)$.

Lemma 3. ([19]) *Let X be a $CAT(0)$ space, $p_1, p_2, z \in X$ and $t \in [0,1]$. Then*

(i) $d(tp_1 \oplus (1-t)p_2, \gamma p_1 \oplus (1-\gamma)p_2) = |t-\gamma| d(p_1,p_2)$,
(ii) $d(tp_1 \oplus (1-t)p_2, tp_1 \oplus (1-t)z) \leq (1-t)d(p_2,z)$.

Now, we give the concept of $\triangle$-convergence and its some basic properties.

Kirk and Panyanak [20] insisted the concept of $\triangle$-convergence in $CAT(0)$ spaces that was introduced by Lim [21] in 1976 is very similar to the weak convergence in a Banach space setting.

Let $\{x_n\}$ be a bounded sequence in $CAT(0)$ space X. For $p \in X$, we set

$$r(p,\{x_n\}) = \limsup_{n\to\infty} d(p,x_n).$$

The asymptotic radius $A_r(\{x_n\})$ of $\{x_n\}$ is given by

$$A_r(\{x_n\}) = \inf\{r(p,\{x_n\}) : p \in X\},$$

and the asymptotic center $A_c(\{x_n\})$ of $\{x_n\}$ is the set

$$A_c(\{x_n\}) = \{p \in X : r(p, \{x_n\}) = A_r(\{x_n\})\}.$$

It is well known that asymptotic center $A_c(\{x_n\})$ consists of exactly one point (see, e.g., [22], Proposition 7, p. 767) in a complete $CAT(0)$ space.

Definition 2. ([20]) *A sequence $\{x_n\}$ in a complete $CAT(0)$ space X is said to $\triangle$-converge to $x \in X$ if x is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$, i.e., $A_c(\{u_n\}) = \{x\}$. In this case one can write*

$$x_n \xrightarrow{\triangle} x \quad \text{or} \quad \triangle - \lim_{n\to\infty} x_n = x$$

and call x the $\triangle$-limit of $\{x_n\}$.

Remark 3. *In a $CAT(0)$ space, strong convergence in the metric implies $\triangle$-convergence (see, [23,24]).*

For any bounded sequence $\{z_n\}$ in a $CAT(0)$ space X, there exists $x^* \in X$ such that

$$\varphi(x^*) = \inf\{\varphi(x) : x \in X\},$$

where

$$\varphi(x) = \limsup_{n\to\infty} d(z_n, x), \quad x \in X.$$

Lemma 4. ([20]) *Every bounded sequence in a complete $CAT(0)$ space always has a $\triangle$-convergent subsequence.*

Now, we shall give the existence of a fixed point for asymptotically nonexpansive nonself mapping $T : C \to X$ in a complete $CAT(0)$ space.

Lemma 5. ([18]) *Let C be a nonempty closed convex subset of a complete $CAT(0)$ space X and let $T : C \to X$ be an asymptotically nonexpansive nonself mapping with a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$. Then T has a unique fixed point in C. Moreover, the set $\mathcal{F}(T)$ is a closed and convex subset of X.*

Before we state the next lemma, we need the following notation

$$\{z_n\} \rightharpoonup x^* \quad \Longleftrightarrow \quad \varphi(x^*) = \inf\{\varphi(x) : x \in C\},$$

where C is a nonempty closed convex subset that contains the bounded sequence $\{z_n\}$ and $\varphi(x) = \limsup_{n\to\infty} d(z_n, x)$.

Lemma 6. *Let X be a $CAT(0)$ space and C be a nonempty closed convex subset of X. Let $T : C \to X$ be an asymptotically nonexpansive nonself mapping with a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$. If*

$$\lim_{n\to\infty} d(z_n, Tz_n) = 0 \;\; \text{and} \;\; \{z_n\} \rightharpoonup x^*,$$

then we have

$$T(x^*) = x^*.$$

Proof. Since $\lim_{n\to\infty} d(z_n, Tz_n) = 0$, we have

$$\varphi(x) = \limsup_{n\to\infty} d(T(PT)^{m-1}z_n, x), \ \ \forall\, m \geq 1.$$

Hence

$$\begin{aligned}\varphi(T(PT)^{m-1}x) &= \limsup_{n\to\infty} d(T(PT)^{m-1}z_n, T(PT)^{m-1}x) \\ &\leq \limsup_{n\to\infty} k_m d(z_n, x) = k_m \limsup_{n\to\infty} d(z_n, x) \\ &= k_m \varphi(x), \ \ \forall\, x \in C.\end{aligned}$$

In particular, we have

$$\lim_{m\to\infty} \varphi(T(PT)^{m-1}x^*) = \lim_{m\to\infty} k_m \varphi(x^*) = \varphi(x^*). \tag{6}$$

From Lemma 2-(ii),

$$\begin{aligned}d^2\left(z_n, \frac{x^* \oplus T(PT)^{m-1}x^*}{2}\right) \leq \frac{1}{2}d^2(z_n, x^*) + \frac{1}{2}d^2(z_n, T(PT)^{m-1}x^*) \\ - \frac{1}{4}d^2(x^*, T(PT)^{m-1}x^*), \ \ \forall\, m, n \geq 1.\end{aligned}$$

Taking m as fixed and $\limsup_{n\to\infty}$ on both sides, we have

$$\begin{aligned}\varphi^2\left(\frac{x^* \oplus T(PT)^{m-1}x^*}{2}\right) \leq \frac{1}{2}\varphi^2(x^*) + \frac{1}{2}\varphi^2(T(PT)^{m-1}x^*) \\ - \frac{1}{4}d^2(x^*, T(PT)^{m-1}x^*), \ \ \forall\, m \geq 1.\end{aligned}$$

From the definition of x^*, we obtain

$$\begin{aligned}\varphi^2(x^*) \leq \frac{1}{2}\varphi^2(x^*) + \frac{1}{2}\varphi^2(T(PT)^{m-1}x^*) \\ - \frac{1}{4}d^2(x^*, T(PT)^{m-1}x^*), \ \ \forall\, m \geq 1,\end{aligned}$$

which implies

$$d^2(x^*, T(PT)^{m-1}x^*) \leq 2\varphi^2(T(PT)^{m-1}x^*) - 2\varphi^2(x^*).$$

Taking $\lim_{m\to\infty}$ on both sides, from Equation (6), we get

$$\lim_{m\to\infty} d(x^*, T(PT)^{m-1}x^*) = 0,$$

that is

$$\lim_{m\to\infty} T(PT)^{m-1}x^* = x^*.$$

Since TP is a continuous mapping, we obtain

$$\begin{aligned}x^* &= \lim_{m\to\infty} T(PT)^{m-1}x^* = \lim_{m\to\infty} TP(T(PT)^{m-2}x^*) \\ &= (TP)x^* = Tx^*.\end{aligned}$$

□

Lemma 7. ([23], Theorem 2.6) *Let X be a complete $CAT(0)$ space, $\{x_n\}$ be a sequence in X and $x \in X$. Then $\{x_n\}$ $\triangle$-converges to x if and only if*

$$\limsup_{n\to\infty} \langle \overrightarrow{xx_n}, \overrightarrow{xy} \rangle \leq 0, \ \ \forall y \in X.$$

Lemma 8. ([25]) *Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ and $\{\lambda_n\}$ be nonnegative sequences such that*

$$a_{n+1} \leq (1-\lambda_n)a_n + \lambda_n b_n + c_n, \quad n \geq 0,$$

with $\{\lambda_n\} \subset [0,1]$, $\sum_{n=0}^{\infty} \lambda_n = \infty$, $\lim_{n\to\infty} b_n = 0$ and $\sum_{n=0}^{\infty} c_n < \infty$. Then $\lim_{n\to\infty} a_n = 0$.

The following two useful lemmas can be found in [13].

Lemma 9. ([13]) *Let X be a complete $CAT(0)$ space. Then the following inequality holds*

$$d^2(p,r) \leq d^2(q,r) + 2\langle \overrightarrow{pq}, \overrightarrow{pr} \rangle, \ \ \forall p,q,r \in X.$$

Lemma 10. ([13]) *Let X be a $CAT(0)$ space. For any $l \in (0,1)$ and $x, y \in X$, let*

$$x_l = lx \oplus (1-l)y.$$

Then, for all $u, v \in X$,

(i) $\langle \overrightarrow{x_l u}, \overrightarrow{x_l v} \rangle \leq l \langle \overrightarrow{xu}, \overrightarrow{x_l v} \rangle + (1-l)\langle \overrightarrow{yu}, \overrightarrow{x_l v} \rangle$,

(ii) $\langle \overrightarrow{x_l u}, \overrightarrow{xv} \rangle \leq l \langle \overrightarrow{xu}, \overrightarrow{xv} \rangle + (1-l)\langle \overrightarrow{yu}, \overrightarrow{xv} \rangle$ *and*
$\langle \overrightarrow{x_l u}, \overrightarrow{yv} \rangle \leq l \langle \overrightarrow{xu}, \overrightarrow{yv} \rangle + (1-l)\langle \overrightarrow{yu}, \overrightarrow{yv} \rangle$.

3. Main Results

In this section, we study the convergence theorems of Moudafi's viscosity approximation methods for asymptotically nonexpansive nonself mapping $T : C \to X$ in a complete $CAT(0)$ space.

Theorem 2. *Let C be a nonempty closed convex subset of a complete $CAT(0)$ space X and let $T : C \to X$ be an asymptotically nonexpansive nonself mapping with a sequence $\{k_n\} \subset [1,\infty)$ with $\lim_{n\to\infty} k_n = 1$. Let f be a contraction mapping defined on C with coefficient $\alpha \in (0,1)$. Let $\{a_n\}$ be a real valued sequence with $0 < a_n < 1$. If it satisfies the following conditions*

(i) $\frac{k_n - 1}{a_n} < 1 - \alpha < a_n(k_n - \alpha), \ \ \forall n \in \mathbb{N}$,

(ii) $a_n \to 0$, $\frac{k_n-1}{a_n} \to 0$ *and* $\frac{|a_n - a_{n-1}|}{a_n} \to 0$ *as* $n \to \infty$,

then the following statements hold.

(1) *There exists x_n such that*

$$x_n = a_n f(x_n) \oplus (1-a_n)T(PT)^{n-1}x_n, \ \ \forall n \in \mathbb{N}. \tag{7}$$

(2) *The sequence $\{x_n\}$ converges strongly to x^* as $n \to \infty$ such that*

$$x^* = P_{\mathcal{F}(T)} f(x^*),$$

which is equivalent to the following variational inequality:

$$\langle \overrightarrow{x^* f(x^*)}, \overrightarrow{xx^*} \rangle \geq 0, \ \ \forall x \in \mathcal{F}(T). \tag{8}$$

Proof. **I.** For each integer $n \geq 1$, we shall define a mapping $F_n : C \to X$ by

$$F_n(x) = a_n f(x) \oplus (1 - a_n)T(PT)^{n-1}x, \ \ \forall x \in C.$$

First, we show that F_n is a contraction mapping. For any $x, y \in C$, by Lemma 1

$$\begin{aligned}
&d(F_n(x), F_n(y)) \\
&= d(a_n f(x) \oplus (1 - a_n)T(PT)^{n-1}x, a_n f(y) \oplus (1 - a_n)T(PT)^{n-1}y) \\
&\leq a_n d(f(x), f(y)) + (1 - a_n)d(T(PT)^{n-1}x, T(PT)^{n-1}y) \\
&\leq a_n \alpha d(x, y) + (1 - a_n)k_n d(x, y) \\
&= ((1 - a_n)k_n + \alpha a_n)d(x, y).
\end{aligned}$$

From the condition (i), we have $(1 - a_n)k_n + \alpha a_n < 1$. So F_n is a contraction mapping. Thus there exists a unique $z_n \in C$ such that

$$z_n = F_n(z_n),$$

that is

$$z_n = a_n f(z_n) \oplus (1 - a_n)T(PT)^{n-1}z_n. \tag{9}$$

II. Next, we show that $\{z_n\}$ is bounded. From Lemma 5, there exists $v \in C$ which is a fixed point of T with

$$v = Tv = (TP)v.$$

Taking TP mapping on the both sides, we have

$$v = (TP)v = (TP)(TP)v = T(PT)Pv = T(PT)v.$$

Continuing this process, we obtain

$$v = T(PT)^{n-1}v, \ \ n \in \mathbb{N}. \tag{10}$$

For any $v \in \mathcal{F}(T)$, we have

$$\begin{aligned}
d(z_n, v) &= d(a_n f(z_n) \oplus (1 - a_n)T(PT)^{n-1}z_n, v) \\
&\leq a_n[d(f(z_n), f(v)) + d(f(v), v)] + (1 - a_n)d(T(PT)^{n-1}z_n, v) \\
&\leq \alpha a_n d(z_n, v) + a_n d(f(v), v) + (1 - a_n)k_n d(z_n, v) \\
&= (k_n - (k_n - \alpha)a_n)d(z_n, v) + a_n d(f(v), v).
\end{aligned}$$

Then

$$\begin{aligned}
d(z_n, v) &\leq \frac{a_n}{(k_n - \alpha)a_n - (k_n - 1)} d(f(v), v) = \frac{1}{k_n - \alpha - \frac{k_n - 1}{a_n}} d(f(v), v) \\
&\leq \frac{1}{k_n - \alpha - (k_n - 1)} d(f(v), v) = \frac{1}{1 - \alpha} d(f(v), v).
\end{aligned}$$

Hence $\{z_n\}$ is bounded. So $\{Tz_n\}$ and $\{f(z_n)\}$ are bounded. For $v \in \mathcal{F}(T)$ and Equation (10),

$$\begin{aligned}d(T(PT)^{n-1}z_n, v) &= d(T(PT)^{n-1}z_n, T(PT)^{n-1}v)\\ &\leq k_n d(z_n, v)\\ &\leq L \cdot d(z_n, v),\end{aligned}$$

where $L = \sup_n k_n$. It follows that the sequence $\{T(PT)^{n-1}z_n\}$ is bounded.
III. We shall claim that

$$\lim_{n\to\infty} d(z_n, Tz_n) = 0.$$

III-1. From Equation (9) and Lemma 2-(i), we get

$$\begin{aligned}d(z_n, T(PT)^{n-1}z_n) &= d(a_n f(z_n) \oplus (1-a_n)T(PT)^{n-1}z_n, T(PT)^{n-1}z_n)\\ &\leq a_n d(f(z_n), T(PT)^{n-1}z_n).\end{aligned} \tag{11}$$

On the other hand, since

$$\begin{aligned}&d(f(z_n), T(PT)^{n-1}z_n)\\ &\leq d(f(z_n), z_n) + d(z_n, T(PT)^{n-1}z_n)\\ &= d(f(z_n), a_n f(z_n) \oplus (1-a_n)T(PT)^{n-1}z_n) + d(z_n, T(PT)^{n-1}z_n)\\ &\leq (1-a_n)d(f(z_n), T(PT)^{n-1}z_n) + d(z_n, T(PT)^{n-1}z_n),\end{aligned}$$

we obtain

$$a_n d(f(z_n), T(PT)^{n-1}z_n) \leq d(z_n, T(PT)^{n-1}z_n). \tag{12}$$

Since $a_n \to 0$, from Equations (11) and (12), we have

$$\lim_{n\to\infty} a_n d(f(z_n), T(PT)^{n-1}z_n) = \lim_{n\to\infty} d(z_n, T(PT)^{n-1}z_n) = 0. \tag{13}$$

III-2. By condition $\lim_{n\to\infty} \frac{k_n - 1}{a_n} = 0$, for any $0 < \varepsilon < 1 - \alpha$, there exists a sufficiently large $n \geq 0$, and we have

$$k_n - 1 \leq a_n \varepsilon. \tag{14}$$

From Equation (9) and Lemma 3, we have

$$\begin{aligned}&d(z_n, z_{n-1})\\ &= d(a_n f(z_n) \oplus (1-a_n)T(PT)^{n-1}z_n, a_{n-1}f(z_{n-1}) \oplus (1-a_{n-1})T(PT)^{n-1}z_{n-1})\\ &\leq d(a_n f(z_n) \oplus (1-a_n)T(PT)^{n-1}z_n, a_n f(z_n) \oplus (1-a_n)T(PT)^{n-1}z_{n-1})\\ &\quad + d(a_n f(z_n) \oplus (1-a_n)T(PT)^{n-1}z_{n-1}, a_n f(z_{n-1}) \oplus (1-a_n)T(PT)^{n-1}z_{n-1})\\ &\quad + d(a_n f(z_{n-1}) \oplus (1-a_n)T(PT)^{n-1}z_{n-1}, a_{n-1}f(z_{n-1}) \oplus (1-a_{n-1})T(PT)^{n-1}z_{n-1})\\ &\leq (1-a_n)d(T(PT)^{n-1}z_n, T(PT)^{n-1}z_{n-1}) + a_n d(f(z_n), f(z_{n-1})\\ &\quad + |a_n - a_{n-1}|d(f(z_{n-1}), T(PT)^{n-1}z_{n-1})\\ &\leq (1-a_n)k_n d(z_n, z_{n-1}) + a_n \alpha d(z_n, z_{n-1}) + |a_n - a_{n-1}|M^*,\end{aligned}$$

where $M^* = \sup_{n\geq 1} d(f(z_{n-1}), T(PT)^{n-1}z_{n-1})$. This implies that

$$(1-(1-a_n)k_n - \alpha a_n)d(z_n, z_{n-1}) \leq |a_n - a_{n-1}|M^*.$$

From condition (i), we know

$$1-(1-a_n)k_n - \alpha a_n = a_n(k_n-\alpha) - (k_n - 1) > 0$$

and from Equation (14), we have

$$\begin{aligned} a_n(k_n-\alpha) - (k_n-1) &\geq a_n(k_n-\alpha) - a_n\varepsilon \\ &= (k_n - \alpha - \varepsilon)a_n \\ &\geq (1-\alpha-\varepsilon)a_n. \end{aligned}$$

Thus

$$\begin{aligned} d(z_n, z_{n-1}) &\leq \frac{|a_n - a_{n-1}|}{1-(1-a_n)k_n - \alpha a_n} M^* \\ &\leq \frac{1}{1-\alpha-\varepsilon} \cdot \frac{|a_n - a_{n-1}|}{a_n} M^* \\ &\to 0, \quad \text{as } \; n \to \infty. \end{aligned} \tag{15}$$

III-3. Therefore, from Equations (13) and (15), we get

$$\begin{aligned} d(z_n, Tz_n) \leq\, & d(z_n, T(PT)^{n-1}z_n) + d(T(PT)^{n-1}z_n, T(PT)^{n-1}z_{n-1}) \\ & + d(T(PT)^{n-1}z_{n-1}, Tz_n) \\ \leq\, & d(z_n, T(PT)^{n-1}z_n) + k_n d(z_n, z_{n-1}) \\ & + d(T(PT)^{1-1}(PT)^{n-1}z_{n-1}, T(PT)^{1-1}z_n) \\ \leq\, & d(z_n, T(PT)^{n-1}z_n) + k_n d(z_n, z_{n-1}) \\ & + k_1 d((PT)^{n-1}z_{n-1}, z_n) \\ \leq\, & d(z_n, T(PT)^{n-1}z_n) + k_n d(z_n, z_{n-1}) \\ & + k_1[d(T(PT)^{n-2}z_{n-1}, z_{n-1}) + d(z_{n-1}, z_n)] \\ \to\, & 0, \quad \text{as } \; n \to \infty. \end{aligned}$$

IV. Finally, we will show that $\{z_n\}$ contains a subsequence converge strongly to x^* such that

$$x^* = P_{\mathcal{F}(T)} f(x^*),$$

which is equivalent to the following variational inequality

$$\langle \overrightarrow{x^* f(x^*)}, \overrightarrow{xx^*} \rangle \geq 0, \;\; \forall\, x \in \mathcal{F}(T).$$

IV-1. Since $\{z_n\}$ is bounded, there exists a subsequence $\{z_{n_i}\}$ of $\{z_n\}$ which $\triangle$-converges to x^*. By Lemmas 4 and 6, we may assume that $\{z_{n_i}\}$ $\triangle$-converges to a point x^* and $x^* \in \mathcal{F}(T)$. It follows from Lemma 10-(i) and Equations (1) and (10) that

$$
\begin{aligned}
d^2(z_{n_i}, x^*) &= \langle \overrightarrow{z_{n_i}x^*}, \overrightarrow{z_{n_i}x^*} \rangle \\
&\leq a_{n_i} \langle \overrightarrow{f(z_{n_i})x^*}, \overrightarrow{z_{n_i}x^*} \rangle + (1 - a_{n_i}) \langle \overrightarrow{T(PT)^{n_i - 1}z_{n_i}x^*}, \overrightarrow{z_{n_i}x^*} \rangle \\
&\leq a_{n_i} \langle \overrightarrow{f(z_{n_i})x^*}, \overrightarrow{z_{n_i}x^*} \rangle + (1 - a_{n_i}) d(T(PT)^{n_i - 1}z_{n_i}, x^*) d(z_{n_i}, x^*) \\
&\leq a_{n_i} \langle \overrightarrow{f(z_{n_i})x^*}, \overrightarrow{z_{n_i}x^*} \rangle + (1 - a_{n_i}) k_{n_i} d^2(z_{n_i}, x^*).
\end{aligned} \tag{16}
$$

Since

$$
\begin{aligned}
\langle \overrightarrow{f(z_{n_i})x^*}, \overrightarrow{z_{n_i}x^*} \rangle &= \langle \overrightarrow{f(z_{n_i})f(x^*)} + \overrightarrow{f(x^*)x^*}, \overrightarrow{z_{n_i}x^*} \rangle \\
&= \langle \overrightarrow{f(z_{n_i})f(x^*)}, \overrightarrow{z_{n_i}x^*} \rangle + \langle \overrightarrow{f(x^*)x^*}, \overrightarrow{z_{n_i}x^*} \rangle \\
&\leq d(f(z_{n_i}), f(x^*)) d(z_{n_i}, x^*) + \langle \overrightarrow{f(x^*)x^*}, \overrightarrow{z_{n_i}x^*} \rangle \\
&\leq \alpha d^2(z_{n_i}, x^*) + \langle \overrightarrow{f(x^*)x^*}, \overrightarrow{z_{n_i}x^*} \rangle,
\end{aligned}
$$

combining Equation (16), it follows that

$$
d^2(z_{n_i}, x^*) \leq a_{n_i} \alpha d^2(z_{n_i}, x^*) + a_{n_i} \langle \overrightarrow{f(x^*)x^*}, \overrightarrow{z_{n_i}x^*} \rangle + (1 - a_{n_i}) k_{n_i} d^2(z_{n_i}, x^*).
$$

Hence

$$
\begin{aligned}
d^2(z_{n_i}, x^*) &\leq \frac{a_{n_i}}{a_{n_i}(k_{n_i} - \alpha) - (k_{n_i} - 1)} \langle \overrightarrow{f(x^*)x^*}, \overrightarrow{z_{n_i}x^*} \rangle \\
&\leq \frac{1}{1 - \alpha} \langle \overrightarrow{f(x^*)x^*}, \overrightarrow{z_{n_i}x^*} \rangle.
\end{aligned} \tag{17}
$$

Since $\{z_{n_i}\}$ $\triangle$-converges to x^*, by Lemma 7, we have

$$
\limsup_{i \to \infty} \langle \overrightarrow{f(x^*)x^*}, \overrightarrow{z_{n_i}x^*} \rangle \leq 0.
$$

It follows from Equation (17) that $\{z_{n_i}\}$ converges strongly to x^*.

IV-2. Next, we will show that x^* solves the variational inequality of Equation (8). Applying Lemma 2-(ii), for any $z \in \mathcal{F}(T)$,

$$
\begin{aligned}
d^2(z_{n_i}, z) &= d^2(a_{n_i} f(z_{n_i}) \oplus (1 - a_{n_i}) T(PT)^{n_i - 1} z_{n_i}, z) \\
&\leq a_{n_i} d^2(f(z_{n_i}), z) + (1 - a_{n_i}) d^2(T(PT)^{n_i - 1} z_{n_i}, z) \\
&\quad - a_{n_i}(1 - a_{n_i}) d^2(f(z_{n_i}), T(PT)^{n_i - 1} z_{n_i}) \\
&\leq a_{n_i} d^2(f(z_{n_i}), z) + (1 - a_{n_i}) k_{n_i}^2 d^2(z_{n_i}, z) \\
&\quad - a_{n_i}(1 - a_{n_i}) d^2(f(z_{n_i}), T(PT)^{n_i - 1} z_{n_i}).
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
&a_{n_i}(1 - a_{n_i}) d^2(f(z_{n_i}), T(PT)^{n_i - 1} z_{n_i}) + a_{n_i} k_{n_i}^2 d^2(z_{n_i}, z) \\
&\leq a_{n_i} d^2(f(z_{n_i}), z) + (k_{n_i}^2 - 1) d^2(z_{n_i}, z),
\end{aligned}
$$

so

$$(1-a_{n_i})d^2(f(z_{n_i}),T(PT)^{n_i-1}z_{n_i})+k_{n_i}^2 d^2(z_{n_i},z)\leq d^2(f(z_{n_i}),z)+\frac{k_{n_i}^2-1}{a_{n_i}}d^2(z_{n_i},z)$$
$$\leq d^2(f(z_{n_i}),z)+\frac{k_{n_i}-1}{a_{n_i}}\overline{M}, \tag{18}$$

where $\overline{M}=(L+1)d^2(z_{n_i},z)$, $L=\sup_{i\geq 1}k_{n_i}$. Since $z_{n_i}\xrightarrow{\triangle}x^*$ and by Equation (13), we have

$$T(PT)^{n_i-1}z_{n_i}\xrightarrow{\triangle}x^*. \tag{19}$$

From the conditions $k_n\to 1$, $a_n\to 0$, $\frac{k_n-1}{a_n}\to 0$, continuity of the metric d and Equation (19), we have Equation (18) as follows

$$d^2(f(x^*),x^*)+d^2(x^*,z)\leq d^2(f(x^*),z).$$

Therefore

$$\begin{aligned}0&\leq\frac{1}{2}(d^2(x^*,x^*)+d^2(f(x^*),z)-d^2(x^*,z)-d^2(f(x^*),x^*))\\&=\langle\overrightarrow{x^*f(x^*)},\overrightarrow{zx^*}\rangle,\quad\forall z\in\mathcal{F}(T),\end{aligned}$$

that is, x^* solves Equation (8).

IV-3. Finally, we will show the uniqueness of the solution of the variational inequality of Equation (8). Assume there exists a subsequence $\{z_{n_j}\}$ of $\{z_n\}$ which $\triangle$-converges to ω by the same argument. We know that $\omega\in\mathcal{F}(T)$ and solves the variational inequality of Equation (8), i.e.,

$$\langle\overrightarrow{x^*f(x^*)},\overrightarrow{x^*\omega}\rangle\leq 0 \tag{20}$$

and

$$\langle\overrightarrow{\omega f(\omega)},\overrightarrow{\omega x^*}\rangle\leq 0. \tag{21}$$

From Equations (20) and (21), we can obtain

$$\begin{aligned}0&\geq\langle\overrightarrow{x^*f(x^*)},\overrightarrow{x^*\omega}\rangle-\langle\overrightarrow{\omega f(\omega)},\overrightarrow{x^*\omega}\rangle\\&=\langle\overrightarrow{x^*f(\omega)},\overrightarrow{x^*\omega}\rangle+\langle\overrightarrow{f(\omega)f(x^*)},\overrightarrow{x^*\omega}\rangle-\langle\overrightarrow{\omega x^*},\overrightarrow{x^*\omega}\rangle-\langle\overrightarrow{x^*f(\omega)},\overrightarrow{x^*\omega}\rangle\\&=\langle\overrightarrow{x^*\omega},\overrightarrow{x^*\omega}\rangle-\langle\overrightarrow{f(\omega)f(x^*)},\overrightarrow{\omega x^*}\rangle\\&\geq\langle\overrightarrow{x^*\omega},\overrightarrow{x^*\omega}\rangle-d(f(\omega),f(x^*))d(\omega,x^*)\\&\geq d^2(x^*,\omega)-\alpha d^2(\omega,x^*)\\&=(1-\alpha)d^2(x^*,\omega).\end{aligned}$$

Since $0<\alpha<1$, we have

$$d(x^*,\omega)=0,$$

so

$$x^*=\omega.$$

Hence $\{z_n\}$ converges strongly to x^*, which solves the variational inequality of Equation (8). □

Now, we explain a strong convergence theorem for an asymptotically nonexpansive nonself mapping.

Theorem 3. *Let C be a nonempty closed convex subset of a complete $CAT(0)$ space X and let $T : C \to X$ be an asymptotically nonexpansive nonself mapping with a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$. Let f be a contraction mapping defined on C with coefficient $\alpha \in (0, 1)$. For the arbitrarily given initial point $x_0 \in C$, let $\{x_n\}$ be a sequence generated by*

$$x_{n+1} = a_n f(x_n) \oplus (1 - a_n)T(PT)^{n-1}x_n, \quad \forall n \geq 0$$

where $\{a_n\} \subset (0, 1)$ satisfies the following conditions:

(i) $\lim_{n\to\infty} a_n = 0$,
(ii) $\lim_{n\to\infty} \frac{k_n - 1}{a_n} = 0$.

Then the sequence $\{x_n\}$ converges strongly to x^* as $n \to \infty$ such that

$$x^* = P_{\mathcal{F}(T)} f(x^*),$$

which is equivalent to the variational inequality of Equation (8).

Proof. I. First, we show that the sequence $\{x_n\}$ is bounded. From Lemma 5, there exists $q \in C$ which is a fixed point of T with

$$q = Tq = (TP)q = T(PT)^{n-1}q.$$

Since $\lim_{n\to\infty} \frac{k_n - 1}{a_n}$, for any $0 < \varepsilon < 1 - \alpha$, there exists a sufficiently large $n \geq 0$, we have

$$k_n - 1 \leq a_n \varepsilon. \tag{22}$$

For any $q \in \mathcal{F}(T)$, from Equations (13) and (22), we get

$$\begin{aligned}
d(x_{n+1}, q) &= d(a_n f(x_n) \oplus (1 - a_n)T(PT)^{n-1}x_n, q) \\
&\leq a_n d(f(x_n), q) + (1 - a_n)d(T(PT)^{n-1}x_n, q) \\
&\leq a_n(d(f(x_n), f(q)) + d(f(q), q)) + (1 - a_n)k_n d(x_n, q) \\
&\leq a_n \alpha d(x_n, q) + a_n d(f(q), q) + (1 - a_n)k_n d(x_n, q) \\
&= (1 + (k_n - 1) - a_n(k_n - \alpha))d(x_n, q) + a_n d(f(q), q) \\
&\leq (1 - a_n(k_n - \alpha - \varepsilon))d(x_n, q) + a_n d(f(q), q) \\
&\leq (1 - a_n(1 - \alpha - \varepsilon))d(x_n, q) + a_n d(f(q), q) \\
&\leq \max\left\{d(x_n, q), \frac{1}{1 - \alpha - \varepsilon} d(f(q), q)\right\},
\end{aligned}$$

for $0 \leq a_n(1 - \alpha - \varepsilon) \leq 1$. Similarly, we can get

$$d(x_n, q) \leq \max\left\{d(x_{n-1}, q), \frac{1}{1 - \alpha - \varepsilon} d(f(q), q)\right\}.$$

Continuing this process, we obtain that

$$d(x_{n+1}, q) \leq \max\left\{d(x_0, q), \frac{1}{1 - \alpha - \varepsilon} d(f(q), q)\right\}, \quad \forall n \geq 0.$$

Thus, the sequence $\{x_n\}$ is bounded. So $\{f(x_n)\}$ and $\{Tx_n\}$ are also bounded. From the fact that $\{x_n\}$ is bounded and from Lemmas 4 and 6, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ which $\triangle$-converges to $q \in \mathcal{F}(T)$.

II. Next, we prove that $x_n \to q$ as $n \to \infty$. For any $n \in \mathbb{N}$, we set

$$y_n = a_n q + (1-a_n)T(PT)^{n-1}x_n.$$

It follows from Lemmas 9 and 10 that

$$\begin{aligned}
d^2(x_{n+1},q) &\le d^2(y_n,q) + 2\langle \overrightarrow{x_{n+1}y_n}, \overrightarrow{x_{n+1}q}\rangle \\
&\le (a_n d(q,q) + (1-a_n)d(T(PT)^{n-1}x_n,q))^2 \\
&\quad + 2\Big[a_n\langle \overrightarrow{f(x_n)y_n}, \overrightarrow{x_{n+1}q}\rangle + (1-a_n)\langle \overrightarrow{T(PT)^{n-1}x_n y_n}, \overrightarrow{x_{n+1}q}\rangle\Big] \\
&\le (1-a_n)^2 k_n^2 d^2(x_n,q) \\
&\quad + 2\Big[a_n^2\langle \overrightarrow{f(x_n)q}, \overrightarrow{x_{n+1}q}\rangle + a_n(1-a_n)\langle \overrightarrow{f(x_n)T(PT)^{n-1}x_n}, \overrightarrow{x_{n+1}q}\rangle \\
&\quad + (1-a_n)a_n\langle \overrightarrow{T(PT)^{n-1}x_n q}, \overrightarrow{x_{n+1}q}\rangle \\
&\quad + (1-a_n)^2\langle \overrightarrow{T(PT)^{n-1}x_n T(PT)^{n-1}x_n}, \overrightarrow{x_{n+1}q}\rangle\Big] \\
&= (1-a_n)^2 k_n^2 d^2(x_n,q) \\
&\quad + 2\Big[a_n^2\langle \overrightarrow{f(x_n)q}, \overrightarrow{x_{n+1}q}\rangle + a_n(1-a_n)\langle \overrightarrow{f(x_n)T(PT)^{n-1}x_n}, \overrightarrow{x_{n+1}q}\rangle \\
&\quad + (1-a_n)a_n\langle \overrightarrow{T(PT)^{n-1}x_n q}, \overrightarrow{x_{n+1}q}\rangle\Big] \\
&= (1-a_n)^2 k_n^2 d^2(x_n,q) + 2\Big[a_n^2\langle \overrightarrow{f(x_n)q}, \overrightarrow{x_{n+1}q}\rangle + a_n(1-a_n)\langle \overrightarrow{f(x_n)q}, \overrightarrow{x_{n+1}q}\rangle\Big] \\
&= (1-a_n)^2 k_n^2 d^2(x_n,q) + 2a_n\langle \overrightarrow{f(x_n)q}, \overrightarrow{x_{n+1}q}\rangle \\
&= (1-a_n)^2 k_n^2 d^2(x_n,q) + 2a_n\langle \overrightarrow{f(x_n)f(q)}, \overrightarrow{x_{n+1}q}\rangle + 2a_n\langle \overrightarrow{f(q)q}, \overrightarrow{x_{n+1}q}\rangle \\
&\le (1-a_n)^2 k_n^2 d^2(x_n,q) + 2a_n\alpha d(x_n,q)d(x_{n+1},q) + 2a_n d(f(q),q)d(x_{n+1},q) \\
&\le (1-a_n)^2 k_n^2 d^2(x_n,q) + a_n\alpha(d^2(x_n,q) + d^2(x_{n+1},q)) + 2a_n d(f(q),q)d(x_{n+1},q)
\end{aligned}$$

which implies

$$(1-\alpha a_n)d^2(x_{n+1},q) \le ((1-a_n)^2 k_n^2 + \alpha a_n)d^2(x_n,q) + 2a_n d(f(q),q)d(x_{n+1},q),$$

$$\begin{aligned}
d^2(x_{n+1},q) &\le \frac{(1-a_n)^2 k_n^2 + \alpha a_n}{1-\alpha a_n} d^2(x_n,q) + \frac{2a_n}{1-\alpha a_n} d(f(q),q)d(x_{n+1},q) \\
&\le \frac{(1-a_n)k_n^2 + \alpha}{1-\alpha a_n} d^2(x_n,q) + \frac{2a_n}{1-\alpha a_n} d(f(q),q)d(x_{n+1},q) \\
&= \left(1 - \frac{a_n(k_n^2-\alpha) + (1-k_n^2-\alpha)}{1-\alpha a_n}\right) d^2(x_n,q) \\
&\quad + \frac{2a_n}{1-\alpha a_n} d(f(q),q)d(x_{n+1},q).
\end{aligned}$$

Now, taking

$$\lambda_n = \frac{a_n(k_n^2 - \alpha) + (1 - k_n^2 - \alpha)}{1 - \alpha a_n},$$
$$b_n = \frac{2a_n}{a_n(k_n^2 - \alpha) + (1 - k_n^2 - \alpha)} d(f(q), q) d(x_{n+1}, q),$$

by Lemma 8, we can conclude that

$$\lim_{n\to\infty} x_n = q.$$

III. Finally, from the proof of IV-2 and IV-3 in Theorem 2, we can easily show that $q \in \mathcal{F}(T)$ is the unique solution satisfying the variational inequality of Equation (8). This completes the proof of Theorem 3. □

If a mapping $T : C \to C$ is a self mapping, then P becomes the identity mapping. Thus we have the following corollaries (cf. [13,26]).

Corollary 1. *Let C be a nonempty closed convex subset of a complete $CAT(0)$ space X and let $T : C \to C$ be an asymptotically nonexpansive mapping with a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$. Let f be a contraction defined on C with coefficient $0 < \alpha < 1$. Let $\{a_n\}$ be a sequence of real numbers with $0 < a_n < 1$. If it satisfies the following conditions*

(i) $\frac{k_n - 1}{a_n} < 1 - \alpha < a_n(k_n - \alpha),\ \forall n \in \mathbb{N}$,
(ii) $a_n \to 0$, $\frac{k_n-1}{a_n} \to 0$ *and* $\frac{|a_n - a_{n-1}|}{a_n} \to 0$ *as* $n \to \infty$,

then the following statements hold.

(1) *There exists x_n such that*

$$x_n = a_n f(x_n) \oplus (1 - a_n) T^n x_n, \quad \forall n \in \mathbb{N}.$$

(2) *The sequence $\{x_n\}$ converges strongly to x^* as $n \to \infty$ such that*

$$x^* = P_{\mathcal{F}(T)} f(x^*),$$

which is equivalent to the following variational inequality:

$$\langle \overrightarrow{x^* f(x^*)}, \overrightarrow{x x^*} \rangle \geq 0, \quad \forall x \in \mathcal{F}(T).$$

Corollary 2. *Let C be a nonempty closed convex subset of a complete $CAT(0)$ space X and let $T : C \to C$ be an asymptotically nonexpansive mapping with a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$. Let f be a contraction defined on C with coefficient $0 < \alpha < 1$. For the arbitrary initial point $x_0 \in C$, let $\{x_n\}$ be generated by*

$$x_{n+1} = a_n f(x_n) \oplus (1 - a_n) T^n x_n, \quad \forall n \geq 0$$

where $\{a_n\} \subset (0, 1)$ satisfies the following conditions:

(i) $\lim_{n\to\infty} a_n = 0$,
(ii) $\lim_{n\to\infty} \frac{k_n - 1}{a_n} = 0$.

Then the sequence $\{x_n\}$ converges strongly to x^* as $n \to \infty$ such that

$$x^* = P_{\mathcal{F}(T)} f(x^*),$$

which is equivalent to the variational inequality of Equation (8).

4. Conclusions

Theorems 2 and 3 generalize and improve the results which are discussed in Wangkeeree et al. [13], Shi et al. [14], Kim et al. [17], Kim [18] and others.

The strong convergence theorems of the Moudafi's viscosity approximation methods apply various classes of variational inequalities and optimization problems, its results proved in this paper continue to hold for these problems. It is expected that this class will inspire and motivate further research in this area.

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