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Odd Cycles and Hilbert Functions of Their Toric Rings

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Abstract: Studying Hilbert functions of concrete examples of normal toric rings, it is demonstrated that for each $1 \leq s \leq 5$, an O-sequence $(h_0, h_1, \ldots, h_{2s-1}) \in \mathbb{Z}^{2s}_{\geq 0}$ satisfying the properties that (i) $h_0 \leq h_1 \leq \cdots \leq h_{s-1}$, (ii) $h_{2s-1} = h_0$, $h_{2s-2} = h_1$ and (iii) $h_{2s-1-i} = h_i + (-1)^i$, $2 \leq i \leq s-1$, can be the h-vector of a Cohen-Macaulay standard G-domain.

Keywords: O-sequence; h-vector; flawless; toric ring; stable set polytope

MSC: 13A02; 13H10

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1. Background

In the paper [1] published in 1989, several conjectures on Hilbert functions of Cohen-Macaulay integral domains are studied.

Let $A = \bigoplus_{n=0}^{\infty} A_n$ be a standard G-algebra [2]. Thus A is a Noetherian commutative graded ring for which (i) $A_0 = K$ a field, (ii) $A = K[A_1]$ and (iii) $\dim_K A_1 < \infty$. The Hilbert function of A is defined by

$$H(A, n) = \dim_K A_n, \quad n = 0, 1, 2, \ldots$$

Let $\dim A = d$ and $v = H(A, 1) = \dim_K A_1$. A classical result ([3], Chapter 5, Section 13) says that $H(A, n)$ is a polynomial for n sufficiently large and its degree is $d - 1$. It follows that the sequence $h(A) = (h_0, h_1, h_2, \ldots)$, called the h-vector of A, defined by the formula

$$(1-\lambda)^d \sum_{n=0}^{\infty} H(A, n)\lambda^n = \sum_{i=0}^{\infty} h_i \lambda^i$$

has finitely many non-zero terms with $h_0 = 1$ and $h_1 = v - d$. If $h_i = 0$ for $i > s$ and $h_s \neq 0$, then we write $h(A) = (h_0, h_1, \ldots, h_s)$.

Let $Y_1, \ldots, Y_r$ be indeterminates. A non-empty set M of monomials $Y_1^{a_1} \cdots Y_r^{a_r}$ in the variables $Y_1, \ldots, Y_r$ is said to be an order ideal of monomials if, whenever $m \in M$ and m' divides m, then $m' \in M$. Equivalently, if $Y_1^{a_1} \cdots Y_r^{a_r} \in M$ and $0 \leq b_i \leq a_i$, then $Y_1^{b_1} \cdots Y_r^{b_r} \in M$. In particular, since M is non-empty, $1 \in M$. A finite sequence $(h_0, h_1, \ldots, h_s)$ of non-negative integers is said to be an O-sequence if there exists an order ideal M of monomials in $Y, \ldots, Y_r$ with each $\deg Y_i = 1$ such that $h_j = |\{m \in M | \deg m = j\}|$ for any $0 \leq j \leq s$. In particular, $h_0 = 1$. If A is Cohen-Macaulay, then $h(A) = (h_0, h_1, \ldots, h_s)$ is an O-sequence ([2], p. 60). Furthermore, a finite sequence $(h_0, h_1, \ldots, h_s)$ of integers with $h_0 = 1$ and $h_s \neq 0$ is the h-vector of a Cohen-Macaulay standard G-algebra if and only if $(h_0, h_1, \ldots, h_s)$ is an O-sequence ([2], Corollary 3.11).

An O-sequence $(h_0, h_1, \ldots, h_s)$ with $h_s \neq 0$ is called flawless ([1], p. 245) if (i) $h_i \leq h_{s-i}$ for $0 \leq i \leq [s/2]$ and (ii) $h_0 \leq h_1 \leq \cdots \leq h_{[s/2]}$. A standard G-domain is a standard G-algebra which is an integral domain. It was conjectured ([1], Conjecture 1.4) that the h-vector of a Cohen-Macaulay standard G-domain is flawless. Niesi and Robbiano ([4], Example 2.4) succeeded in constructing a Cohen-Macaulay standard G-domain with $(1,3,5,4,4,1)$ its h-vector. Thus, in general, the h-vector of a Cohen-Macaulay standard G-domain is not flawless.

In the present paper, it is shown that, for each $1 \leq s \leq 5$, an O-sequence

$$(h_0, h_1, \ldots, h_{s-1}, h_s, \ldots, h_{2s-2}, h_{2s-1}) \in \mathbb{Z}_{\geq 0}^{2s}$$

satisfying the properties that

(i) $h_0 \leq h_1 \leq \cdots \leq h_{s-1}$,
(ii) $h_{2s-1} = h_0, \ \ h_{2s-2} = h_1$,
(iii) $h_{2s-1-i} = h_i + (-1)^i, 2 \leq i \leq s-1$

can be the h-vector of a normal toric ring arising from a cycle of odd length. In particular, the above O-sequence, which is non-flawless for each of $s = 4$ and $s = 5$, can be the h-vector of a Cohen-Macaulay standard G-domain.

2. Toric Rings Arising from Odd Cycles

Let C_{2s+1} denote a cycle of length $2s+1$, where $s \geq 1$, on $[2s+1] = \{1, 2, \ldots, 2s+1\}$ with the edges

$$\{1,2\}, \{2,3\}, \ldots, \{2s-1, 2s\}, \{2s, 2s+1\}, \{2s+1, 1\}. \qquad (1)$$

A finite set $W \subset [2s+1]$ is called *stable* in C_{2s+1} if none of the sets of (1) is a subset of W. In particular, the empty set $\emptyset$ and $\{1\}, \{2\}, \ldots, \{2s+1\}$ are stable. Let $S = K[x_1, \ldots, x_{2s+1}, y]$ denote the polynomial ring in $2s+2$ variables over K. The *toric ring* of C_{2s+1} is the subring $K[C_{2s+1}]$ of S which is generated by those squarefree monomials $(\prod_{i \in W} x_i)y$ for which $W \subset [2s+1]$ is stable in C_{2s+1}. It follows that $K[C_{2s+1}]$ can be a standard G-algebra with each $\deg(\prod_{i \in W} x_i)y = 1$. It is shown ([5], Theorem 8.1) that $K[C_{2s+1}]$ is normal. In particular, $K[C_{2s+1}]$ is a Cohen-Macaulay standard G-domain. Now, we discuss when $K[C_{2s+1}]$ is Gorenstein. Here a Cohen-Macaulay ring is called Gorenstein if it has finite injective dimension.

Theorem 1. *The toric ring $K[C_{2s+1}]$ is Gorenstein if and only if either $s = 1$ or $s = 2$.*

Proof. Since the h-vector of $K[C_3]$ is $(1,1)$ and since the h-vector of $K[C_5]$ is $(1,6,6,1)$, it follows from ([2], Theorem 4.4) that each of $K[C_3]$ and $K[C_5]$ is Gorenstein.

Now, we show that $K[C_{2s+1}]$ is not Gorenstein if $s \geq 3$. Let $s \geq 3$. Write $\mathcal{Q}_{C_{2s+1}} \subset \mathbb{R}^{2s+1}$ for the stable set polytope of C_{2s+1}. Thus $\mathcal{Q}_{C_{2s+1}}$ is the convex hull of the finite set

$$\left\{ \sum_{i \in W} \mathbf{e}_i : W \text{ is a stable set of } G \right\} \subset \mathbb{R}^{2s+1},$$

where $\mathbf{e}_1, \ldots, \mathbf{e}_{2s+1} \in \mathbb{R}^{2s+1}$ are the canonical unit coordinate vectors of $\mathbb{R}^{2s+1}$ and where $\sum_{i \in \emptyset} \mathbf{e}_i = (0, \ldots, 0) \in \mathbb{R}^{2s+1}$. One has $\dim \mathcal{Q}_{2s+1} = 2s+1$. Then ([6], Theorem 4) says that $\mathcal{Q}_{C_{2s+1}}$ is defined by the following inequalities:

- $0 \leq x_i \leq 1$ for all $1 \leq i \leq 2s+1$;
- $x_i + x_{i+1} \leq 1$ for all $1 \leq i \leq 2s$;
- $x_1 + x_{2s+1} \leq 1$;
- $x_1 + \cdots + x_{2s+1} \leq s$.

It then follows that each of $\mathcal{Q}_{C_{2s+1}}$ and $2\mathcal{Q}_{C_{2s+1}}$ has no interior lattice points and that $(1, \ldots, 1)$ is an interior lattice point of $3\mathcal{Q}_{C_{2s+1}}$. Furthermore, (Ref. [7], Theorem 4.2) guarantees that the inequality

$$x_1 + \cdots + x_{2s+1} \leq s$$

defines a facet of $\mathcal{Q}_{C_{2s+1}}$. Let $\mathcal{P}_s = 3\mathcal{Q}_{C_{2s+1}} - (1, \ldots, 1)$. Thus the origin of $\mathbb{R}^{2s+1}$ is an interior lattice point of $\mathcal{P}_s$ and the inequality

$$x_1 + \cdots + x_{2s+1} \leq s - 1$$

defines a facet of $\mathcal{P}_s$. This fact together with [8] implies that $\mathcal{P}_s$ is not reflexive. In other words, the dual polytope $\mathcal{P}_s^\vee$ of $\mathcal{P}_s$ defined by

$$\mathcal{P}_s^\vee = \{\mathbf{y} \in \mathbb{R}^{2s+1} : \langle \mathbf{x}, \mathbf{y} \rangle \leq 1 \text{ for all } \mathbf{x} \in \mathcal{P}_s\}$$

is not a lattice polytope, where $\langle \mathbf{x}, \mathbf{y} \rangle$ is the usual inner product of $\mathbb{R}^{2s+1}$. It then follows from ([9], Theorem (1.1)) (and also from ([5], Theorem 8.1)) that $K[C_{2s+1}]$ is not Gorenstein, as desired. □

It is known ([2], Theorem 4.4) that a Cohen-Macaulay standard G-domain A is Gorenstein if and only if the h-vector $h(A) = (h_0, \ldots, h_s)$ is symmetric, i.e., $h_i = h_{s-i}$ for $0 \leq i \leq [s/2]$. Hence the h-vector of the toric ring $K[C_{2s+1}]$ is not symmetric when $s \geq 3$.

Example 1. *By using Normaliz [10], the h-vector of the toric ring* $K[C_7]$ *is* $(1, 21, 84, 85, 21, 1)$.

3. Non-Flawless *O*-Sequences of Normal Toric Rings

We now come to concrete examples of non-flawless O-sequences which can be the h-vectors of normal toric rings.

Example 2. *The h-vector of the toric ring* $K[C_9]$ *is*

$$(1, 66, 744, 2305, 2304, 745, 66, 1).$$

Furthermore,

$$(1, 187, 5049, 37247, 96448, 96449, 37246, 5050, 187, 1)$$

is the h-vector of the toric ring $K[C_{11}]$.

We conclude the present paper with the following

Conjecture 1. *The h-vector of the toric ring* $K[C_{2s+1}]$ *of* C_{2s+1} *is of the form*

$$(1, h_1, h_2, h_3, \ldots, h_i, \ldots, h_{s-1}, h_{s-1} + (-1)^{s-1}, \ldots, h_i + (-1)^i, \ldots, h_3 - 1, h_2 + 1, h_1, 1).$$

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