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The Regularity of Edge Rings and Matching Numbers

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Abstract: Let $K[G]$ denote the edge ring of a finite connected simple graph G on $[d]$ and $\mathrm{mat}(G)$ the matching number of G. It is shown that $\mathrm{reg}(K[G]) \leq \mathrm{mat}(G)$ if G is non-bipartite and $K[G]$ is normal, and that $\mathrm{reg}(K[G]) \leq \mathrm{mat}(G) - 1$ if G is bipartite.

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Let G be a finite connected simple graph on the vertex set $[d] = \{1, \ldots, d\}$ and let $E(G)$ be its edge set. Let $S = K[x_1, \ldots, x_d]$ denote the polynomial ring in d variables over a field K. The *edge ring* of G is the toric ring $K[G] \subset S$ which is generated by those monomials $x_i x_j$ with $\{i, j\} \in E(G)$. The systematic study of edge rings originated in [1]. It has been shown that $K[G]$ is normal if and only if G satisfies the odd cycle condition ([2], p. 131). Thus, particularly if G is bipartite, $K[G]$ is normal.

Let $\mathbf{e}_1, \ldots, \mathbf{e}_d$ denote the canonical unit coordinate vectors of $\mathbb{R}^d$. The *edge polytope* is the lattice polytope $\mathcal{P}_G \subset \mathbb{R}^d$ which is the convex hull of the finite set $\{\, \mathbf{e}_i + \mathbf{e}_j \;:\; \{i, j\} \in E(G) \,\}$. One has $\dim \mathcal{P}_G = d - 1$ if G is non-bipartite and $\dim \mathcal{P}_G = d - 2$ if G is bipartite. We refer the reader to ([2], Chapter 5) for the fundamental materials on edge rings and edge polytopes.

A *matching* of G is a subset $M \subset E(G)$ for which $e \cap e' = \emptyset$ for $e \neq e'$ belonging to M. The *matching number* is the maximal cardinality of matchings of G. Let $\mathrm{mat}(G)$ denote the matching number of G.

When $K[G]$ is normal, the upper bound of regularity of $K[G]$ can be explicitly described in terms of $\mathrm{mat}(G)$. Our main result in the present paper is as follows:

Theorem 1. *Let G be a finite connected simple graph. Then*

(a) *If G is non-bipartite and $K[G]$ is normal, then* $\mathrm{reg}\, K[G] \leq \mathrm{mat}(G)$;
(b) *If G is bipartite, then* $\mathrm{reg}\, K[G] \leq \mathrm{mat}(G) - 1$.

Lemma 1 stated below, which provides information on lattice points belonging to the interiors of dilations of edge polytopes, is indispensable for the proof of Theorem 1.

Lemma 1. *Suppose that $(a_1, \ldots, a_d) \in \mathbb{Z}^d$ belongs to the interior $q(\mathcal{P}_G \setminus \partial\mathcal{P}_G)$ of the dilation $q\mathcal{P}_G = \{q\alpha : \alpha \in \mathcal{P}_G\}$, where $q \geq 1$, of $\mathcal{P}_G$. Then $a_i \geq 1$ for each $1 \leq i \leq d$.*

Proof. The facets of $\mathcal{P}_G$ are described in ([1], Theorem 1.7). When $W \subset [d]$, we write G_W for the induced subgraph of G on W. Since $K[G]$ is normal, it follows that $\mathcal{P}_G$ possesses the integer decomposition property ([2], p. 91). In other words, each $\mathbf{a} \in q\mathcal{P}_G \cap \mathbb{Z}^d$ is of the form

$$\mathbf{a} = (\mathbf{e}_{i_1} + \mathbf{e}_{j_1}) + \cdots + (\mathbf{e}_{i_q} + \mathbf{e}_{j_q}),$$

where $\{i_1, j_1\}, \ldots, \{i_q, j_q\}$ are edges of G.

(First Step) Let G be non-bipartite. Let $i \in [d]$. Let $H_1, \ldots, H_s$ and $H'_1, \ldots, H'_{s'}$ denote the connected components of $G_{[d]\setminus\{i\}}$, where each H_j is bipartite and where each $H'_{j'}$ is non-bipartite. If $s = 0$, then $i \in [d]$ is regular ([1], p. 414) and the hyperplane of $\mathbb{R}^d$ defined by the equation $x_i = 0$ is a facet of $q\mathcal{P}_G$. Hence $a_i > 0$.

Let $s \geq 1$ and $s' \geq 0$. For each $1 \leq j \leq s$, we write $W_j \cup U_j$ for the vertex set of the bipartite graph H_j for which there is $a \in W_j$ with $\{a, i\} \in E(G)$, where $U_j = \emptyset$ if H_j is a graph consisting of a single vertex. Then $T = W_1 \cup \cdots \cup W_s$ is independent ([1], p. 414). In other words, no edge $e \in E(G)$ satisfies $e \subset T$. Let G' denote the bipartite graph induced by T. Thus the edges of G' are $\{b, c\} \in E(G)$ with $b \in T$ and $c \in T' = U_1 \cup \cdots \cup U_s \cup \{i\}$. Since each induced subgraph $G_{W_j \cup U_j \cup \{i\}}$ is connected, it follows that G' is connected with $V(G') = T \cup T'$ as its vertex set. Since the connected components of $G_{[d]\setminus V(G')}$ are $H'_1, \ldots, H'_{s'}$, it follows that T is fundamental ([1], p. 415) and the hyperplane of $\mathbb{R}^d$ defined by $\sum_{\xi \in T} x_\xi = \sum_{\xi' \in T'} x_{\xi'}$ is a facet of $q\mathcal{P}_G$. Now, suppose that $a_i = 0$. Since $\mathcal{P}_G$ possesses the integer decomposition property, one has $\sum_{\xi \in T} a_\xi = \sum_{\xi' \in T'} a_{\xi'}$. Hence $(a_1, \ldots, a_d) \in \mathbb{Z}^d$ cannot belong to $q(\mathcal{P}_G \setminus \partial\mathcal{P}_G)$. Thus $a_i > 0$, as desired.

(Second Step) Let G be bipartite. If G is a star graph with, say, $E(G) = \{\{1,2\}, \{1,3\}, \ldots, \{1,d\}\}$, then $\mathcal{P}_G$ can be regarded to be the $(d-2)$ simplex of $\mathbb{R}^{d-1}$ with the vertices $(1,0,\ldots,0), (0,1,0,\ldots,0), \ldots, (0,\ldots,0,1)$. Thus, since each $(a_1, \ldots, a_d) \in q\mathcal{P}_G \cap \mathbb{Z}^d$ satisfies $a_1 = q$, the assertion follows immediately. In the argument below, one will assume that G is not a star graph.

Let $i \in [d]$ and $H_1, \ldots, H_s$ be the connected components of $G_{[d]\setminus\{i\}}$. If $s = 1$, then $i \in [d]$ is ordinary ([1], p. 414) and the hyperplane of $\mathbb{R}^d$ defined by the equation $x_i = 0$ is a facet of $q\mathcal{P}_G$. Hence $a_i > 0$.

Let $s \geq 2$. Let $W_j \cup U_j$ denote the vertex set of H_j for which there is $a \in W_j$ with $\{a, i\} \in E(G)$. Since G is not a star graph, one can assume that $U_1 \neq \emptyset$. Then $T = W_2 \cup \cdots \cup W_s$ is independent and the bipartite graph induced by T is $G_{[d]\setminus(W_1 \cup U_1)}$. Hence T is acceptable ([1], p. 415) and the hyperplane of $\mathbb{R}^d$ defined by $\sum_{\xi \in W_1} x_\xi = \sum_{\xi' \in U_1} x_{\xi'}$ is a facet of $q\mathcal{P}_G$. Now, suppose that $a_i = 0$. Since $\mathcal{P}_G$ possesses the integer decomposition property, one has $\sum_{\xi \in W_1} a_\xi = \sum_{\xi' \in U_1} a_{\xi'}$. Hence $(a_1, \ldots, a_d) \in \mathbb{Z}^d$ cannot belong to $q(\mathcal{P}_G \setminus \partial\mathcal{P}_G)$. Thus $a_i > 0$, as required. □

We say that a finite subset $L \subset E(G)$ is an *edge cover* of G if $\cup_{e \in L} e = [d]$. Let $\mu(G)$ denote the minimal cardinality of edge covers of G.

Corollary 1. *When $K[G]$ is normal, one has $q \geq \mu(G)$ if $q(\mathcal{P}_G \setminus \partial\mathcal{P}_G) \cap \mathbb{Z}^d \neq \emptyset$.*

Proof. Since $\mathcal{P}_G$ possesses the integer decomposition property, Lemma 1 guarantees that, if $\mathbf{a} \in q(\mathcal{P}_G \setminus \partial\mathcal{P}_G) \cap \mathbb{Z}^d$, one has $q \geq \mu(G)$. □

Once Corollary 1 is established, to complete the proof of Theorem 1 is a routine job on computing the regularity of normal toric rings.

Proof of Theorem 1. In each of the cases (a) and (b), since the edge ring $K[G]$ is normal, it follows that the Hilbert function of $K[G]$ coincides the Ehrhart function ([2], p. 100) of the edge polytope $\mathcal{P}_G$, which says that the Hilbert series of $K[G]$ is of the form

$$(h_0 + h_1\lambda + \cdots + h_s\lambda^s)/(1-\lambda)^{(\dim \mathcal{P}_G)+1}$$

with each $h_i \in \mathbb{Z}$ and $h_s \neq 0$. One has

$$s = (\dim \mathcal{P}_G + 1) - \min\{\, q \geq 1 \,:\, q(\mathcal{P}_G \setminus \partial\mathcal{P}_G) \cap \mathbb{Z}^d \neq \emptyset \,\}.$$

Now, Corollary 1 guarantees that

$$s \leq (\dim \mathcal{P}_G + 1) - \mu(G).$$

Finally, since $\mu(G) = d - \mathrm{mat}(G)$ ([3], Lemma 2.1), one has

$$\mathrm{reg}\, K[G] = s \leq \dim \mathcal{P}_G - (d-1) + \mathrm{mat}(G),$$

as required. □

Rafael H. Villarreal informed us that part (b) of Theorem 1 can also be deduced from ([4], Theorem 14.4.19).

When $K[G]$ is non-normal, the behavior of regularity is curious.

Proposition 1. *For given integers $0 \leq r \leq m$, there exists a finite connected simple graph G such that $\mathrm{reg}\, K[G] = r$, and*

$$\mathrm{mat}(G) = \begin{cases} m, \text{if } G \text{ is non-bipartite}, \\ m+1, \text{if } G \text{ is bipartite}. \end{cases}$$

Proof. In the non-bipartite case, let H be the complete graph with $2r$ vertices. Its matching number is r. We know from ([5], Corollary 2.12) that $\mathrm{reg}\, K[H] = r$. At one vertex of H we attach a path graph of length $2(m-r)$ and call this new graph G. Then $\mathrm{mat}(G) = m$ and $\mathrm{reg}\, K[G] = \mathrm{reg}\, K[H] = r$, as $K[G]$ is just a polynomial extension of $K[H]$.

In the bipartite case, let H be the bipartite graph of type $(r+1, r+1)$. The matching number is $r+1$. Indeed, $K[H]$ may be viewed as a Hibi ring whose regularity is well-known, see for example ([6], Theorem 1.1). At one vertex of H we attach a path graph of length $2(m-r)$ and call this new graph G. Then $\mathrm{mat}(G) = m+1$ and $\mathrm{reg}\, K[G] = \mathrm{reg}\, K[H] = r$, for the same reason as before. □

These bounds for the regularity of $K[G]$ are generally only valid if $K[G]$ is normal. Consider, for example, the graph G which consists of two disjoint triangles combined as a path of length ℓ. Then the defining ideal of $K[G]$ is generated by a binomial of degree $\ell + 3$, and hence $\mathrm{reg}\, K[G] = \ell + 2$, while the matching number of G is $2 + \lceil \ell/2 \rceil$.

Question 1. Let m be a positive integer, and consider the set $\mathcal{S}_m$ of finite connected simple graphs with matching number m.

- Is there a bound for $\mathrm{reg}\, K[G]$ with $G \in \mathcal{S}_m$?
- If such a bound exists, is it a linear function of m?

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References

1. Ohsugi, H.; Hibi, T. Normal polytopes arising from finite graphs. *J. Algebra* **1998**, *207*, 409–426. [CrossRef]
2. Herzog, J.; Hibi, T.; Ohsugi, H. *Binomial Ideals*; GTM 279; Springer: New York, NY, USA, 2018.
3. Herzog, J.; Hibi, T. Matching numbers and the regularity of the Rees algebra of an edge ideal. *arXiv* **2019**, arXiv:1905.02141.

4. Villarreal, R.H. *Monomial Algebras*, 2nd ed.; Monographs and Research Notes in Mathematics, Taylor & Francis Group: Abingdon, UK, 2015.
5. Bruns, W.; Vasoncelos, W.V.; Villarreal, R.H. Degree bounds in monomial subrings. *Ill. J. Math.* **1997**, *41*, 341–353. [CrossRef]
6. Ene, V.; Herzog, J.; Madani, S.S. A note on the regularity of Hibi rings. *Manuscripta Math.* **2015**, *148*, 501–506. [CrossRef]