mathematics

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Fixed Points of Kannan Maps in the Variable Exponent Sequence Spaces $\ell_{p(\cdot)}$

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Received: 5 December 2019; Accepted: 26 December 2019; Published: 3 January 2020

Abstract: Kannan maps have inspired a branch of metric fixed point theory devoted to the extension of the classical Banach contraction principle. The study of these maps in modular vector spaces was attempted timidly and was not successful. In this work, we look at this problem in the variable exponent sequence spaces $\ell_{p(\cdot)}$. We prove the modular version of most of the known facts about these maps in metric and Banach spaces. In particular, our results for Kannan nonexpansive maps in the modular sense were never attempted before.

Keywords: electrorheological fluids; fixed point; Kannan contraction mapping; Kannan nonexpansive mapping; modular vector spaces; Nakano

MSC: primary 47H09; 47H10

1. Introduction

The variable exponent sequence spaces find their roots in the celebrated work by Orlicz [1] where he introduced the vector space

$$\ell_{p(\cdot)} = \left\{ \{x_n\} \subset \mathbb{R}^{\mathbb{N}};\ \sum_{n=0}^{\infty} |\lambda\, x_n|^{p(n)} < \infty \ \text{ for some } \lambda > 0 \right\},$$

where $\{p(n)\} \subset [1, \infty)$. These spaces were extensively studied [2–6]. They inspired the formal definition of a modular introduced by Nakano [7]. These vector spaces are a special case of the variable exponent spaces $L^{p(\cdot)}$. Toward the second half of the twentieth century, it was realized that these variable exponent spaces constituted the right framework for the mathematical formulation of a number of problems for which the classical Lebesgue spaces were inadequate. The applicability of these spaces and their properties made them a popular and efficient tool in the treatment of a variety of situations; today the area of $L^{p(\cdot)}(\Omega)$ spaces is a prolific field of research with ramifications reaching into very diverse mathematical specialties [8]. The study of the variable exponent Lebesgue spaces $L^{p(\cdot)}$ received further impetus from the mathematical description of the hydrodynamics of non-Newtonian fluids [9,10]. Applications of non-Newtonian fluids, also called electrorheological, range from their use in military science to civil engineering and orthopedics.

This work is devoted to the investigation of the fixed point problem for Kannan maps defined within the spaces $\ell_{p(\cdot)}$. Some new results are presented. Metric fixed point theory constitutes an essential part of our work. Due to the vastness of the topic it would be impossible to include the background of metric fixed point theory here—the reader is referred to Reference [11] and to Reference [12] for the necessary elements on this topic.

2. Basic Notation and Terminology

We open the discussion by presenting some definitions and basic facts about the space $\ell_{p(\cdot)}$.

Definition 1 ([1]). *Define the sequence vector space*

$$\ell_{p(\cdot)} = \Big\{\{x_n\} \subset \mathbb{R}^{\mathbb{N}};\ \sum_{n=0}^{\infty} \frac{1}{p(n)}\ |\lambda\ x_n|^{p(n)} < \infty \ \textit{for some}\ \lambda > 0\Big\},$$

where $p : \mathbb{N} \to [1, \infty)$.

Orlicz [1] did not use the terminology variable exponent sequence spaces for $\ell_{p(\cdot)}$. It was at a later stage that these spaces played a major role in the new theory of the more general concept of variable exponent spaces. Inspired by the structure of these spaces, Nakano [4,7] introduced the concept of modular vector spaces.

Proposition 1 ([3,5,7]). *Consider the vector space* $\ell_{p(\cdot)}$. *Define the function* $v : \ell_{p(\cdot)} \to [0, \infty]$ *by*

$$v(x) = v((x_n)) = \sum_{n=0}^{\infty} \frac{1}{p(n)}\ |x_n|^{p(n)}.$$

Then the following properties hold:

(i) $v(x) = 0$ *if and only if* $x = 0$;

(ii) $v(\gamma x) = v(x)$, *if* $|\gamma| = 1$;

(iii) $v(s\ x + (1-s)\ y) \leq s\ v(x) + (1-s)\ v(y)$, *for any* $s \in [0, 1]$,

for any $x, y \in \ell_{p(\cdot)}$. *In this case, we say that* v *is a convex modular.*

Note that v is left-continuous, that is, $\lim\limits_{\alpha \to 1-} v(\alpha x) = v(x)$, for any $x \in \ell_{p(\cdot)}$. Next, we introduce the modular version of some metric known properties.

Definition 2 ([13]).
1. *A sequence* $\{x_n\} \subset \ell_{p(\cdot)}$ *is* v*-convergent to* $x \in \ell_{p(\cdot)}$ *if and only if* $v(x_n - x) \to 0$. *Note that the* v*-limit is unique if it exists.*
2. *A sequence* $\{x_n\} \subset \ell_{p(\cdot)}$ *is* v*-Cauchy if* $v(x_n - x_m) \to 0$ *as* $n, m \to \infty$.
3. *A set* $C \subset \ell_{p(\cdot)}$ *is* v*-closed if for any* v*-converging sequence* $\{x_n\} \subset C$ *to* x *one has* $x \in C$.
4. *A set* $C \subset \ell_{p(\cdot)}$ *is* v*-bounded if* $\delta_v(C) = \sup\{v(x-y); x, y \in C\} < \infty$.
5. *The* x*-centered* v*-ball of radius* r *is defined as*

$$B_v(x, r) = \{y \in \ell_{p(\cdot)};\ v(x-y) \leq r\},$$

for any $x \in \ell_{p(\cdot)}$ *and* $r \geq 0$.

For the sake of completeness, note that v satisfies what is known as the Fatou's property which says that for any sequence $\{y_n\} \subseteq \ell_{p(\cdot)}$ which v-converges to y and any $x \in \ell_{p(\cdot)}$, we have

$$v(x-y) \leq \liminf_{n \to \infty} v(x - y_n).$$

Observe that this property implies the v-closedness of the v-balls. The next property, called the Δ_2-condition, is central in investigating modular vector spaces.

Definition 3. v *satisfies the* Δ_2*-condition if there exists* $K \geq 0$ *such that*

$$v(2x) \leq K\ v(x),$$

for any $x \in \ell_{p(\cdot)}$.

Note that υ satisfies the Δ_2-condition if and only if $p^+ = \sup_{n\in\mathbb{N}} p(n) < \infty$ [3,5,7]. For more on this condition and its impact on the structure of modular vector spaces, the reader may consult References [12,14,15]. The Luxemburg norm on $\ell_{p(\cdot)}$ is defined as the Minkowsky's functional of the unit υ-ball, that is:

$$\|x\|_\upsilon = \inf\left\{\lambda > 0; \upsilon\left(\frac{1}{\lambda}x\right) \le 1\right\}.$$

$(\ell_{p(\cdot)}, \|.\|_\upsilon)$ is a uniformly convex Banach space if and only if $1 < p^- \le p^+ < \infty$ [5], where

$$p^- = \inf_{n\in\mathbb{N}} p(n) \quad and \quad p^+ = \sup_{n\in\mathbb{N}} p(n).$$

3. Modular Kannan Mappings in $\ell_{p(\cdot)}$

Since the publication of the Banach fixed point theorem [16], many mathematicians worked on possible extensions. It is Kannan [17] who is credited for giving an example of a class of mappings with the same fixed point behavior as contractions but that fail to be continuous. The only attempt to define Kannan maps in modular vector spaces was made in Reference [18]; however, the definition given there is not appropriate since the author assumes the mapping to be defined on the entire modular vector space. Furthermore, the authors fail to consider the case of Kannan nonexpansive mappings. Some definitions, which we now present, will be needed in the sequel:

Definition 4. *Let K be a nonempty subset of* $\ell_{p(\cdot)}$. *The map* $T : K \to K$ *is said to be Kannan* υ*-Lipschitzian if there exists* $L \ge 0$ *such that*

$$\upsilon(T(x) - T(y)) \le L\Big(\upsilon(x - T(x)) + \upsilon(y - T(y))\Big),$$

for any $x, y \in K$. *We will say that T is:*

(1) *Kannan* υ*-contraction if* $L < 1/2$;
(2) *Kannan* υ*-nonexpansive if* $L = 1/2$.

$x \in K$ *is called a fixed point of T if* $T(x) = x$.

Note that Kannan υ-Lipschitzian mappings have at most one fixed point.

Next we discuss the fixed point problem for Kannan υ-contraction mappings.

Theorem 1. *Let K be a nonempty* υ*-closed subset of* $\ell_{p(\cdot)}$. *Let* $T : K \to K$ *be a Kannan* υ*-contraction mapping. Let* $z \in K$ *be such that* $\upsilon(z - T(z)) < +\infty$. *Then,* $\{T^n(z)\}$ υ*-converges to some point* $w \in C$. *Furthermore, we have* $\upsilon(w - T(w)) = +\infty$ *or* $\upsilon(w - T(w)) = 0$ *(i.e., w is the fixed point of T).*

Proof. Let $z \in K$ such that $\upsilon(z - T(z)) < +\infty$. Let us prove that $\{T^n(z)\}$ is υ-convergent. Since $\ell_{p(\cdot)}$ is υ-complete, it suffices to prove that $\{T^n(z)\}$ is υ-Cauchy. Since T is a Kannan υ-contraction mapping, there exists $L \in [0, 1/2)$ such that

$$\upsilon(T(x) - T(y)) \le L\Big(\upsilon(x - T(x)) + \upsilon(y - T(y))\Big),$$

for any $x, y \in K$. Set $k = L/(1 - L)$. Note that $k < 1$. Moreover,

$$\upsilon(T^n(z) - T^{n+1}(z)) \le L\Big(\upsilon(T^{n-1}(z) - T^n(z)) + \upsilon(T^n(z) - T^{n+1}(z))\Big),$$

which implies

$$\begin{aligned} v(T^n(z) - T^{n+1}(z)) &\leq \frac{L}{1-L}\, v(T^{n-1}(z) - T^n(z)) \\ &= k\, v(T^{n-1}(z) - T^n(z)), \end{aligned}$$

for any $n \geq 1$. Hence,

$$v(T^n(z) - T^{n+1}(z)) \leq k^n\, v(z - T(z)),$$

for any $n \in \mathbb{N}$. Since T is a Kannan v-contraction mapping, we have

$$v(T^n(z) - T^{n+h}(z)) \leq L \left(v(T^{n-1}(z) - T^n(z)) + v(T^{n+h-1}(z) - T^{n+h}(z)) \right),$$

which implies

$$v(T^n(z) - T^{n+h}(z)) \leq L \left(k^{n-1} + k^{n+h-1} \right) v(z - T(z)), \tag{NL}$$

for $n \geq 1$ and $h \in \mathbb{N}$. Since $k < 1$ and $v(z - T(z)) < +\infty$, we conclude that $\{T^n(z)\}$ is v-Cauchy, as claimed. Let $w \in \ell_{p(\cdot)}$ be the v-limit of $\{T^n(z)\}$. Since K is v-closed, we get $w \in K$. Assume that $v(w - T(w)) < +\infty$; we will prove that $v(w - T(w)) = 0$. Since

$$\begin{aligned} v(T^n(z) - T(w))) &\leq L\Big(v(T^{n-1}(z) - T^n(z)) + v(w - T(w)) \Big) \\ &\leq L\Big(k^{n-1}\, v(z - T(z)) + v(w - T(w)) \Big), \end{aligned}$$

for any $n \geq 1$, using the Fatou's property, we get

$$\begin{aligned} v(w - T(w))) &\leq \liminf_{n \to \infty}\, v(T^n(z) - T(w))) \\ &\leq L\, v(w - T(w)). \end{aligned}$$

Since $L < 1/2$, we conclude that $v(w - T(w)) = 0$, i.e., w is the fixed point of T. □

Note that there is no reason for $v(w - T(w)) < +\infty$ to hold. The proof of the main theorem of Reference [18] fails because it assumes this condition. The next corollary is the analogue to Kannan's extension of the classical Banach fixed point theorem in $\ell_{p(\cdot)}$.

Corollary 1. *Let K be a nonempty v-closed subset of $\ell_{p(\cdot)}$. Let $T : K \to K$ be a Kannan v-contraction mapping such that $v(x - T(x)) < +\infty$, for any $x \in K$. Then for any $z \in K$, $\{T^n(z)\}$ v-converges to the unique fixed point w of T. Moreover, if L is the Kannan constant associated to T, then we have*

$$v(T^n(z) - w) \leq L \left(\frac{L}{1-L} \right)^{n-1} v(z - T(z)),$$

for any $x \in C$ and $n \geq 1$.

Proof. The first part follows directly from Theorem 1. Using the inequality (NL) above combined with the Fatou's property, we get

$$v(T^n(z) - w) \leq L\, k^{n-1}\, v(z - T(z)) = L \left(\frac{L}{1-L} \right)^{n-1} v(z - T(z)),$$

for any $n \geq 1$ and $z \in K$. □

Next we prove a new result that is known in Banach spaces but that is still elusive in the case of modular vector spaces. To prepare the ground for this step, we present some material that, though known in the case of Banach spaces is new in the setting of metric spaces.

Definition 5. *We will say that:*

(i) $\ell_{p(\cdot)}$ *satisfies the property* (R) *if any decreasing sequence of nonempty* v*-bounded and* v*-closed convex subsets have a nonempty intersection;*

(ii) $\ell_{p(\cdot)}$ *satisfies the* v*-quasi-normal property if for any nonempty* v*-bounded and* v*-closed convex subset* K *with more than one point, there exists* $x \in K$ *such that*

$$v(x-y) < \delta_v(K) = \sup\{v(a-b);\ a, b \in K\},$$

for any $y \in K$.

Remark 1. *In Reference [19], the authors proved that if* $p^- = \inf\limits_{n \in \mathbb{N}} p(n) > 1$*, then* $\ell_{p(\cdot)}$ *satisfies a weaker form of modular uniform convexity which implies the property* (R)*. The case when* $p^- = 1$ *remains elusive and hard to tackle. Recently, the authors of Reference [20] proved that if* $\{n \in \mathbb{N}; p(n) = 1\}$ *has at most one element, then* $\ell_{p(\cdot)}$ *satisfies a modular geometric property known as the modular uniform convexity in every direction (in short* v*-UCED). Note that this property implies the modular normal property [20] which in turn implies the* v*-quasi-normal property. Note that in both cases, we may have* $p^+ = \sup\limits_{n \in \mathbb{N}} p(n) = \infty$ *which implies the failure of the* Δ_2*-condition.*

Before we state the main result of this work, we will need the following technical lemma.

Lemma 1. *Assume that* $\ell_{p(\cdot)}$ *satisfy the* (R) *property and the* v*-quasi-normal property. Let* K *be a nonempty,* v*-bounded,* v*-closed and convex subset of* $\ell_{p(\cdot)}$*. Let* $T : K \to K$ *be a Kannan* v*-nonexpansive mapping. Fix* $r > 0$*. Assume that* $A_r = \{x \in K;\ v(x - T(x)) \le r\}$ *is not empty. Set*

$$K_r = \bigcap \{B_v(a,t); T(A_r) \subset B_v(a,t)\} \cap K.$$

Then K_r *is a nonempty,* v*-closed and convex subset of* K *and*

$$T(K_r) \subset K_r \subset A_r \quad \text{and} \quad \delta_v(K_r) \le r.$$

Proof. Note that $T(A_r) \subset K_r$ which implies that K_r is not empty. Since the v-balls are v-closed and convex, we deduce that K_r is a v-closed convex subset of K. Let us prove that $K_r \subset A_r$. Let $x \in K_r$. If $v(x - T(x)) = 0$, then obviously we have $x \in A_r$. Otherwise, assume $v(x - T(x)) > 0$. Set

$$s = \sup\ \{v(T(z) - T(x));\ z \in A_r\}.$$

From the definition of s, we have $T(A_r) \subset B_v(T(x), s)$. Hence $K_r \subset B_v(T(x), s)$, which implies $v(x - T(x)) \le s$. Let $\varepsilon > 0$. Then there exists $z \in A_r$ such that $s - \varepsilon \le v(T(x) - T(z))$. Hence

$$\begin{aligned} v(x - T(x)) - \varepsilon &\le s - \varepsilon \\ &\le v(T(x) - T(z)) \\ &\le \frac{1}{2}\Big(v(x - T(x)) + v(z - T(z))\Big) \\ &\le \frac{1}{2}\Big(v(x - T(x)) + r\Big). \end{aligned}$$

Since ε is taken arbitrarily positive, we get

$$v(x - T(x)) \leq \frac{1}{2}\Big(v(x - T(x)) + r\Big),$$

which implies $v(x - T(x)) \leq r$, that is, $x \in A_r$ as claimed. Since $T(A_r) \subset K_r$, we get $T(K_r) \subset T(A_r) \subset K_r$, that is, K_r is T-invariant. Next we prove that $\delta_v(K_r) \leq r$. First, we notice that

$$v(T(x) - T(y)) \leq \frac{1}{2}\Big(v(x - T(x)) + v(z - T(z))\Big) \leq r,$$

for any $x, y \in A_r$. Fix $x \in A_r$. Then $T(A_r) \subset B_v(T(x), r)$. The definition of K_r implies $K_r \subset B_v(T(x), r)$. Hence $T(x) \in \bigcap\limits_{y \in K_r} B_v(y, r)$, which implies $T(A_r) \subset \bigcap\limits_{y \in K_r} B_v(y, r)$. Again from the definition of K_r, we get $K_r \subset \bigcap\limits_{y \in K_r} B_v(y, r)$. Therefore, we have $v(y - z) \leq r$, for any $y, z \in K_r$, that is, $\delta_v(K_r) \leq r$. The proof of the Lemma 1 is complete. □

Now we are ready to state the main result of our work.

Theorem 2. *Assume that $\ell_{p(\cdot)}$ satisfy the (R) property and the v-quasi-normal property. Let K be a nonempty v-bounded v-closed and convex subset of $\ell_{p(\cdot)}$. Let $T : K \to K$ be a Kannan v-nonexpansive mapping. Then T has a fixed point.*

Proof. Set $r_0 = \inf\{v(x - T(x));\ x \in K\}$ and $r_n = r_0 + 1/n$, for $n \geq 1$. By definition of r_0, the set $A_{r_n} = \{x \in K;\ v(x - T(x)) \leq r_n\}$ is not empty, for any $n \geq 1$. Consider the subset K_{r_n} defined in Lemma 1. It is easy to check that $\{K_{r_n}\}$ is a decreasing sequence of nonempty v-bounded v-closed and convex subsets of K. The property (R) implies that $K_\infty = \bigcap\limits_{n \geq 1} K_{r_n}$ is not empty. Let $x \in K_\infty$. Then we have $v(x - T(x)) \leq r_n$, for any $n \geq 1$. If we let $n \to \infty$, we get $v(x - T(x)) \leq r_0$ which implies $v(x - T(x)) = r_0$. Hence the set $A_{r_0} \neq \emptyset$. We claim that $r_0 = 0$. Otherwise, $r_0 > 0$ which implies that T fails to have a fixed point. Again consider the set K_{r_0} as defined in Lemma 1. Note that since T fails to have a fixed point and K_{r_0} is T-invariant, then K_{r_0} has more than one point, that is, $\delta_v(K_{r_0}) > 0$. It follows from the v-quasi-normal property that there exists $x \in K_{r_0}$ such that

$$v(x - y) < \delta_v(K_{r_0}) \leq r_0,$$

for any $y \in K_{r_0}$. From Lemma 1, we know that $K_{r_0} \subset A_{r_0}$. Hence $T(x) \in T(A_{r_0}) \subset K_{r_0}$ by definition of K_{r_0}. Obviously this will imply

$$v(x - T(x)) < \delta_v(K_{r_0}) \leq r_0,$$

which is a contradiction with the definition of r_0. Hence $r_0 = 0$ which implies that any point in A_{r_0} is a fixed point of T, that is, T has a fixed point in K. □

Remark 2. *In fact, the conclusion of Theorem 2 gives a characterization of the v-quasi-normal property of $\ell_{p(\cdot)}$. Indeed, assume that $\ell_{p(\cdot)}$ fails the v-quasi-normal property. Then there exists a nonempty v-bounded v-closed and convex subset K of $\ell_{p(\cdot)}$ not reduced to one point such that for any $x \in K$, there exists $z_x \in K$ such that $v(x - z_x) = \delta_v(K) > 0$. The mapping $T : K \to K$ defined by $T(x) = z_x$ is a Kannan v-nonexpansive mapping. Indeed, we have*

$$v(T(x) - T(y)) \leq \delta_v(K) = \frac{1}{2}\Big(v(x - T(x) + v(y - T(y))\Big),$$

for any $x, y \in K$. Since $v(x - T(x)) = \delta_v(K) > 0$, for any $x \in K$, then T fails to have a fixed point.

Using Remark 1 combined with Theorem 2, we get the following corollary:

Corollary 2. *Assume that* $p^- = \inf_{n\in\mathbb{N}} p(n) > 1$*. Let* K *be a nonempty* v*-bounded* v*-closed and convex subset of* $\ell_{p(\cdot)}$*. Let* $T : K \to K$ *be a Kannan* v*-nonexpansive mapping. Then* T *has a fixed point.*

Author Contributions: A.A.N.A. and M.A.K. contributed equally on the development of the theory and their respective analysis. All authors have read and agreed to the published version of the manuscript.

Funding: This project was funded by the Deanship of Scientific Research (DSR), King Abdulaziz University, Jeddah, under grant No. (D-037-363-1441). The authors, therefore, gratefully acknowledge DSR for their technical and financial support.

Conflicts of Interest: The authors declare no conflict of interest.

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