On the Asymptotic Behavior of $D$-Solutions to the Displacement Problem of Linear Elastostatics in Exterior Domains

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Abstract: We study the asymptotic behavior of solutions with finite energy to the displacement problem of linear elastostatics in a three-dimensional exterior Lipschitz domain.

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1. Introduction

The displacement problem of elastostatics in an exterior Lipschitz domain $\Omega$ of $\mathbb{R}^3$ consists of finding a solution to the equations \[ (\text{div } C[\nabla u]) = \partial_j(C_{ijkl}\partial_k u_l), \text{ Lin is the space of second–order tensors (linear maps from } \mathbb{R}^3 \text{ into itself) and Sym, Skw are the spaces of the symmetric and skew elements of Lin respectively; if } E \in \text{ Lin and } v \in \mathbb{R}^3, Ev \text{ is the vector with components } E_{ij}v_j. B_R = \{ x \in \mathbb{R}^3 : r = |x| < R \}, T_R = B_{2R} \setminus B_R, \mathcal{C}_{BR} = B_R \setminus B_{R_0} \text{ and } B_{R_0} \text{ is a fixed ball containing } \partial \Omega. \]

If $f(x)$ and $\phi(r)$ are functions defined in a neighborhood of infinity, then $f(x) = o(\phi(r))$ means that $\lim_{R \to +\infty} (f/\phi) = 0. To lighten up the notation, we do not distinguish between scalar, vector, and second–order tensor space functions; $c$ will denote a positive constant whose numerical value is not essential to our purposes.)

\[
\begin{align*}
&\text{div } C[\nabla u] = 0 \quad \text{in } \Omega, \\
&u = \hat{u} \quad \text{on } \partial \Omega, \\
&\lim_{R \to +\infty} \int_{\partial B} u(R, \sigma)d\sigma = 0, \\
\end{align*}
\]

where $u$ is the (unknown) displacement field, $\hat{u}$ is an (assigned) boundary displacement, $B$ is the unit ball, $C = [C_{ijkl}]$ is the (assigned) elasticity tensor, i.e., a map from $\Omega \times \text{ Lin} \to \text{ Sym}$, linear on Sym and vanishing in $\Omega \times \text{ Skw}$. We shall assume $C$ to be symmetric, i.e.,

\[ E \cdot C[L] = L \cdot C[E] \quad \forall E, L \in \text{ Lin}, \]

and positive definite, i.e., there exists positive scalars $\mu_0$ and $\mu_e$ (minimum and maximum elastic moduli [1]) such that

\[ \mu_0 |E|^2 \leq E \cdot C[E] \leq \mu_e |E|^2, \quad \forall E \in \text{ Sym}, \quad a.e. \text{ in } \Omega. \]
Let $D^{1,q}_0(\Omega)$, $D^{1,q}_1(\Omega)$ ($q \in [1, +\infty)$) be the completion of $C^\infty_0(\overline{\Omega})$ and $C^\infty_0(\Omega)$, respectively, with respect to the norm $\|\nabla u\|_{L^q(\Omega)}$.

We consider solutions $u$ to equations (1) with finite Dirichlet integral (or with finite energy) that we call $D$-solutions analogous with the terminology used in viscous fluid dynamics (see [2]). More precisely, we say that $u \in D^{1,2}_0(\Omega)$ is a $D$-solution to equation (1)

$$\int_{\Omega} \nabla \varphi \cdot C[\nabla u] = 0, \quad \forall \varphi \in D^{1,2}_0(\Omega).$$

A $D$-solution to system (1) is a $D$-solution to equation (1)$_1$, which satisfies the boundary condition in the sense of trace in Sobolev’s spaces and tends to zero at infinity in a mean square sense [2]

$$\lim_{R \to +\infty} \int_{\partial B} |u(R, \sigma)|^2 d\sigma = 0.$$ (5)

If $u$ is a $D$-solution to (1)$_1$, then the traction field on the boundary is

$$s(u) = C[\nabla u]n$$

where a well defined field of $W^{-1/2,2}(\partial \Omega)$ exists and the following generalized work and energy relation [1] holds

$$\int_{\Omega \cap B_R} \nabla u \cdot C[\nabla u] = \int_{\partial \Omega} u \cdot s(u),$$

where abuse of notation $\int_{\partial \Omega} u \cdot s(u)$ means the value of the functional $s(u) \in W^{-1/2,2}(\partial \Omega)$ at $u \in W^{1/2,2}(\partial \Omega)$, and $n$ is the unit outward (with respect to $\Omega$) normal to $\partial \Omega$.

If $\hat{u} \in W^{1/2,2}(\partial \Omega)$, denoting by $u_0 \in D^{1,2}(\Omega)$ an extension of $\hat{u}$ in $\Omega$ vanishing outside a ball, then (1)$_{1,2}$ is equivalent to finding a field $u \in D^{1,2}_0(\Omega)$ such that

$$\int_{\Omega} \nabla \varphi \cdot C[\nabla v] = -\int_{\Omega} \nabla \varphi \cdot C[\nabla u_0], \quad \forall \varphi \in D^{1,2}_0(\Omega).$$ (6)

Since the right-hand side of (6) defines a linear and continuous functional on $D^{1,2}_0(\Omega)$, and by the first Korn inequality (see [1] Section 13)

$$\int_{\Omega} |\nabla v|^2 \leq \frac{2}{\mu_0} \int_{\Omega} \nabla v \cdot C[\nabla v],$$

by the Lax–Milgram lemma, (6) has a unique solution $v$, and the field $u = v + u_0$ is a $D$-solution to (1)$_{1,2}$. It satisfies (1)$_3$ in the following sense (see Lemma 1)

$$\int_{\partial B} |u(R, \sigma)|^2 d\sigma = o(R^{-1}).$$ (7)

Moreover, $u$ exhibits more regularity properties provided $C$, $\partial \Omega$ and $\hat{u}$ are more regular. In particular, if $C$, $\hat{u}$ and $\partial \Omega$ are of class $C^\infty$, then $u \in C^\infty(\overline{\Omega})$ [3].

If $C$ is constant, then existence and regularity hold under the weak assumption of strong ellipticity [1], i.e.,

$$\mu_0 |a|^2 |b|^2 \leq a \cdot C[a \otimes b]b, \quad \forall a, b \in \mathbb{R}^3.$$

As far as we are aware, except for the property (7), little is known about the convergence at infinity of a $D$-solution and, in particular, whether or under what additional conditions (7) can be improved.
The main purpose of this paper is just to determine reasonable conditions on \( C \) assuring that (7) can be improved.

We say that \( C \) is regular at infinity if there is a constant elasticity tensor \( C_0 \) such that

\[
\lim_{|x| \to +\infty} C(x) = C_0. \tag{9}
\]

Let \( C_0 \) and \( C \) denote the linear spaces of the \( D \)-solutions to the equations

\[
\text{div} \, C[\nabla \tau] = 0 \quad \text{in } \Omega, \\
\tau = \tau \quad \text{on } \partial \Omega,
\]

\[
\lim_{R \to +\infty} \int_{\partial B} |h(R, \sigma)|^2 d\sigma = 0, \tag{10}
\]

for all \( \tau \in \mathbb{R}^3 \) and

\[
\text{div} \, C[\nabla \tau] = 0 \quad \text{in } \Omega, \\
\tau = \tau + Ax \quad \text{on } \partial \Omega,
\]

\[
\lim_{R \to +\infty} \int_{\partial B} |h(R, \sigma)|^2 d\sigma = 0, \tag{11}
\]

for all \( \tau \in \mathbb{R}^3, A \in \text{Lin} \), respectively.

The following theorem holds.

**Theorem 1.** Let \( u \) be the \( D \)-solution to (1). There is \( q < 2 \) depending only on \( C \) such that

\[
\int_{\partial B} |u(R, \sigma)|^q d\sigma = o(R^{q-3}). \tag{12}
\]

If \( C \) is regular at infinity, then

\[
\int_{\partial B} |u(R, \sigma)|^q d\sigma = o(R^{q-3}), \quad \forall q \in (3/2, +\infty), \tag{13}
\]

and

\[
\int_{\partial B} |u(R, \sigma)|^q d\sigma = o(R^{q-3}), \quad \forall q \in (1, 2] \iff \int_{\partial \Omega} \hat{u} \cdot s(\tau) = 0, \quad \forall \tau \in C_0. \tag{14}
\]

Moreover, if

\[
\int_{\partial \Omega} C[\hat{u} \otimes n] = 0, \quad \int_{\partial \Omega} \hat{u} \cdot s(\tau) = 0, \quad \forall \tau \in C,
\]

then

\[
\int_{\partial B} |u(R, \sigma)|^2 d\sigma = o(R^{-2}). \tag{16}
\]

2. Preliminary Results

In this section, we collect the main tools we need to prove Theorem 1.

**Lemma 1.** If \( u \in D^{1,q}(\Omega) \), for \( q \in [1, 2] \), then

\[
\int_{\partial B} |u(R, \sigma)|^q d\sigma \leq c(q) R^{q-3} \int_{CB_R} |
abla u|^q. \tag{17}
\]
Moreover, if \( q = 2 \), then, for \( R \gg R_0 \),
\[
\int_{\mathcal{C}_B R} \frac{|u|^2}{r^2} \leq 4 \int_{\mathcal{C}_B R} |\nabla u|^2. \tag{18}
\]

**Proof.** Lemma 1 is well-known (see, e.g., [2,4] and [5] Chapter II). We propose a simple proof for the sake of completeness. Since \( D^{1,2}(\Omega) \) is the completion of \( C_\infty^0(\Omega) \) with respect to the norm \( \| \nabla u \|_{L^2(\Omega)} \), it is sufficient to prove (17) and (18) for a regular field \( u \) vanishing outside a ball. By basic calculus and Hölder inequality,
\[
\int_{\partial B} \left| u(R, \sigma) \right|^q d\sigma = \int_{\partial B} \left| \frac{\partial}{\partial r} u(r, \sigma) dr \right|^q d\sigma = \int_{\partial B} r^{2q/\theta - 2q} \left| \partial_r u(r, \sigma) dr \right|^q d\sigma
\]
\[
\leq \left\{ \int_{\partial B} \left| \nabla u(r, \sigma) \right|^q r^{2q/\theta} dr \right\} \left\{ \int_{\partial B} r^{-2q/\theta} dr \right\}^{q-1}.
\]
Hence, (17) follows by a simple integration.

From
\[
\int_{\mathcal{C}_B R} \frac{|u|^2}{r^2} = \int_{R} \frac{u}{r} \partial_r \left( r \int_{\partial B} |u(r, \sigma)|^2 d\sigma \right) = 2 \int_{\mathcal{C}_B R} \frac{u}{r} \partial_r u
\]
by Schwarz’s inequality, one gets
\[
\int_{\mathcal{C}_B R} \frac{|u|^2}{r^2} \leq 2 \left\{ \int_{\mathcal{C}_B R} \frac{|u|^2}{r^2} \right\} \left\{ \int_{\mathcal{C}_B R} |\nabla u|^2 \right\}^{1/2}.
\]
Hence, (18) follows at once. \( \Box \)

Let \( C_0 \) be a constant and strongly elliptic elasticity tensor. The equation
\[
\text{div} \ C_0 |\nabla u| = 0 \tag{19}
\]
admits a fundamental solution \( \mathcal{U}(x - y) \) [6] that enjoys the same qualitative properties as the well-known ones of homogeneous and isotropic elastostatics, defined by
\[
\mathcal{U}_{ij}(x - y) = \frac{1}{8\pi\mu(1 - \nu)|x - y|} \left( (3 - 4\nu)\delta_{ij} + \frac{(x_i - y_i)(x_j - y_j)}{|x - y|^2} \right),
\]
where \( \mu \) is the shear modulus and \( \nu \) the Poisson ratio (see [1] Section 51). In particular, \( \mathcal{U}(x) = O(r^{-1}) \) and for \( f \) with compact support (say) the volume potential
\[
\mathcal{V}[f](x) = \int_{\mathbb{R}^3} \mathcal{U}(x - y) f(y) dy
\]
is a solution (in a sense depending on the regularity of \( f \)) to the system
\[
\text{div} \ C_0 |\nabla u| + f = 0 \quad \text{in} \ \mathbb{R}^3. \tag{20}
\]

Let \( \mathcal{H}^1 \) denote the Hardy space on \( \mathbb{R}^3 \) (see [7] Chapter III). The following result is classical (see, e.g., [7]).

**Lemma 2.** \( \nabla_2 \mathcal{V} \) maps boundedly \( L^q \) into itself for \( q \in (1, +\infty) \) and \( \mathcal{H}^1 \) into itself.
Lemma 3. Let \( u \) be the D-solution to (1), then, for \( R \gg R_0 \),
\[
\int_{\partial \Omega} s(u) = \int_{\partial \Omega} s(u),
\]
and
\[
\int_{\partial \Omega} h \cdot s(u) = \int_{\partial \Omega} \hat{u} \cdot s(h), \quad \forall h \in \mathcal{C},
\]
where \( \mathcal{C} \) is the space of D-solutions to system (11).

Proof. Let
\[
g(x) = \begin{cases} 
0, & |x| > 2R, \\
1, & |x| < R, \\
R^{-1}(R - |x|), & R \leq |x| \leq 2R,
\end{cases}
\]
with \( R \gg R_0 \). Scalar multiplication of both sides of (1) by \( gh \), (2) and an integration by parts yield
\[
\int_{\partial \Omega} h \cdot s(u) - \int_{\partial \Omega} \hat{u} \cdot s(h) = \frac{1}{R} \int_{\mathbb{T}_R} \left( u \cdot C[\nabla h]e_r - h \cdot C[\nabla u]e_r \right),
\]
where \( e_r = x/r \). Since \( R \leq |x| \leq 2R \), by Schwarz’s inequality,
\[
\left| \frac{1}{R} \int_{\mathbb{T}_R} h \cdot C[\nabla u]e_r \right| \leq 2 \int_{\mathbb{T}_R} \left| r^{-1}h \cdot C[\nabla u]e_r \right| \leq c \|r^{-1}h\|_{L^2(\mathbb{T}_R)} \|\nabla u\|_{L^2(\mathbb{T}_R)},
\]
\[
\left| \frac{1}{R} \int_{\mathbb{T}_R} u \cdot C[\nabla h]e_r \right| \leq 2 \int_{\mathbb{T}_R} \left| r^{-1}u \cdot C[\nabla h]e_r \right| \leq c \|r^{-1}u\|_{L^2(\mathbb{T}_R)} \|\nabla h\|_{L^2(\mathbb{T}_R)}.
\]
Hence, letting \( R \to +\infty \) and taking into account Lemma 1, (23) follows. \( \square \)

Lemma 4. Let \( u \) be the D-solution to (1); then, for \( R \gg R_0 \),
\[
\int_{\partial \Omega} (x \otimes s(u) - C[\hat{u} \otimes u]) = \int_{\partial \Omega} (x \otimes s(u) - C[u \otimes e_R]),
\]
where \( \mathcal{C} \) is the space of D-solutions to system (11).

Proof. (25) is easily obtained by integrating the identity
\[
0 = x \otimes \text{div} \ C[\nabla u] = \text{div} (x \otimes C[\nabla u]) - C[\nabla u]
\]
over \( B_R \) and using the divergence theorem. \( \square \)

3. Proof of Theorem 1

Let \( \vartheta(r) \) be a regular function, vanishing in \( B_R \) and equal to 1 outside \( B_{2R} \), for \( R \gg R_0 \). The field \( v = \vartheta u \) is a D-solution to the equation
\[
\text{div} C[\nabla v] + f = 0 \quad \text{in} \ \mathbb{R}^3,
\]
with
\[
f = -C[\nabla u] \nabla \vartheta - \text{div} C[u \otimes \nabla \vartheta].
\]
Of course, \( f \in L^2(\mathbb{R}^3) \) vanishes outside \( T_R \). Let \( C_0 \) be a strongly elliptic elasticity tensor. Clearly, \( v \) is a \( D \)-solution to the system

\[
\operatorname{div} C_0 [\nabla v] + \operatorname{div} (C - C_0) [\nabla v] + f = 0 \quad \text{in} \mathbb{R}^3,
\]

which coincides with \( u \) outside \( B_{2R} \). Since

\[
\nabla_k V[f](x) = O(r^{-1-k}), \quad k \in \mathbb{N}, \quad \nabla_k = \nabla \cdots \nabla \quad \text{\( k \)-times}
\]

by Lemma 2, the map

\[
w'(x) = \nabla V[(C - C_0) [\nabla w]](x) + V[f](x)
\]

is continuous from \( D^{1,q} \) into itself, for \( q \in (3/2, +\infty) \). Choose

\[
C_{0ijkl} = \mu_e \delta_{ij} \delta_{kl}.
\]

Since

\[
\|\nabla V[(C - C_0) [\nabla w]]\|_{D^{1,q}} \leq c(q) \frac{\mu_e - \mu_0}{\mu_e} \|w\|_{D^{1,q}}
\]

and [7]

\[
\lim_{q \to 2} c(q) = 1,
\]

the map (30) is contractive in a neighborhood of 2 and its fixed point must coincide with \( v \). Hence, there is \( q \in (1, 2) \) such that \( u \in D^{1,q}(\Omega) \) and (12) is proved.

If \( C \) is regular at infinity, then by Lemma 1 and the property of \( \vartheta \),

\[
\|u' - z'\|_{D^{1,q}} \leq c(q) \|C - C_0\|_{L^\infty(\mathbb{R}^3)} \|v - z\|_{D^{1,q}}.
\]

Since the constant \( c(q) \) is uniformly bounded in every interval \([a, b]\) and \( \|C - C_0\|_{L^\infty(\mathbb{R}^3)} \) is sufficiently small, \( u \in D^{1,q} \) for \( q \in (3/2, +\infty) \).

Assume that

\[
\int_{\partial \Omega} \tilde{u} \cdot s(h) = 0, \quad \forall h \in \mathfrak{C}_0.
\]

By Lemma 3, for \( R \gg R_0 \),

\[
\int_{\partial B_R} s(u) = \int_{\partial B_R} C[\nabla u] \epsilon_R = 0.
\]

Therefore, taking into account (27),

\[
\int f = \int_{\mathbb{R}^3} f = \int_{\mathbb{R}^3} \vartheta'(r) dr \int_{\partial B_r} C[\nabla u] \epsilon_r = 0,
\]

(33)

Since

\[
V[f](x) = \int_{\mathbb{R}^3} [\mathcal{U}(x - y) - \mathcal{U}(y)] f(y) dv + \mathcal{U}(x) \int_{\mathbb{R}^3} f,
\]

by (33), Lagrange’s theorem and (29)

\[
\nabla V[f](x) = O(r^{-3}),
\]

so that

\[
\nabla V[f] \in L^q, \quad q \in (1, 2].
\]

(34)
Then, by (31), the map (30) is contractive for \( q \) in a right neighborhood of 1 so that
\[ u \in D^{1,q}(\Omega), \quad q \in (1,2]. \tag{35} \]

Conversely, if (35) holds, then a simple computation yields
\[ \int_{\partial\Omega} s(u) = \int_{\mathbb{R}^3} C[\nabla u] \nabla g = -\frac{1}{R} \int_{\mathbb{R}^3} C[\nabla u] e_r, \tag{36} \]
where \( g \) is the function (24). By Hölder’s inequality,
\[ \frac{1}{R} \left| \int_{\mathbb{R}^3} C[\nabla u] e_r \right| \leq \frac{c}{R} \left\{ \int_{\mathbb{R}^3} |\nabla u|^{3/2} \right\}^{2/3} \left\{ \int_{\mathbb{R}^3} d\nu \right\}^{1/3} \leq \left\{ \int_{\mathbb{R}^3} |\nabla u|^{3/2} \right\}^{2/3}. \]

Therefore, letting \( R \to +\infty \) in (36) yields
\[ \int_{\partial\Omega} s(u) = 0 \]
and this implies (32).

From
\[ \nabla \nabla [f](x) = \int_{\mathbb{R}^3} [u_{ij}(x - y) - u(x) - y_i \partial_i u_{ij}(y)] f_j(y) d\nu_y + u_{ij}(x) \int_{\mathbb{R}^3} f_j + \partial_i u(x) \int_{\mathbb{R}^3} u_{ij}(y) f_j(y) d\nu_y \]
by (33), Lemma 4, Lagrange’s theorem and (29)
\[ \nabla \nabla [f](x) = O(r^{-4}), \]
so that \( \nabla \nabla [f] \in L^1 \). Since \( f \in L^2(\mathbb{R}^3) \) has compact support and satisfies (33), it belongs to \( \mathcal{H}^1 \) (see [7] p. 92) and by Lemma 2 \( \nabla [f] \in \mathcal{H}^1 \). Hence, it follows that (30) maps \( \mathcal{H}^1 \) into itself and
\[ ||v' - z'||_{\mathcal{H}^1} \leq ||C - C_0||_{L^\infty(\mathcal{G}_R^\beta)} ||v - z||_{\mathcal{H}^1}. \]

Since, by assumptions, \( ||C - C_0||_{L^\infty(\mathcal{G}_R^\beta)} \) is small, (30) is a contraction and by the above argument its (unique) fixed point must coincide with \( v \) so that \( \nabla u \in L^1(\Omega) \).

We aim at concluding the paper with the following remarks.

(i) It is evident that the hypothesis that \( C \) is regular at infinity can be replaced by the weaker one that \( \|C - C_0\| \) is suitably small at a large spatial distance.

(ii) The operator \( \nabla \) maps boundedly the Hardy space \( \mathcal{H}^q \) (\( q \in (0,1) \)) into itself [7]. Hence, the argument in the proof of (16) can be used to show that \( \nabla v \in \mathcal{H}^q, q > 3/4 \). We can then use the Sobolev–Poincaré (see [8] p. 255) to see that \( u \in L^q(\Omega) \) for \( q > 1 \).

(iii) Relation (16) is a kind of Stokes’ paradox in nonhomogeneous elastostatics: if \( C \) is regular at infinity, then the system
\[
\begin{align*}
\text{div} \ C[\nabla h] &= 0 \quad \text{in} \ \Omega, \\
h &= \tau \quad \text{on} \ \partial\Omega, \\
\int_{\partial\Omega} h(R,\sigma) d\sigma &= o(R^{-1}),
\end{align*}
\]
with $\tau$ nonzero constant vector, does not admit solutions.

(iv) If $C$ is constant and strongly elliptic, then $u$ is analytic in $\Omega$ and at large spatial distance admits the representation

$$u(x) = U(x) \int_{\partial\Omega} s(u) + \nabla U(x) \int_{\partial\Omega} \left( \xi \otimes s(u) - C[\hat{u} \otimes n] \right)(\xi) + \psi(x)$$

with $|x^3|\psi(x)| \leq c$. Therefore, in the homogeneous case, the conclusions of Theorem 1 hold pointwise:

$$|x^2|u(x)| \leq c \iff \int_{\partial\Omega} \hat{u} \cdot s(h) = 0, \forall h \in C_0,$$

$$\int_{\partial\Omega} C[\hat{u} \otimes n] = 0, \quad \int_{\partial\Omega} \hat{u} \cdot s(h) = 0, \forall h \in C \Rightarrow |x^3|u(x)| \leq c.$$