mathematics

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Argument and Coefficient Estimates for Certain Analytic Functions

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Received: 3 December 2019; Accepted: 2 January 2020; Published: 5 January 2020

Abstract: The aim of the present paper is to introduce a new class $\mathcal{G}(\alpha, \delta)$ of analytic functions in the open unit disk and to study some properties associated with strong starlikeness and close-to-convexity for the class $\mathcal{G}(\alpha, \delta)$. We also consider sharp bounds of logarithmic coefficients and Fekete-Szegö functionals belonging to the class $\mathcal{G}(\alpha, \delta)$. Moreover, we provide some topics related to the results reported here that are relevant to outcomes presented in earlier research.

Keywords: starlike function; subordinate; univalent function

MSC: Primary 30C45; Secondary 30C80

1. Introduction and Preliminaries

Let $\mathbb{U}$ denote the open unit dick in the complex plane $\mathbb{C}$. A function $\omega : \mathbb{U} \to \mathbb{C}$ is called a *Schwarz function* if ω is a analytic function in $\mathbb{U}$ with $\omega(0) = 0$ and $|\omega(z)| < 1$ for all $z \in \mathbb{U}$. Clearly, a Schwarz function ω is the form

$$\omega(z) = w_1 z + w_2 z^2 + \cdots .$$

We denote by Ω the set of all Schwarz functions on $\mathbb{U}$.

Let $\mathcal{A}$ be consisting of all analytic functions of the following normalized form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

in the open unit disk $\mathbb{U}$. An analytic function f is said to be *univalent* in a domain if it provides a one-to-one mapping onto its image: $f(z_1) = f(z_2) \Rightarrow z_1 = z_2$. Geometrically, this means that different points in the domain will be mapped into different points on the image domain. Also, let $\mathcal{S}$ be the class of functions $f \in \mathcal{A}$ which are univalent in $\mathbb{U}$. A domain D in the complex plane $\mathbb{C}$ is called *starlike* with respect to a point $w_0 \in D$, if the line segment joining w_0 to every other point $w \in D$ lies in the interior of D. In other words, for any $w \in D$ and $0 \leq t \leq 1$, $tw_0 + (1-t)w \in D$. A function $f \in \mathcal{A}$ is starlike if the image $f(D)$ is starlike with respect to the origin.

For two analytic functions f and F in $\mathbb{U}$, we say that the function f *is subordinate to* the function F in $\mathbb{U}$ and we write $f(z) \prec F(z)$, if there exists a Schwarz function ω such that $f(z) = F(\omega(z))$ for all $z \in \mathbb{U}$. Specifically, if the function F is univalent in $\mathbb{U}$, then we have the next equivalence:

$$f(z) \prec F(z) \iff f(0) = F(0) \quad \text{and} \quad f(\mathbb{U}) \subset F(\mathbb{U}).$$

The logarithmic coefficients γ_n of $f \in \mathcal{S}$ are defined with the following series expansion:

$$\log\left(\frac{f(z)}{z}\right) = 2\sum_{n=1}^{\infty} \gamma_n(f) z^n, \; z \in \mathbb{U}. \tag{2}$$

These coefficients are an important factor in studying diverse estimates in the theory of univalent functions. Note that we use γ_n instead of $\gamma_n(f)$. The concept of logarithmic coefficients inspired Kayumov [1] to solve Brennan's conjecture for conformal mappings. The importance of the logarithmic coefficients follows from Lebedev-Milin inequalities [2] (Chapter 2), see also [3,4], where estimates of the logarithmic coefficients were used to find bounds on the coefficients of f. Milin [2] conjectured the inequality

$$\sum_{m=1}^{n}\sum_{k=1}^{m}\left(k|\gamma_k|^2 - \frac{1}{k}\right) \le 0 \quad (n = 1, 2, 3, \cdots),$$

which implies Robertson's conjecture [5], and hence, Bieberbach's conjecture [6]. This is the famous coefficient problem in univalent function theory. L. de Branges [7] established Bieberbach's conjecture by proving Milin's conjecture.

Definition 1. *Let $q, n \in \mathbb{N}$. The q^{th} Hankel determinant is denote by $H_q(n)$ and defined by*

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}, \tag{3}$$

where a_k $(k = 1, 2, \ldots)$ are the coefficients of the Taylor series expansion of a function f of the form (1)*. Note that $a_1 = 1$.*

The Hankel determinant $H_q(n)$ was defined by Pommerenke [8,9] and for fixed q, n the bounds of $|H_q(n)|$ have been studied for several subfamilies of univalent functions. Different properties of these determinants can be observed in [10] (Chapter 4). The Hankel determinants $H_2(1) = a_3 - a_2^2$ and $H_2(2) = a_2a_4 - a_3^2$, are well-known as *Fekete-Szegö* and *second Hankel determinant functionals*, respectively. In addition, Fekete and Szegö [11] introduced the generalized functional $a_3 - \lambda a_2^2$, where λ is a real number. Recently, Hankel determinants and other problems for various classes of bi-univalent functions have been studied, see [12–16].

For $\alpha \in [0, 1)$, we denote by $\mathcal{S}^*(\alpha)$ the subclass of $\mathcal{A}$ including of all $f \in \mathcal{A}$ for which f is a *starlike function of order* α in $\mathbb{U}$, with

$$\operatorname{Re}\frac{zf'(z)}{f(z)} > \alpha \quad (z \in \mathbb{U}).$$

Also, for $\alpha \in (0, 1]$, we denote by $\widetilde{\mathcal{S}}^*(\alpha)$ the subclass of $\mathcal{A}$ consisting of all $f \in \mathcal{A}$ for which f is a *strongly starlike function of order* α in $\mathbb{U}$, with

$$\left|\operatorname{Arg}\left(\frac{zf'(z)}{f(z)}\right)\right| < \frac{\alpha\pi}{2} \quad (z \in \mathbb{U}).$$

Note that $\widetilde{\mathcal{S}}^*(1) = \mathcal{S}^*(0) = \mathcal{S}^*$, the class of *starlike functions* in $\mathbb{U}$.

For $\alpha \in (0,1]$, we denote by $\widetilde{\mathcal{C}}(\alpha)$ the subclass of $\mathcal{A}$ including all of $f \in \mathcal{A}$ for which

$$\left|\operatorname{Arg}\left(f'(z)\right)\right| < \frac{\alpha\pi}{2} \qquad (z \in \mathbb{U}).$$

Note that $\widetilde{\mathcal{C}}(1) = \mathcal{C}$, the subclass of *close-to-convex functions* in $\mathbb{U}$. Here we understand that Arg w is a number in $(-\pi, \pi]$.

For $\alpha \in (0,1]$, Nunokawa and Saitoh in [17] defined the more general class $\mathcal{G}(\alpha)$ consisting of all $f \in \mathcal{A}$ satisfying

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) < 1 + \frac{\alpha}{2} \qquad (z \in \mathbb{U}).$$

They proved that $\mathcal{G}(\alpha)$ is a subclass of $\mathcal{S}^*$. Ozaki in [18] showed that every function $\mathcal{G}(1)$ is univalent in the unit disk $\mathbb{U}$. In the following, Umezawa [19], Sakaguchi [20] and Singh and Singh [21] obtained some geometric properties of $\mathcal{G}(1)$ including, convex in one direction, close-to-convex and starlike, respectively. Obradović et al. in [22] proved the sharp coefficient bounds for the moduli of the Taylor coefficients a_n of $f \in \mathcal{G}(\alpha)$ and determined the sharp bound for the Fekete-Szegö functional for functions in $\mathcal{G}(\alpha)$ with complex parameter λ. Also, Ponnusamy et al. [22,23] studied bounds for the logarithmic coefficients for functions in $\mathcal{G}(\alpha)$.

Here, we introduce a class as follows:

Definition 2. *For $\alpha, \delta \in (0,1]$, we define the subclass $\mathcal{G}(\alpha,\delta)$ of $\mathcal{A}$ as the following:*

$$\mathcal{G}(\alpha,\delta) := \left\{ f \in \mathcal{A} : \left| \operatorname{Arg}\left(\frac{2+\alpha}{\alpha} - \frac{2}{\alpha}\left(1 + \frac{zf''(z)}{f'(z)}\right)\right)\right| < \frac{\delta\pi}{2} \quad (z \in \mathbb{U}\,) \right\}.$$

It is clear that $\mathcal{G}(\alpha,1) = \mathcal{G}(\alpha)$ for $\alpha \in (0,1]$. Let $\alpha, \delta \in (0,1]$, identity function on $\mathbb{U}$ belongs to $\mathcal{G}(\alpha,\delta)$ which implies that $\mathcal{G}(\alpha,\delta) \neq \emptyset$. By means of the principle of subordination between analytic functions, we deduce

$$\mathcal{G}(\alpha,\delta) := \left\{ f \in \mathcal{A} : 1 + \frac{zf''(z)}{f'(z)} \prec -\frac{\alpha}{2}\left(\frac{1+z}{1-z}\right)^{\delta} + \frac{2+\alpha}{2} := \phi(z) \quad (z \in \mathbb{U}) \right\}. \tag{4}$$

Since the function f defined by

$$f(z) = \int_0^z \exp\left(\int_0^x \frac{-\frac{\alpha}{2}\left(\frac{1+t}{1-t}\right)^{\delta} + \frac{\alpha}{2}}{t}\, \mathrm{d}t \right) \mathrm{d}x \qquad (z \in \mathbb{U})$$

satisfies

$$1 + \frac{zf''(z)}{f'(z)} = \phi(z) \prec \phi(z),$$

we deduce $f \in \mathcal{G}(\alpha,\delta)$.

The aim of the present paper is to study some geometric properties for the class $\mathcal{G}(\alpha,\delta)$ such as strongly starlikeness and close-to-convexity. Also we investigate sharp bounds on logarithmic coefficients and Fekete-Szegö functionals for functions belonging to the class $\mathcal{G}(\alpha,\delta)$, which incorporate some known results as the special cases.

2. Some Properties of the Class $\mathcal{G}(\alpha,\delta)$

We denote by Q the class of all complex-valued functions q for which q is univalent at each $\overline{\mathbb{U}} \setminus \mathrm{E}(q)$ and $q'(\xi) \neq 0$ for all $\xi \in \partial\mathbb{U} \setminus \mathrm{E}(q)$ where

$$\mathrm{E}(q) = \left\{ \xi \in \partial\mathbb{U} : \lim_{z \to \xi} q(z) = \infty \right\}.$$

The following lemmas will be required to establish our main results.

Lemma 1 ([24] (Lemma 2.2d (i))). *Let $q \in Q$ with $q(0) = a$ and let $p(z) = a + p_n z^n + \ldots$ be analytic in $\mathbb{U}$ with $p(z) \not\equiv 1$ and $n \geq 1$. If p is not subordinate to q in $\mathbb{U}$ then there exist $z_0 \in \mathbb{U}$ and $\xi_0 \in \partial\mathbb{U} \setminus E(q)$ such that $\{p(z) : z \in \mathbb{U},\ |z| < |z_0|\} \subset q(\mathbb{U})$,*

$$p(z_0) = q(\xi_0).$$

Lemma 2. (see [25,26]) *Let the function p given by*

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$$

be analytic in $\mathbb{U}$ with $p(0) = 1$ and $p(z) \neq 0$ for all $z \in \mathbb{U}$. If there exists a point $z_0 \in \mathbb{U}$ with

$$|\arg(p(z))| < \frac{\beta\pi}{2} \qquad (|z| < |z_0|)$$

and

$$|\arg(p(z_0))| = \frac{\beta\pi}{2},$$

for some $\beta > 0$, then

$$\frac{z_0 p'(z_0)}{p(z_0)} = ik\beta \qquad (i = \sqrt{-1}),$$

where

$$k \geq \frac{a + a^{-1}}{2} \geq 1 \quad \text{when} \quad \arg(p(z_0)) = \frac{\beta\pi}{2} \tag{5}$$

and

$$k \leq -\frac{a + a^{-1}}{2} \leq -1 \quad \text{when} \quad \arg(p(z_0)) = -\frac{\beta\pi}{2}, \tag{6}$$

where

$$[p(z_0)]^{1/\beta} = \pm ia \quad \text{and} \quad a > 0.$$

Theorem 1. *Let $\alpha, \beta \in (0,1]$. If $f \in \mathcal{A}$ satisfies the condition*

$$\left| \operatorname{Arg}\left(\frac{2+\alpha}{\alpha} - \frac{2}{\alpha}\left(1 + \frac{zf''(z)}{f'(z)}\right)\right)\right| < \operatorname{Arctan}\left(\frac{4\beta}{2+\alpha}\right), \tag{7}$$

then

$$\left| \operatorname{Arg}\left(\frac{zf'(z)}{f(z)}\right)\right| < \frac{\beta\pi}{2} \quad (z \in \mathbb{U}).$$

Proof. Let $f \in \mathcal{A}$ and define the function $p : \mathbb{U} \to \mathbb{C}$ by

$$p(z) = \frac{zf'(z)}{f(z)} = 1 + \sum_{n=1}^{\infty} c_n z^n \quad (z \in \mathbb{U}).$$

Then it follows that p is analytic in $\mathbb{U}$, $p(0) = 1$,

$$1 + \frac{zf''(z)}{f'(z)} = p(z) + \frac{zp'(z)}{p(z)} \quad (z \in \mathbb{U})$$

and $p(z) \neq 0$ for all $z \in \mathbb{U}$. In fact, if p has a zero $z_0 \in \mathbb{U}$ of order m, then we may write

$$p(z) = (z - z_0)^m p_1(z) \quad (m \in \mathbb{N} = 1, 2, 3, \cdots),$$

where p_1 is analytic in $\mathbb{U}$ with $p_1(z_0) \neq 0$. Then

$$\frac{2+\alpha}{\alpha} - \frac{2}{\alpha}\left(p(z) + \frac{zp'(z)}{p(z)}\right) = \frac{2+\alpha}{\alpha} - \frac{2}{\alpha}\left(p(z) + \frac{zp_1'(z)}{p_1(z)} + \frac{mz}{z - z_0}\right).$$

Thus, choosing $z \to z_0$, suitably the argument of the right-hand of the above equality can take any value between $-\pi$ and π, which contradicts (7).

Define the function $q : \overline{\mathbb{U}} \setminus \{1\} \to \mathbb{C}$ by

$$q(z) = \left(\frac{1+z}{1-z}\right)^{\beta} \quad (z \in \overline{\mathbb{U}} \setminus \{1\}).$$

Then $q \in Q$, $q(0) = 1$ and $\mathrm{E}(q) = \{1\}$. It is clear that $\left|\operatorname{Arg}(p(z))\right| < \frac{\beta\pi}{2}$ for all $z \in \mathbb{U}$ if and only if $p \prec q$ on $\mathbb{U}$. Let $\left|\operatorname{Arg}(p(z_1))\right| \geq \frac{\beta\pi}{2}$ for some $z_1 \in \mathbb{U}$. Then p is not subordinate to q. By Lemma 1 there exists $z_0 \in \mathbb{U}$ and $\zeta_0 \in \partial\mathbb{U} \setminus \{1\}$ such that $\{p(z) : z \in \mathbb{U}, |z| < |z_0|\} \subset q(\mathbb{U})$ and $p(z_0) = q(\zeta_0)$. Therefore,

$$\left|\operatorname{Arg}(p(z))\right| < \frac{\beta\pi}{2},$$

for all $z \in \mathbb{U}$ with $|z| < |z_0|$ and

$$\left|\operatorname{Arg}(p(z_0))\right| = \frac{\beta\pi}{2}.$$

Then, Lemma 2, gives us that

$$\frac{z_0 p'(z_0)}{p(z_0)} = ik\beta,$$

where $[p(z_0)]^{\frac{1}{\beta}} = \pm ia$ $(a > 0)$ and k is given by (5) or (6).

Define the function $g : (0, a) \to \mathbb{R}$ by

$$g(t) = \frac{\frac{2}{2+\alpha}\left(t^{\beta}\sin(\frac{\beta\pi}{2}) + \beta\right)}{1 - \frac{2}{2+\alpha}t^{\beta}\cos\frac{\beta\pi}{2}} \quad t \in (0, a).$$

Then g is a differentiable function on $(0, a)$ and $g'(t) > 0$ for all $t \in (0, a)$. This implies that the function $h : (0, a) \to \mathbb{R}$ defined by

$$h(t) = \operatorname{Arctan}(g(t)) \quad t \in (0, a),$$

is a non-decreasing function on $(0, a)$. Thus

$$h(a) \geq \lim_{t \to 0^+} h(t) = \operatorname{Arctan}\left(\frac{2\beta}{2+\alpha}\right).$$

Therefore, we have

$$\operatorname{Arctan}\left(\frac{\frac{2}{2+\alpha}\left(a^{\beta}\sin\frac{\beta\pi}{2}+\beta\right)}{1-\frac{2}{2+\alpha}a^{\beta}\cos\frac{\beta\pi}{2}}\right)\geq \operatorname{Arctan}\left(\frac{2\beta}{2+\alpha}\right). \tag{8}$$

Now we consider six cases for estimation of $\operatorname{Arg}\left(p(z_0)\right)$ as follows:

Case 1. $\operatorname{Arg}\left(p(z_0)\right)=\dfrac{\beta\pi}{2}$ and $1-\frac{2}{2+\alpha}a^{\beta}\cos\frac{\beta\pi}{2}>0$. In this case we have $[p(z_0)]^{\frac{1}{\beta}}=ia$ $(a>0)$, and $k\geq 1$. Therefore,

$$\begin{aligned}\operatorname{Arg}\left(\frac{2+\alpha}{\alpha}\left(1-\frac{2}{2+\alpha}\left(p(z_0)+\frac{z_0p'(z_0)}{p(z_0)}\right)\right)\right)&=\operatorname{Arg}\left(1-\frac{2}{2+\alpha}a^{\beta}\cos\frac{\beta\pi}{2}-i\frac{2}{2+\alpha}\left(a^{\beta}\sin\frac{\beta\pi}{2}+k\beta\right)\right)\\&=\operatorname{Arctan}\left(\frac{-\frac{2}{2+\alpha}\left(a^{\beta}\sin\frac{\beta\pi}{2}+k\beta\right)}{1-\frac{2}{2+\alpha}a^{\beta}\cos\frac{\beta\pi}{2}}\right)\\&\leq \operatorname{Arctan}\left(\frac{-\frac{2}{2+\alpha}\left(a^{\beta}\sin\frac{\beta\pi}{2}+\beta\right)}{1-\frac{2}{2+\alpha}a^{\beta}\cos\frac{\beta\pi}{2}}\right)\\&=-\operatorname{Arctan}\left(\frac{\frac{2}{2+\alpha}\left(a^{\beta}\sin\frac{\beta\pi}{2}+\beta\right)}{1-\frac{2}{2+\alpha}a^{\beta}\cos\frac{\beta\pi}{2}}\right)\\&=-h(a)\\&\leq -\operatorname{Arctan}\left(\frac{2\beta}{2+\alpha}\right).\end{aligned} \tag{9}$$

Now applying (8) and (9) we get

$$\begin{aligned}\operatorname{Arg}\left(\frac{2+\alpha}{\alpha}\left(1-\frac{2}{2+\alpha}\left(p(z_0)+\frac{z_0p'(z_0)}{p(z_0)}\right)\right)\right)&=\operatorname{Arg}\left(1-\frac{2}{2+\alpha}\left(p(z_0)+\frac{z_0p'(z_0)}{p(z_0)}\right)\right)\\&=\operatorname{Arg}\left(1-\frac{2}{2+\alpha}\left(1+\frac{z_0f''(z_0)}{f'(z_0)}\right)\right)\\&\leq -\operatorname{Arctan}\left(\frac{\frac{2}{2+\alpha}\left(a^{\beta}\sin\frac{\beta\pi}{2}+\beta\right)}{1-\frac{2}{2+\alpha}a^{\beta}\cos\frac{\beta\pi}{2}}\right)\\&\leq -\operatorname{Arctan}\left(\frac{2\beta}{2+\alpha}\right),\end{aligned}$$

which contradicts (7).

Case 2. $\operatorname{Arg}\left(p(z_0)\right)=\dfrac{\beta\pi}{2}$ and $1-\frac{2}{2+\alpha}a^{\beta}\cos\frac{\beta\pi}{2}=0$. In this case, we have $p(z_0)=a^{\beta}(\cos\frac{\beta\pi}{2}+i\sin\frac{\beta\pi}{2})$ and $k\geq 1$. Thus $-\frac{2}{2+\alpha}\left(a^{\beta}\sin\frac{\beta\pi}{2}+k\beta\right)<0$ and so

$$\begin{aligned}\operatorname{Arg}\left(\frac{2+\alpha}{\alpha}\left(1-\frac{2}{2+\alpha}\left(p(z_0)+\frac{z_0p'(z_0)}{p(z_0)}\right)\right)\right)&=\operatorname{Arg}\left(-i\frac{2}{2+\alpha}\left(a^{\beta}\sin\frac{\beta\pi}{2}+k\beta\right)\right)\\&=-\frac{\pi}{2}<-\operatorname{Arctan}\left(\frac{2\beta}{2+\alpha}\right),\end{aligned}$$

which contradicts (7).

Case 3. $\operatorname{Arg}\left(p(z_0)\right)=\dfrac{\beta\pi}{2}$ and $1-\frac{2}{2+\alpha}a^{\beta}\cos\frac{\beta\pi}{2}<0$. In this case, we have $p(z_0)=a^{\beta}(\cos\frac{\beta\pi}{2}+i\sin\frac{\beta\pi}{2})$ and $k\geq 1$. Thus

$$\frac{-\frac{2}{2+\alpha}\left(a^{\beta}\sin\frac{\beta\pi}{2}+k\beta\right)}{1-\frac{2}{2+\alpha}a^{\beta}\cos\frac{\beta\pi}{2}}>0.$$

Therefore,

$$\begin{aligned}\operatorname{Arg}\left(\frac{2+\alpha}{\alpha}\left(1-\frac{2}{2+\alpha}\left(p(z_0)+\frac{z_0p'(z_0)}{p(z_0)}\right)\right)\right) &= \operatorname{Arg}\left(1-\frac{2}{2+\alpha}a^\beta\cos\frac{\beta\pi}{2}-i\frac{2}{2+\alpha}\left(a^\beta\sin\frac{\beta\pi}{2}+k\beta\right)\right)\\ &= -\pi+\operatorname{Arctan}\left(\frac{-\frac{2}{2+\alpha}\left(a^\beta\sin\frac{\beta\pi}{2}+k\beta\right)}{1-\frac{2}{2+\alpha}a^\beta\cos\frac{\beta\pi}{2}}\right)\\ &< -\pi+\frac{\pi}{2}\\ &= -\frac{\pi}{2}\\ &< -\operatorname{Arctan}\left(\frac{2\beta}{2+\alpha}\right),\end{aligned}$$

which contradicts (7).

Case 4. $\operatorname{Arg}\left(p(z_0)\right)=-\dfrac{\beta\pi}{2}$ and $1-\frac{2}{2+\alpha}a^\beta\cos\frac{\beta\pi}{2}>0$. In this case we have $p(z_0)=a^\beta(\cos\frac{\beta\pi}{2}-i\sin\frac{\beta\pi}{2})$ and $k\le -1$. Thus $-\frac{2}{2+\alpha}\left(-a^\beta\sin\frac{\beta\pi}{2}+k\beta\right)<0$. Now , applying (8) we get

$$\begin{aligned}\operatorname{Arg}\left(\frac{2+\alpha}{\alpha}\left(1-\frac{\alpha}{2+\alpha}\left(p(z_0)+\frac{z_0p'(z_0)}{p(z_0)}\right)\right)\right) &= \operatorname{Arg}\left(1-\frac{2}{2+\alpha}\left(a^\beta e^{\frac{-i\beta\pi}{2}}+ik\beta\right)\right)\\ &= \operatorname{Arctan}\left(\frac{-\frac{2}{2+\alpha}\left(-a^\beta\sin\frac{\beta\pi}{2}+k\beta\right)}{1-\frac{2}{2+\alpha}a^\beta\cos\frac{\beta\pi}{2}}\right)\\ &\ge \operatorname{Arctan}\left(\frac{-\frac{2}{2+\alpha}\left(-a^\beta\sin\frac{\beta\pi}{2}-\beta\right)}{1-\frac{2}{2+\alpha}a^\beta\cos\frac{\beta\pi}{2}}\right)\\ &= \operatorname{Arctan}\left(\frac{\frac{2}{2+\alpha}\left(a^\beta\sin\frac{\beta\pi}{2}+\beta\right)}{1-\frac{2}{2+\alpha}a^\beta\cos\frac{\beta\pi}{2}}\right)\\ &\ge \operatorname{Arctan}\left(\frac{2\beta}{2+\alpha}\right),\end{aligned}$$

which contradicts (7).

For other cases applying the same method in Case 2. and Case 3. with $k\le -1$ we obtain

$$\operatorname{Arg}\left(\frac{2+\alpha}{\alpha}\left(1-\frac{2}{2+\alpha}\left(p(z_0)+\frac{z_0p'(z_0)}{p(z_0)}\right)\right)\right)\ge \operatorname{Arctan}\left(\frac{2\beta}{2+\alpha}\right),$$

which contradicts (7). Hence the proof is completed. □

Corollary 1. *Let* $\alpha,\beta\in(0,1]$ *and* $\delta=\frac{2}{\pi}\operatorname{Arctan}\left(\frac{2\beta}{2+\alpha}\right)$. *If* $f\in\mathcal{G}(\alpha,\delta)$, *then* $f\in\widetilde{\mathcal{S}}^*(\beta)$.

Theorem 2. *Let* $\alpha,\beta\in(0,1]$. *If* $f\in\mathcal{A}$ *and*

$$\left|\operatorname{Arg}\left(\frac{2+\alpha}{\alpha}-\frac{2}{\alpha}\left(1+\frac{zf''(z)}{f'(z)}\right)\right)\right|<\operatorname{Arctan}\left(\frac{2\beta}{\alpha}\right), \tag{10}$$

then

$$\left|\operatorname{Arg}\left(f'(z)\right)\right|<\frac{\beta\pi}{2}\quad(z\in\mathbb{U}).$$

Proof. Define the function $p : \mathbb{U} \to \mathbb{C}$ by

$$p(z) = f'(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \quad (z \in \mathbb{U}).$$

Then p is analytic in $\mathbb{U}$, $p(0) = 1$,

$$1 + \frac{zf''(z)}{f'(z)} = 1 + \frac{zp'(z)}{p(z)}.$$

and $p(z) \neq 0$ for all $z \in \mathbb{U}$. If there exists a point $z_0 \in \mathbb{U}$ such that

$$\left| \operatorname{Arg}\left(p(z)\right) \right| < \frac{\beta\pi}{2},$$

for all $z \in \mathbb{U}$ with $|z| < |z_0|$ and

$$\left| \operatorname{Arg}\left(p(z_0)\right) \right| = \frac{\beta\pi}{2}.$$

Then, Lemma 2, gives us that

$$\frac{z_0 p'(z_0)}{p(z_0)} = ik\beta,$$

where $[p(z_0)]^{\frac{1}{\beta}} = \pm ia \ (a > 0)$ and k is given by (5) or (6).

For the case $\operatorname{Arg}\left(p(z_0)\right) = \dfrac{\alpha\pi}{2}$ when

$$p(z_0)]^{\frac{1}{\beta}} = ia \qquad (a > 0)$$

and $k \geq 1$, we have

$$\begin{aligned}
\operatorname{Arg}\left(\frac{2+\alpha}{\alpha}\left(1 - \frac{2}{2+\alpha}\left(1 + \frac{z_0 p'(z_0)}{p(z_0)}\right)\right)\right) &= \operatorname{Arg}\left(1 - \frac{2}{2+\alpha}\left(1 + \frac{z_0 p'(z_0)}{p(z_0)}\right)\right) \\
&= \operatorname{Arg}\left(1 - \frac{2}{2+\alpha}\left(1 + ik\beta\right)\right) \\
&= \operatorname{Arctan}\left(\frac{-2k\beta}{\alpha}\right) \\
&\leq -\operatorname{Arctan}\left(\frac{2\beta}{\alpha}\right),
\end{aligned}$$

which contradicts (10).

Next, for the case $\operatorname{Arg}\left(p(z_0)\right) = -\dfrac{\alpha\pi}{2}$ when

$$p(z_0) = -ia \qquad (a > 0)$$

and $k \leq -1$, using the same method as before, we can obtain

$$\begin{aligned}
\operatorname{Arg}\left(\frac{2+\alpha}{\alpha}\left(1 - \frac{2}{2+\alpha}\left(1 + \frac{z_0 p'(z_0)}{p(z_0)}\right)\right)\right) &= \operatorname{Arg}\left(1 - \frac{2}{2+\alpha}\left(1 + \frac{z_0 p'(z_0)}{p(z_0)}\right)\right) \\
&= \operatorname{Arg}\left(1 - \frac{2}{2+\alpha}\left(1 + ik\beta\right)\right) \\
&= \operatorname{Arctan}\left(\frac{-2k\beta}{\alpha}\right) \\
&\geq \operatorname{Arctan}\left(\frac{2\beta}{\alpha}\right),
\end{aligned}$$

which is a contradicts (10).

Consequently, from the two above-discussed contradictions, it follows that

$$|\operatorname{Arg}(f'(z))| < \frac{\beta\pi}{2} \quad (z \in \mathbb{U}).$$

and hence the proof is completed. □

Corollary 2. *Let $\alpha, \beta \in (0,1]$ and $\delta = \frac{2}{\pi}\operatorname{Arctan}\left(\frac{2\beta}{\alpha}\right)$. If $f \in \mathcal{G}(\alpha,\delta)$, then $f \in \widetilde{\mathcal{C}}(\beta)$. In other words, if $f \in \mathcal{G}(\alpha,\delta)$, then $f(z)$ is close-to-convex (univalent) in $\mathbb{U}$.*

3. Coefficient Bounds

In this section, we give a the general problem of coefficients in the class $\mathcal{G}(\alpha,\delta)$ like the estimates of coefficients for membership of this, bounds of logarithmic coefficients and the Fekete-Szegö problem with sharp inequalities. In order to achieve our aim we need to establish some knowledge.

Lemma 3 ([27] (p. 172))**.** *Let $\omega \in \Omega$ with $\omega(z) = \sum_{n=1}^{\infty} w_n z^n$ for all $z \in \mathbb{U}$. Then $|w_1| \le 1$ and*

$$|w_n| \le 1 - |w_1|^2 \quad \text{for all } n \in \mathbb{N} \text{ with } n \ge 2.$$

Lemma 4 ([28] (Inequality 7, p. 10))**.** *Let $\omega \in \Omega$ with $\omega(z) = \sum_{n=1}^{\infty} w_n z^n$ for all $z \in \mathbb{U}$. Then*

$$|w_2 - t w_1^2| \le \max\{1, |t|\} \quad \text{for all } t \in \mathbb{C}.$$

The inequality is sharp for the functions $\omega(z) = z^2$ or $\omega(z) = z$.

Lemma 5 ([29])**.** *If $\omega \in \Omega$ with $\omega(z) = \sum_{n=1}^{\infty} w_n z^n$ $(z \in \mathbb{U})$, then for any real numbers q_1 and q_2, we have the following sharp estimate:*

$$|p_3 + q_1 w_1 w_2 + q_2 w_1^3| \le H(q_1; q_2),$$

where

$$H(q_1;q_2) = \begin{cases} 1 & \text{if} \quad (q_1,q_2) \in D_1 \cup D_2 \cup \{(2,1)\}, \\ |q_2| & \text{if} \quad (q_1,q_2) \in \cup_{k=3}^{7} D_k, \\ \frac{2}{3}(|q_1|+1)\left(\frac{|q_1|+1}{3(|q_1|+1+q_2)}\right)^{\frac{1}{2}} & \text{if} \quad (q_1,q_2) \in D_8 \cup D_9, \\ \frac{q_2}{3}\left(\frac{q_1^2-4}{q_1^2-4q_2}\right)\left(\frac{q_1^2-4}{3(q_2-1)}\right)^{\frac{1}{2}} & \text{if} \quad (q_1,q_2) \in D_{10} \cup D_{11} \setminus \{(2,1)\}, \\ \frac{2}{3}(|q_1|-1)\left(\frac{|q_1|-1}{3(|q_1|-1-q_2)}\right)^{\frac{1}{2}} & \text{if} \quad (q_1,q_2) \in D_{12}, \end{cases}$$

and the sets D_k, $k = 1, 2, \ldots, 12$ are stated as given below:

$$D_1 = \left\{(q_1,q_2) : |q_1| \le \frac{1}{2},\ |q_2| \le 1\right\},$$

$$D_2 = \left\{(q_1,q_2) : \frac{1}{2} \le |q_1| \le 2,\ \frac{4}{27}\left((|q_1|+1)^3\right) - (|q_1|+1) \le q_2 \le 1\right\},$$

$$D_3 = \left\{(q_1,q_2) : |q_1| \le \frac{1}{2},\ q_2 \le -1\right\},$$

$$D_4 = \left\{(q_1, q_2) : |q_1| \geq \frac{1}{2},\ |q_2| \leq -\frac{2}{3}(|q_1|+1)\right\},$$
$$D_5 = \{(q_1, q_2) : |q_1| \leq 2,\ q_2 \geq 1\},$$
$$D_6 = \left\{(q_1, q_2) : 2 \leq |q_1| \leq 4,\ q_2 \geq \frac{1}{12}(q_1^2+8)\right\},$$
$$D_7 = \left\{(q_1, q_2) : |q_1| \geq 4,\ q_2 \geq \frac{2}{3}(|q_1|-1)\right\},$$
$$D_8 = \left\{(q_1, q_2) : \frac{1}{2} \leq |q_1| \leq 2,\ -\frac{2}{3}(|q_1|+1) \leq q_2 \leq \frac{4}{27}\left((|q_1|+1)^3\right) - (|q_1|+1)\right\},$$
$$D_9 = \left\{(q_1, q_2) : |q_1| \geq 2,\ -\frac{2}{3}(|q_1|+1) \leq q_2 \leq \frac{2|q_1|(|q_1|+1)}{q_1^2+2|q_1|+4}\right\},$$
$$D_{10} = \left\{(q_1, q_2) : 2 \leq |q_1| \leq 4,\ \frac{2|q_1|(|q_1|+1)}{q_1^2+2|q_1|+4} \leq q_2 \leq \frac{1}{12}(q_1^2+8)\right\},$$
$$D_{11} = \left\{(q_1, q_2) : |q_1| \geq 4,\ \frac{2|q_1|(|q_1|+1)}{q_1^2+2|q_1|+4} \leq q_2 \leq \frac{2|q_1|(|q_1|-1)}{q_1^2-2|q_1|+4}\right\},$$
$$D_{12} = \left\{(q_1, q_2) : |q_1| \geq 4,\ \frac{2|q_1|(|q_1|-1)}{q_1^2-2|q_1|+4} \leq q_2 \leq \frac{2}{3}(|q_1|-1)\right\}.$$

We assume that φ is a univalent function in the unit disk $\mathbb{U}$ satisfying $\varphi(0) = 1$ such that it has the power series expansion of the following form

$$\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \ldots,\ z \in \mathbb{U}, \quad \text{with} \quad B_1 \neq 0. \tag{11}$$

Lemma 6 ([30] (Theorem 2))**.** *Let the function $f \in \mathcal{K}(\varphi)$. Then the logarithmic coefficients of f satisfy the inequalities*

$$|\gamma_1| \leq \frac{|B_1|}{4}, \tag{12}$$

$$|\gamma_2| \leq \begin{cases} \dfrac{|B_1|}{12} & \text{if} \quad |4B_2 + B_1^2| \leq 4|B_1|, \\[2ex] \dfrac{|4B_2 + B_1^2|}{48} & \text{if} \quad |4B_2 + B_1^2| > 4|B_1|, \end{cases} \tag{13}$$

and if B_1, B_2, and B_3 are real values,

$$|\gamma_3| \leq \frac{|B_1|}{24} H(q_1; q_2), \tag{14}$$

where $H(q_1; q_2)$ is given by Lemma 5, $q_1 = \frac{B_1 + \frac{4B_2}{B_1}}{2}$ and $q_2 = \frac{B_2 + \frac{2B_3}{B_1}}{2}$. The bounds (12) and (13) are sharp.

Theorem 3. *Let $f \in \mathcal{G}(\alpha, \delta)$. Then*

$$|a_2| \leq \frac{\alpha\delta}{2}, \quad |a_3| \leq \frac{\alpha\delta}{6}, \quad |a_4| \leq \frac{\alpha\delta}{12} H(q_1; q_2),$$

where $H(q_1; q_2)$ is given by Lemma 5,

$$q_1 = \frac{-3\alpha\delta}{2} + 2\delta \quad \text{and} \quad q_2 = \delta^2\left(\frac{-3\alpha}{2} + \frac{\alpha^2}{2} + \frac{2}{3}\right) + \frac{1}{3}.$$

The first two bounds are sharp.

Proof. Set $g(z) =: zf'(z)$, where $f \in \mathcal{G}(\alpha, \delta)$ and suppose that $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$. Hence $b_n = na_n$ for $n \geq 1$. Then from (4), it follows that

$$\begin{aligned}\frac{zg'(z)}{g(z)} &\prec -\frac{\alpha}{2}\left(\frac{1+z}{1-z}\right)^{\delta} + \frac{2+\alpha}{2} =: \phi(z)\\ &= 1 - \alpha\delta z - \alpha\delta^2 z^2 - \frac{1}{3}\alpha\delta(2\delta^2+1)z^3 + \cdots\\ &:= 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots .\end{aligned}$$

Now, by the definition of the subordination, there is a $\omega \in \Omega$ with $\omega(z) = \sum_{n=1}^{\infty} w_n z^n$ so that

$$\begin{aligned}\frac{zg'(z)}{g(z)} =&\phi(\omega(z))\\ =&1 + B_1 w_1 z + (B_1 w_2 + B_2 w_1^2)z^2 + (B_1 w_3 + 2w_1 w_2 B_2 + B_3 w_1^3)z^3 + \cdots .\end{aligned}$$

From the above equality, it concludes that

$$\begin{cases} b_2 = B_1 w_1 \\ 2b_3 - b_2^2 = B_1 w_2 + B_2 w_1^2 \\ 3b_4 - 3b_2 b_3 + b_2^3 = B_1 w_3 + 2w_1 w_2 B_2 + B_3 w_1^3. \end{cases}$$

First, for b_2, from Lemma 3 we get $|b_2| \leq \alpha\delta$, and so $|a_2| \leq \frac{\alpha\delta}{2}$. Next, utilizing Lemma 3 for b_3 and using $|B_2 + B_1^2| \leq |B_1|$, we have

$$\begin{aligned}|b_3| \leq&\frac{|B_1|(1-|w_1|^2) + |B_2 + B_1^2||w_1|^2}{2}\\ =&\frac{|B_1| + \left[|B_2 + B_1^2| - |B_1|\right]|w_1|^2}{2}\\ \leq&\frac{|B_1|}{2} = \frac{\alpha\delta}{2}.\end{aligned}$$

Ultimately, utilizing Lemma 5 for a_4, we have

$$\begin{aligned}|b_4| \leq&\frac{B_1}{3}\left|c_3 + \left(\frac{3}{2}B_1 + \frac{2B_2}{B_1}\right)w_1 w_2 + \left(\frac{3}{2}B_2 + \frac{1}{2}B_1^2 + \frac{B_3}{B_1}\right)w_1^3\right|\\ \leq&\frac{B_1}{3}H(q_1; q_2),\end{aligned}$$

where

$$q_1 = \frac{3}{2}B_1 + \frac{2B_2}{B_1} = \frac{-3\alpha\delta}{2} + 2\delta \quad \text{and} \quad q_2 = \frac{3}{2}B_2 + \frac{1}{2}B_1^2 + \frac{B_3}{B_1} = \delta^2\left(\frac{-3\alpha}{2} + \frac{\alpha^2}{2} + \frac{2}{3}\right) + \frac{1}{3}.$$

The extremal functions for the initial coefficients a_n $(n = 2, 3)$ are of the form:

$$f_n(z) = \int_0^z \exp\left(\int_0^x \frac{\phi(t^n) - 1}{t}\,\mathrm{d}t\right)\mathrm{d}x = z - \frac{\alpha\beta}{n(n+1)}z^{n+1} + \frac{\alpha\beta^2(\alpha/n - 1)}{2n(2n+1)}z^{2n+1} + \cdots ,$$

obtained by taking $\omega(z) = z^n$ in (4). Therefore, this completes the proof. □

Theorem 4. *Let $f \in \mathcal{G}(\alpha, \delta)$. Then*

$$|\gamma_1| \leq \frac{\alpha\delta}{4}, \quad |\gamma_2| \leq \frac{\alpha\delta}{12}, \quad |\gamma_3| \leq \frac{\alpha\delta}{24}H(q_1; q_2),$$

where $H(q_1;q_2)$ is given by Lemma 5, $q_1 = \frac{-\alpha\delta+4\delta}{2}$, and $q_2 = \frac{-\alpha\delta^2+\frac{2(2\delta^2+1)}{3}}{2}$. The first two bounds are sharp.

Proof. The results are concluded from Theorem 6 by setting $\varphi := \phi$. Also, two first bounds are sharp for $f_n(z)$ for $n = 1,2$, respectively. Therefore, this completes the proof. □

Theorem 5. *Let $f \in \mathcal{G}(\alpha,\delta)$. Then we have sharp inequalities for complex parameter μ*

$$\left|a_3 - \mu a_2^2\right| \leq \begin{cases} \frac{\alpha\delta^2}{6}\left|1-\alpha+\frac{3\mu}{2}\alpha\right| & for \quad \left|\mu+\frac{2}{3\alpha}(1-\alpha)\right| \geq \frac{2}{3\alpha\delta}, \\ \frac{\alpha\delta}{6} & for \quad \left|\mu+\frac{2}{3\alpha}(1-\alpha)\right| < \frac{2}{3\alpha\delta}. \end{cases}$$

Proof. Let $f \in \mathcal{G}(\alpha,\delta)$, then from (4), by the definition of the subordination, there is a $\omega \in \Omega$ with $\omega(z) = \sum_{n=1}^{\infty} w_n z^n$ so that

$$1 + \frac{zf''(z)}{f'(z)} = \phi(\omega(z)) = 1 + B_1 w_1 z + (B_1 w_2 + B_2 w_1^2)z^2 + \cdots .$$

Therefore, we get that

$$2a_2 = B_1 w_1 \qquad \text{and} \qquad 6a_3 - 4a_2^2 = B_1 w_2 + B_2 w_1^2.$$

Form the above equalities, we have

$$\left|a_3 - \mu a_2^2\right| = \frac{1}{6}|B_1|\left|w_2 + \nu w_1^2\right|.$$

The results are obtained by the application of Lemma 4 with $\nu = \left[\frac{B_2}{B_1} + B_1(1-\frac{3\mu}{2})\right]$, where $B_1 = -\alpha\delta$ and $B_2 = -\alpha\delta^2$. Equality is attained in the first inequality by the function $f = f_1$ and in the second inequality for $f = f_2$. □

Remark 1.

(i) *Taking into account $\delta = 1$ in Theorem 3, we get the result obtained in [31] (Theorem 1) for $n = 2,3,4$.*
(ii) *Setting $\delta = 1$ in Theorem 3, we have the result obtained in [23] (Theorem 2.10).*
(iii) *Letting $\delta = 1$ in Theorem 4, we obtain a correction of the result presented in [31] (Theorem 2).*

Author Contributions: Investigation, D.A., N.E.C., E.A.A. and A.M.; Writing—original draft, E.A.A.; Writing—review and editing, N.E.C. The authors contributed equally to this work. All authors have read and agreed to the published version of the manuscript.

Funding: The second author was supported by the Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (No. 2019R1I1A3A01050861).

Acknowledgments: The authors would like to express their gratitude to the referees for many valuable suggestions regarding a previous version of this paper.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Kayumov, I.R. On Brennan's conjecture for a special class of functions. *Math. Notes* **2005**, *78*, 498–502.
2. Milin, I.M. *Univalent Functions and Orthonormal Systems*; Amer Mathematical Society, Translations of Mathematical Monographs: Proovidence, RI, USA, 1977; Volume 49.
3. Milin, I.M. On a property of the logarithmic coefficients of univalent functions. In *Metric Questions in the Theory of Functions*; Naukova Dumka: Kiev, Ukraine, 1980; pp. 86–90. (In Russian)

4. Milin, I.M. On a conjecture for the logarithmic coefficients of univalent functions. *Zap. Nauch. Semin. Leningr. Otd. Mat. Inst. Steklova* **1983**, *125*, 135–143. (In Russian)
5. Robertson, M.S. A remark on the odd-schlicht functions. *Bull. Am. Math. Soc.* **1936**, *42*, 366–370.
6. Bieberbach, L. Über die Koeffizienten derjenigen Potenzreihen, welche eine schlichte Abbildung des Einheitkreises vermitteln. *Sitz. Preuss. Akad. Wiss.* **1916**, *138*, 940–955.
7. De Branges, L. A proof of Bieberbach conjecture. *Acta Math.* **1985**, *154*, 137–152.
8. Pommerenke, C. On the coefficients and Hankel determinant of univalent functions. *J. Lond. Math. Soc.* **1966**, *41*, 111–112.
9. Pommerenke, C. On the Hankel determinants of univalent functions. *Mathematika* **1967**, *14*, 108–112.
10. Vein, R.; Dale, P. Determinants and Their Applications in Mathematical Physics. In *Applied Mathematical Sciences*; Springer: New York, NY, USA, 1999; Volume 134.
11. Fekete, M.; Szegö, G. Eine Bemerkung Über Ungerade Schlichte Funktionen. *J. Lond. Math. Soc.* **1933**, *8*, 85–89.
12. Analouei Adegani, E.; Cho, N.E.; Motamednezhad, A.; Jafari, M. Bi-univalent functions associated with Wright hypergeometric functions. *J. Comput. Anal. Appl.* **2020**, *28*, 261–271.
13. Cho, N.E.; Analouei Adegani, E.; Bulut, S.; Motamednezhad, A. The second Hankel determinant problem for a class of bi-close-to-convex functions. *Mathematics* **2019**, 7, 986.
14. Deniz, E.; Çağlar, M.; Orhan, H. Second Hankel determinant for bi-starlike and bi-convex functions of order β. *Appl. Math. Comput.* **2015**, 271, 301–307.
15. Kanas, S.; Analouei Adegani, E.; Zireh, A. An unified approach to second Hankel determinant of bi-subordinate functions. *Mediterr. J. Math.* **2017**, *14*, 233.
16. Motamednezhad, A.; Bulboacă, T.; Adegani, E.A.; Dibagar, N. Second Hankel determinant for a subclass of analytic bi-univalent functions defined by subordination. *Turk. J. Math.* **2018**, *42*, 2798–2808.
17. Nunokawa, M.; Saitoh, H. On certain starlike functions. *Srikaisekikenkysho Kkyroku* **1996**, *963*, 74–77.
18. Ozaki, S. On the theory of multivalent functions, II. *Sci. Rep. Tokyo Bunrika Daigaku. Sect. A* **1941**, *4*, 45–87.
19. Umezawa, T. Analytic functions convex in one direction. *J. Math. Soc. Jpn.* **1952**, *4*, 194–202.
20. Sakaguchi, K. A property of convex functions and an application to criteria for univalence. *Bull. Nara Univ. Ed. Nat. Sci.* **1973**, *22*, 1–5.
21. Singh, R.; Singh, S. Some sufficient conditions for univalence and starlikeness. *Colloq. Math.* **1982**, *47*, 309–314.
22. Obradović, M.; Ponnusamy, S.; Wirths, K.-J. Logarithmic coeffcients and a coefficient conjecture for univalent functions. *Monatshefte Math.* **2017**, doi:10.1007/s00605-017-1024-3.
23. Ponnusamy, S.; Sharma, N.L.; Wirths. K.-J. Logarithmic coefficients problems in families related to starlike and convex functions. *J. Aust. Math. Soc.* **2019**, in press.
24. Miller, S.S.; Mocanu, P.T. *Differential Subordinations. Theory and Applications*; Marcel Dekker Inc.: New York, NY, USA, 2000; ISBN 0-8247-0029-5.
25. Nunokawa, M. On properties of non-Carathéodory functions. *Proc. Jpn. Acad. Ser. A Math. Sci.* **1992**, *68*, 152–153.
26. Nunokawa, M. On the order of strongly starlikeness of strongly convex functions. *Proc. Jpn. Acad. Ser. A Math. Sci.* **1993** *69*, 234–237.
27. Nehari, Z. *Conformal Mapping*; McGraw-Hill: New York, NY, USA, 1952.
28. Keogh, F.R.; Merkes, E.P. A coefficient inequality for certain classes of analytic functions. *Proc. Am. Math. Soc.* **1969**, *20*, 8–12.
29. Prokhorov, D.V.; Szynal, J. Inverse coefficients for $(\alpha;\beta)$-convex functions. *Ann. Univ. Mariae Curie-Sklodowska Sect. A* **1984**, *35*, 125–143.
30. Analouei Adegani, E.; Cho, N.E.; Jafari M. Logarithmic coefficients for univalent functions defined by subordination. *Mathematics* **2019**, *7*, 1–12.
31. Obradović, M.; Ponnusamy, S.; Wirths, K.-J. Coefficient characterizations and sections for some univalent functions. *Sib. Math. J.* **2013**, *54*, 679–696.