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Article

Fractional q-Difference Inclusions in Banach Spaces

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Abstract: In this paper, we study a class of Caputo fractional q-difference inclusions in Banach spaces. We obtain some existence results by using the set-valued analysis, the measure of noncompactness, and the fixed point theory (Darbo and Mönch's fixed point theorems). Finally we give an illustrative example in the last section. We initiate the study of fractional q-difference inclusions on infinite dimensional Banach spaces.

Keywords: fractional q-difference inclusion; measure of noncompactness; solution; fixed point

1. Introduction

Fractional differential equations and inclusions have attracted much more interest of mathematicians and physicists which provides an efficiency for the description of many practical dynamical arising in engineering, vulnerability of networks (fractional percolation on random graphs), and other applied sciences [1–8]. Recently, Riemann–Liouville and Caputo fractional differential equations with initial and boundary conditions are studied by many authors; [2,9–14]. In [15–18] the authors present some interesting results for classes of fractional differential inclusions.

q-calculus (quantum calculus) has a rich history and the details of its basic notions, results and methods can be found in [19–21]. The subject of q-difference calculus, initiated in the first quarter of 20th century, has been developed over the years. Some interesting results about initial and boundary value problems of ordinary and fractional q-difference equations can be found in [22–27].

Difference inclusions arise in the mathematical modeling of various problems in economics, optimal control, and stochastic analysis, see for instance [28–30]. However q-difference inclusions are studied in few papers; see for example [31,32]. In this article we consider the Caputo fractional q-difference inclusion

$$({}^cD_q^\alpha u)(t) \in F(t, u(t)),\ t \in I := [0, T], \tag{1}$$

with the initial condition

$$u(0) = u_0 \in E, \tag{2}$$

where $(E, \|\cdot\|)$ is a real or complex Banach space, $q \in (0,1)$, $\alpha \in (0,1]$, $T > 0$, $F : I \times E \to \mathcal{P}(E)$ is a multivalued map, $\mathcal{P}(E) = \{Y \subset E : y \neq \emptyset\}$, and ${}^cD_q^\alpha$ is the Caputo fractional q-difference derivative of order α.

This paper initiates the study of fractional q-difference inclusions on Banach spaces.

2. Preliminaries

Consider the Banach space $C(I) := C(I, E)$ of continuous functions from I into E equipped with the supremum (uniform) norm

$$\|u\|_\infty := \sup_{t\in I} \|u(t)\|.$$

As usual, $L^1(I)$ denotes the space of measurable functions $v : I \to E$ which are Bochner integrable with the norm

$$\|v\|_1 = \int_I \|v(t)\| dt.$$

For $a \in \mathbb{R}$, we set

$$[a]_q = \frac{1-q^a}{1-q}.$$

The q-analogue of the power $(a-b)^n$ is

$$(a-b)^{(0)} = 1,\ (a-b)^{(n)} = \Pi_{k=0}^{n-1}(a-bq^k);\ a,b \in \mathbb{R},\ n \in \mathbb{N}.$$

In general,

$$(a-b)^{(\alpha)} = a^\alpha \Pi_{k=0}^{\infty}\left(\frac{a-bq^k}{a-bq^{k+\alpha}}\right);\ a,b,\alpha \in \mathbb{R}.$$

Definition 1 ([21]). *The q-gamma function is defined by*

$$\Gamma_q(\xi) = \frac{(1-q)^{(\xi-1)}}{(1-q)^{\xi-1}};\ \xi \in \mathbb{R} - \{0,-1,-2,\ldots\}$$

Notice that $\Gamma_q(1+\xi) = [\xi]_q\Gamma_q(\xi)$.

Definition 2 ([21]). *The q-derivative of order $n \in \mathbb{N}$ of a function $u : I \to E$ is defined by $(D_q^0 u)(t) = u(t)$,*

$$(D_q u)(t) := (D_q^1 u)(t) = \frac{u(t)-u(qt)}{(1-q)t};\ t \neq 0,\ \ (D_q u)(0) = \lim_{t\to 0}(D_q u)(t),$$

and

$$(D_q^n u)(t) = (D_q D_q^{n-1} u)(t);\ t \in I,\ n \in \{1,2,\ldots\}.$$

Set $I_t := \{tq^n : n \in \mathbb{N}\} \cup \{0\}$.

Definition 3 ([21]). *The q-integral of a function $u : I_t \to E$ is defined by*

$$(I_q u)(t) = \int_0^t u(s)d_q s = \sum_{n=0}^{\infty} t(1-q)q^n f(tq^n),$$

provided that the series converges.

We note that $(D_q I_q u)(t) = u(t)$, while if u is continuous at 0, then

$$(I_q D_q u)(t) = u(t) - u(0).$$

Definition 4 ([33]). *The Riemann–Liouville fractional q-integral of order $\alpha \in \mathbb{R}_+ := [0,\infty)$ of a function $u : I \to E$ is defined by $(I_q^0 u)(t) = u(t)$, and*

$$(I_q^\alpha u)(t) = \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} u(s) d_q s;\ t \in I.$$

Lemma 1 ([34]). *For $\alpha \in \mathbb{R}_+$ and $\lambda \in (-1,\infty)$ we have*

$$(I_q^\alpha (t-a)^{(\lambda)})(t) = \frac{\Gamma_q(1+\lambda)}{\Gamma(1+\lambda+\alpha)}(t-a)^{(\lambda+\alpha)};\ 0 < a < t < T.$$

In particular,

$$(I_q^\alpha 1)(t) = \frac{1}{\Gamma_q(1+\alpha)} t^{(\alpha)}.$$

Definition 5 ([35]). *The Riemann–Liouville fractional q-derivative of order $\alpha \in \mathbb{R}_+$ of a function $u : I \to E$ is defined by $(D_q^0 u)(t) = u(t)$, and*

$$(D_q^\alpha u)(t) = (D_q^{[\alpha]} I_q^{[\alpha]-\alpha} u)(t);\ t \in I,$$

where $[\alpha]$ is the integer part of α.

Definition 6 ([35]). *The Caputo fractional q-derivative of order $\alpha \in \mathbb{R}_+$ of a function $u : I \to E$ is defined by $({}^C D_q^0 u)(t) = u(t)$, and*

$$({}^C D_q^\alpha u)(t) = (I_q^{[\alpha]-\alpha} D_q^{[\alpha]} u)(t);\ t \in I.$$

Lemma 2 ([35]). *Let $\alpha \in \mathbb{R}_+$. Then the following equality holds:*

$$(I_q^\alpha\, {}^C D_q^\alpha u)(t) = u(t) - \sum_{k=0}^{[\alpha]-1} \frac{t^k}{\Gamma_q(1+k)} (D_q^k u)(0).$$

In particular, if $\alpha \in (0,1)$, then

$$(I_q^\alpha\, {}^C D_q^\alpha u)(t) = u(t) - u(0).$$

We define the following subsets of $\mathcal{P}(E)$:

$P_{cl}(E) = \{Y \in \mathcal{P}(E) : Y \text{ is closed}\}$,
$P_{bd}(E) = \{Y \in \mathcal{P}(E) : Y \text{ is bounded}\}$,
$P_{cp}(E) = \{Y \in \mathcal{P}(E) : Y \text{ is compact}\}$,
$P_{cv}(E) = \{Y \in \mathcal{P}(E) : Y \text{ is convex}\}$,
$P_{cp,cv}(E) = P_{cp}(E) \cap P_{cv}(E)$.

Definition 7. *A multivalued map $G : E \to \mathcal{P}(E)$ is said to be convex (closed) valued if $G(x)$ is convex (closed) for all $x \in E$. A multivalued map G is bounded on bounded sets if $G(B) = \cup_{x\in B} G(x)$ is bounded in E for all $B \in P_b(E)$ (i.e. $\sup_{x\in B}\{\sup\{|y| : y \in G(x)\}$ exists).*

Definition 8. *A multivalued map $G : E \to \mathcal{P}(E)$ is called upper semi-continuous (u.s.c.) on E if $G(x_0) \in P_{cl}(E)$; for each $x_0 \in E$, and for each open set $N \subset E$ with $G(x_0) \in N$, there exists an open neighborhood N_0 of x_0 such that $G(N_0) \subset N$. G is said to be completely continuous if $G(B)$ is relatively compact for every $B \in P_{bd}(E)$. An element $x \in E$ is a fixed point of G if $x \in G(x)$.*

We denote by $FixG$ the fixed point set of the multivalued operator G.

Lemma 3 ([28]). *Let $G : X \to \mathcal{P}(E)$ be completely continuous with nonempty compact values. Then G is u.s.c. if and only if G has a closed graph, that is,*

$$x_n \to x_*, y_n \to y_*, y_n \in G(x_n) \Longrightarrow y_* \in G(x_*).$$

Definition 9. *A multivalued map $G : J \to P_{cl}(E)$ is said to be measurable if for every $y \in E$, the function*

$$t \to d(y, G(t)) = \inf\{|y - z| : z \in G(t)\}$$

is measurable.

Definition 10. *A multivalued map $F : I \times \mathbb{R} \to \mathcal{P}(E)$ is said to be Carathéodory if:*

(1) $t \to F(t, u)$ is measurable for each $u \in E$;
(2) $u \to F(t, u)$ is upper semicontinuous for almost all $t \in I$.

F is said to be L^1-Carathéodory if Equations (1) and (2) and the following condition holds:

(3) For each $q > 0$, there exists $\varphi_q \in L^1(I, \mathbb{R}_+)$ such that

$$\|F(t, u)\|_{\mathcal{P}} = \sup\{|v| : v \in F(t, u)\} \leq \varphi_q \text{ for all } |u| \leq q \text{ and for a.e. } t \in I.$$

For each $u \in C(I)$, define the set of selections of F by

$$S_{F \circ u} = \{v \in L^1(I) : v(t) \in F(t, u(t)) \text{ a.e. } t \in I\}.$$

Let (E, d) be a metric space induced from the normed space $(E, |\cdot|)$. The function $H_d : \mathcal{P}(E) \times \mathcal{P}(E) \to \mathbb{R}_+ \cup \{\infty\}$ given by:

$$H_d(A, B) = \max\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(A, b)\}.$$

is known as the Hausdorff-Pompeiu metric. For more details on multivalued maps see the books of Hu and Papageorgiou [28].

Let $\mathcal{M}_X$ be the class of all bounded subsets of a metric space X.

Definition 11. *A function $\mu : \mathcal{M}_X \to [0, \infty)$ is said to be a measure of noncompactness on X if the following conditions are verified for all $B, B_1, B_2 \in \mathcal{M}_X$.*

(a) Regularity, i.e., $\mu(B) = 0$ if and only if B is precompact,
(b) invariance under closure, i.e., $\mu(B) = \mu(\overline{B})$,
(c) semi-additivity, i.e., $\mu(B_1 \cup B_2) = \max\{\mu(B_1), \mu(B_2)\}$.

Definition 12 ([36,37]). *Let E be a Banach space and denote by Ω_E the family of bounded subsets of E. the map $\mu : \Omega_E \to [0, \infty)$ defined by*

$$\mu(M) = \inf\{\epsilon > 0 : M \subset \cup_{j=1}^{m} M_j, \text{diam}(M_j) \leq \epsilon\}, \; M \in \Omega_E,$$

is called the Kuratowski measure of noncompactness.

Theorem 1 ([38]). *Let E be a Banach space. Let $C \subset L^1(I)$ be a countable set with $|u(t)| \leq h(t)$ for a.e. $t \in J$ and every $u \in C$, where $h \in L^1(I, \mathbb{R}_+)$. Then $\phi(t) = \mu(C(t)) \in L^1(I, \mathbb{R}_+)$ and verifies*

$$\mu\left(\left\{\int_0^T u(s)\, ds : u \in C\right\}\right) \leq 2\int_0^T \mu(C(s))\, ds,$$

where μ is the Kuratowski measure of noncompactness on the set E.

Lemma 4 ([39]). *Let F be a Carathéodory multivalued map and* $\Theta : L^1(I) \to C(I)$; *be a linear continuous map. Then the operator*

$$\Theta \circ S_{F \circ u} : C(I) \to \mathcal{P}_{cv,cp}(C(I)), \quad u \mapsto (\Theta \circ S_{F \circ u})(u) = \Theta(S_{F \circ u})$$

is a closed graph operator in $C(I) \times C(I)$.

Definition 13. *Let E be Banach space. A multivalued mapping* $T : E \to \mathcal{P}_{cl,b}(E)$ *is called* $k-$*set-Lipschitz if there exists a constant* $k > 0$, *such that* $\mu(T(X)) \leq k\mu(X)$ *for all* $X \in \mathcal{P}_{cl,b}(E)$ *with* $T(X) \in \mathcal{P}_{cl,b}(E)$. *If* $k < 1$, *then T is called a* $k-$*set-contraction on E.*

Now, we recall the set-valued versions of the Darbo and Mönch fixed point theorems.

Theorem 2 ((Darbo fixed point theorem) [40]). *Let X be a bounded, closed, and convex subset of a Banach space E and let* $T : X \to \mathcal{P}_{cl,b}(X)$ *be a closed and* $k-$*set-contraction. Then T has a fixed point.*

Theorem 3 ((Mönch fixed point theorem) [41]). *Let E be a Banach space and* $K \subset E$ *be a closed and convex set. Also, let U be a relatively open subset of K and* $N : \overline{U} \to \mathcal{P}_c(K)$. *Suppose that N maps compact sets into relatively compact sets,* graph(N) *is closed and for some* $x_0 \in U$, *we have*

$$\text{conv}(x_0 \cup N(M)) \supset M \subset \overline{U} \text{ and } \overline{M} = \overline{U} \ (C \subset M \text{ countable}) \text{ imply } \overline{M} \text{ is compact} \tag{3}$$

and

$$x \notin (1-\lambda)x_0 + \lambda N(x) \quad \forall x \in \overline{U} \backslash U, \ \lambda \in (0,1). \tag{4}$$

Then there exists $x \in \overline{U}$ *with* $x \in N(x)$.

3. Existence Results

First, we state the definition of a solution of the problem found in Equations (1) and (2).

Definition 14. *By a solution of the problem in Equations (1) and (2) we mean a function* $u \in C(I)$ *that satisfies the initial condition in Equation (2) and the equation* $({}^C D_q^\alpha u)(t) = v(t)$ *on I, where* $v \in S_{F \circ u}$.

In the sequel, we need the following hypotheses.

Hypothesis 1. (H_1). *The multivalued map* $F : I \times E \to \mathcal{P}_{cp,cv}(E)$ *is Carathéodory.*

Hypothesis 2. (H_2). *There exists a function* $p \in L^\infty(I, \mathbb{R}_+)$ *such that*

$$\|F(t,u)\|_{\mathcal{P}} = \sup\{\|v\|_C : v(t) \in F(t,u)\} \leq p(t);$$

for a.e. $t \in I$, *and each* $u \in E$,

Hypothesis 3. (H_3). *For each bounded set* $B \subset C(I)$ *and for each* $t \in I$, *we have*

$$\mu(F(t, B(t)) \leq p(t)\mu(B(t)),$$

where $B(t) = \{u(t) : u \in B\}$,

Hypothesis 4. (H_4) *The function* $\phi \equiv 0$ *is the unique solution in* $C(I)$ *of the inequality*

$$\Phi(t) \leq 2p^*(I_q^\alpha \Phi)(t),$$

where p is the function defined in (H_3)*, and*

$$p^* = esssup_{t\in I} p(t).$$

Remark 1. *In* (H_3)*,* μ *is the Kuratowski measure of noncompactness on the space* E*.*

Theorem 4. *If the hypotheses* (H_1)*–*(H_3) *and the condition*

$$L := \frac{p^* T^{(\alpha)}}{\Gamma_q(1+\alpha)} < 1$$

hold, then the problem in Equations (1) and (2) has at least one solution defined on I*.*

Proof. Consider the multivalued operator $N : C(I) \to \mathcal{P}(C(I))$ defined by:

$$N(u) = \left\{ h \in C(I) : h(t) = \mu_0 + \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} v(s) d_q s;\ v \in S_{F\circ u} \right\}. \tag{5}$$

From Lemma 2, the fixed points of N are solutions of the problem in Equations (1) and (2). Set

$$R := \|u_0\| + \frac{p^* T^{(\alpha)}}{\Gamma_q(1+\alpha)},$$

and let $B_R := \{u \in C(I) : \|u\|_\infty \leq R\}$ be the bounded, closed and convex ball of $C(I)$. We shall show in three steps that the multivalued operator $N : B_R \to \mathcal{P}_{cl,b}(C(I))$ satisfies all assumptions of Theorem 2.

Step 1. $N(B_R) \in \mathcal{P}(B_R)$.
Let $u \in B_R$, and $h \in N(u)$. Then for each $t \in I$ we have

$$h(t) = u_0 + \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} v(s) d_q s,$$

for some $v \in S_{F\circ u}$. On the other hand,

$$\begin{aligned} \|h(t)\| &\leq \|u_0\| + \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} \|v(s)\| d_q s \\ &\leq \|u_0\| + \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} p(s) d_q s \\ &\leq \|u_0\| + esssup_{t\in I} p(t) \int_0^T \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} d_q s \\ &= \|u_0\| + \frac{p^* T^{(\alpha)}}{\Gamma_q(1+\alpha)}. \end{aligned}$$

Hence
$\|h\|_\infty \leq R$, and so $N(B_R) \in \mathcal{P}(B_R)$.

Step 2. $N(u) \in \mathcal{P}_{cl}(B_R)$ *for each* $u \in B_R$*.*
Let $\{u_n\}_{n\geq 0} \in N(u)$ such that $u_n \longrightarrow \tilde{u}$ in $C(I)$.Then, $\tilde{u} \in B_R$ and there exists $f_n(\cdot) \in S_{F\circ u}$ be such that, for each $t \in I$, we have

$$u_n(t) = u_0 + \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f_n(s) d_q s.$$

From (H_1), and since F has compact values, then we may pass to a subsequence if necessary to get that $f_n(\cdot)$ converges to f in $L^1(I)$, and then $f \in S_{F\circ u}$. Thus, for each $t \in I$, we get

$$u_n(t) \longrightarrow \tilde{u}(t) = u_0 + \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f(s) d_q s.$$

Hence $\tilde{u} \in N(u)$.

Step 3. *N satisfies the Darbo condition.*
Let $U \subset B_R$, then for each $t \in I$, we have

$$\mu((NU)(t)) = \mu(\{(Nu)(t) : u \in U\}).$$

Let $h \in N(u)$. Then, there exists $f \in S_{F\circ u}$ such that for each $t \in I$, we have

$$h(t) = u_0 + \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f(s) d_q s.$$

From Theorem 1 and since $U \subset B_R \subset C(I)$, then

$$\mu((NU)(t)) \leq 2\int_0^t \mu\left(\left\{\frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f(s) : u \in U\right\}\right) d_q s.$$

Now, since $f \in S_{F\circ u}$ and $u(s) \in U(s)$, we have

$$\mu(\{(t-qs)^{(\alpha-1)} f(s)\}) = (t-qs)^{(\alpha-1)} p(s)\mu(U(s)).$$

Then

$$\mu((NU)(t)) \leq 2\int_0^t \mu\left(\left\{\frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f(s)\right\}\right) d_q s.$$

Thus

$$\mu((NU)(t)) \leq 2p^* \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} \mu(U(s)) d_q s.$$

Hence

$$\mu((NU)(t)) \leq \frac{2p^* T^{(\alpha)}}{\Gamma_q(1+\alpha)} \mu(U).$$

Therefore,

$$\mu(N(U)) \leq L\mu(U),$$

which implies the N is a $L-$set-contraction.
As a consequence of Theorem 2, we deduce that N has a fixed point that is a solution of the problem in Equations (1) and (2). □

Now, we prove an other existence result by applying Theorem 3.

Theorem 5. *If the hypotheses $(H_1) - (H_4)$ hold, then there exists at least one solution of our problem in Equations (1) and (2).*

Proof. Consider the multivalued operator $N : C(I) \to \mathcal{P}(C(I))$ defined in Equation (5). We shall show in five steps that the multivalued operator N satisfies all assumptions of Theorem 3.

Step 1. *$N(u)$ is convex for each $u \in C(I)$.*
Let $h_1, h_2 \in N(u)$, then there exist$v_1, v_2 \in S_{F\circ u}$ such that

$$h_i(t) = \mu_0 + \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} v_i(s) d_q s; \; t \in I, \; i = 1, 2.$$

Let $0 \leq \lambda \leq 1$. Then, for each $t \in I$, we have

$$(\lambda h_1 + (1-\lambda) h_2)(t) = \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} (\lambda v_1(s) + (1-\lambda) v_2(s)) d_q s.$$

Since $S_{F\circ u}$ is convex (because F has convex values), we have $\lambda h_1 + (1-\lambda) h_2 \in N(u)$.

Step 2. *For each compact $M \subset C(I)$, $N(M)$ is relatively compact.*
Let (h_n) be any sequence in $N(M)$, where $M \subset C(I)$ is compact.We show that (h_n) has a convergent subsequence from Arzéla–Ascoli compactness criterion in $C(I)$. Since $h_n \in N(M)$ there are $u_n \in M$ and $v_n \in S_{F\circ u_n}$ such that

$$h_n(t) = \mu_0 + \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} v_n(s) d_q s.$$

Using Theorem 1 and the properties of the measure μ, we have

$$\mu(\{h_n(t)\}) \leq 2 \int_0^t \mu\left(\left\{\frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} v_n(s)\right\}\right) d_q s. \tag{6}$$

On the other hand, since M is compact, the set $\{v_n(s) : n \geq 1\}$ is compact. Consequently, $\mu(\{v_n(s) : n \geq 1\}) = 0$ for a.e. $s \in I$. Furthermore

$$\mu(\{(t-qs)^{(\alpha-1)} v_n(s)\}) = (t-qs)^{(\alpha-1)} \mu(\{v_n(s) : n \geq 1\}) = 0.$$

for a.e. $t, s \in I$. Now Equation (6) implies that $\{h_n(t) : n \geq 1\}$ is relatively compact for each $t \in I$. In addition, for each $t_1, t_2 \in I$; with $t_1 < t_2$, we have

$$\begin{aligned}
&\|h_n(t_2) - h_n(t_1)\| \\
&\leq \left\| \int_0^{t_2} \frac{(t_2-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} p(s) d_q s - \int_0^{t_1} \frac{(t_1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} p(s) d_q s \right\| \\
&\leq \int_{t_1}^{t_2} \frac{(t_2-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} p(s) d_q s \\
&\quad + \int_0^{t_1} \frac{|(t_2-qs)^{(\alpha-1)} - (t_1-qs)^{(\alpha-1)}|}{\Gamma_q(\alpha)} p(s) d_q s \\
&\leq \frac{p^* T^\alpha}{\Gamma_q(1+\alpha)} (t_2 - t_1)^\alpha \\
&\quad + p^* \int_0^{t_1} \frac{|(t_2-qs)^{(\alpha-1)} - (t_1-qs)^{(\alpha-1)}|}{\Gamma_q(\alpha)} d_q s \\
&\quad \to 0 \; as \; t_1 \longrightarrow t_2.
\end{aligned} \tag{7}$$

This shows that $\{h_n : n \geq 1\}$ is equicontinuous. Consequently, $\{h_n : n \geq 1\}$ is relatively compact in $C(I)$.

Step 3. *The graph of N is closed.*
Let $(u_n, h_n) \in \text{graph}(N)$, $n \geq 1$, with$(\|u_n - u\|, \|h_n - h\|) \to (0.0)$, as $n \to \infty$. We have to show

that $(u, h) \in \text{graph}(N)$. $(u_n, h_n) \in \text{graph}(N)$ means that $h_n \in N(u_n)$, which implies that there exists $v_n \in S_{F \circ u_n}$, such that for each $t \in I$,

$$h_n(t) = u_0 + \int_0^t \frac{(t - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} v_n(s) d_q s.$$

Consider the continuous linear operator $\Theta : L^1(I) \to C(I)$,

$$\Theta(v)(t) \mapsto h_n(t) = u_0 + \int_0^t \frac{(t - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} v_n(s) d_q s.$$

Clearly, $\|h_n(t) - h(t)\| \to 0$ as as $n \to \infty$. From Lemma 4 it follows that $\Theta \circ S_F$ is a closed graph operator. Moreover, $h_n(t) \in \Theta(S_{F \circ u_n})$. Since $u_n \to u$, Lemma 4 implies

$$h(t) = u_0 + \int_0^t \frac{(t - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} v(s) d_q s.$$

for some $v \in S_{F \circ u}$.

Step 4. *M is relatively compact in* $C(I)$.
Let $M \subset \overline{U}$; with $M \subset \text{conv}(\{0\} \cup N(M))$, and let $\overline{M} = \overline{C}$; for some countable set $C \subset M$. the set $N(M)$ is equicontinuous from Equation (7). Therefore,

$$M \subset \text{conv}(\{0\} \cup N(M)) \Longrightarrow M \textit{ is equicontinuous.}$$

By applying the Arzéla–Ascoli theorem; the set $M(t)$ is relatively compact for each $t \in I$. Since $C \subset M \subset \text{conv}(\{0\} \cup N(M))$, then there exists a countable set $H = \{h_n : n \geq 1\} \subset N(M)$ such that $C \subset \text{conv}(\{0\} \cup H)$. Thus, there exist $u_n \in M$ and $v_n \in S_{F \circ u_n}$ such that

$$h_n(t) = u_0 + \int_0^t \frac{(t - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} v_n(s) d_q s.$$

From Theorem 1, we get

$$M \subset \overline{C} \subset \overline{conv}(\{0\} \cup H)) \Longrightarrow \mu(M(t)) \leq \mu(\overline{C}(t)) \leq \mu(H(t)) = \mu(\{h_n(t) : n \geq 1\}).$$

Using now the inequality Equation (6) in step 2, we obtain

$$\mu(M(t)) \leq 2 \int_0^t \mu \left(\left\{ \frac{(t - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} v_n(s) \right\} \right) d_q s.$$

Since $v_n \in S_{F \circ u_n}$ and $u_n(s) \in M(s)$, we have

$$\mu(M(t)) \leq 2 \int_0^t \mu \left(\left\{ \frac{(t - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} v_n(s) : n \geq 1 \right\} \right) d_q s.$$

Also, since $v_n \in S_{F \circ u_n}$ and $u_n(s) \in M(s)$, then from (H_3) we get

$$\mu(\{(t - qs)^{(\alpha-1)} v_n(s); n \geq 1\}) = (t - qs)^{(\alpha-1)} p(s) \mu(M(s)).$$

Hence

$$\mu(M(t)) \leq 2p^* \int_0^t \frac{(t - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} \mu(M(s)) d_q s.$$

Consequently, from (H_4), the function Φ given by $\Phi(t) = \mu(M(t))$ satisfies $\Phi \equiv 0$; that is, $\mu(M(t)) = 0$ for all $t \in I$. Finally, the Arzéla–Ascoli theorem implies that M is relatively compact in $C(I)$.

Step 5. *The priori estimate.*
Let $u \in C(I)$ such that $u \in \lambda N(u)$ for some $0 < \lambda < 1$. Then

$$u(t) = \lambda u_0 + \lambda \int_0^t \frac{(t - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} v(s) d_q s,$$

for each $t \in I$, where $v \in S_{F\circ u}$. On the other hand,

$$\begin{aligned} \|u(t)\| &\leq \|u_0\| + \int_0^t \frac{(t - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} \|v(s)\| d_q s \\ &\leq \|u_0\| + \int_0^t \frac{(t - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} p(s) d_q s \\ &\leq \|u_0\| + \frac{p^* T^{(\alpha)}}{\Gamma_q(1+\alpha)}. \end{aligned}$$

Then

$$\|u\| \leq \|u_0\| + \frac{p^* T^{(\alpha)}}{\Gamma_q(1+\alpha)} := d.$$

Set

$$U = \{u \in C_\gamma : \|u\| < 1 + d\}.$$

Hence, the condition in Equation (4) is satisfied. Finally, Theorem 3 implies that N has at least one fixed point $u \in C(I)$ which is a solution of our problem in Equations (1) and (2). □

4. An Example

Let

$$E = l^1 = \left\{ u = (u_1, u_2, \dots, u_n, \dots), \sum_{n=1}^{\infty} |u_n| < \infty \right\}$$

be the Banach space with the norm

$$\|u\|_E = \sum_{n=1}^{\infty} |u_n|.$$

Consider now the following problem of fractional $\frac{1}{4}-$difference inclusion

$$\begin{cases} ({}^c D_{\frac{1}{4}}^{\frac{1}{2}} u_n)(t) \in F_n(t, u(t));\ t \in [0, e], \\ u(0) = (1, 0, \dots, 0, \dots), \end{cases} \tag{8}$$

where

$$F_n(t, u(t)) = \frac{t^2 e^{-4-t}}{1 + \|u(t)\|_E} [u_n(t) - 1, u_n(t)];\ t \in [0, e],$$

with $u = (u_1, u_2, \dots, u_n, \dots)$. Set $\alpha = \frac{1}{2}$, and $F = (F_1, F_2, \dots, F_n, \dots)$.
For each $u \in E$ and $t \in [0, e]$, we have

$$\|F(t, u)\|_{\mathcal{P}} \leq p(t),$$

with $p(t) = t^2 e^{-t-4}$. Hence, the hypothesis (H_2) is satisfied with $p^* = e^{-2}$. A simple computation shows that conditions of Theorem 5 are satisfied. Hence, the problem in Equation (8) has at least one solution defined on $[0, e]$.

5. Conclusions

We have provided some sufficient conditions guaranteeing the existence of solutions for some fractional q-difference inclusions involving the Caputo fractional derivative in Banach spaces. The achieved results are obtained using the fixed point theory and the notion of measure of noncompactness. Such notion requires the use of the set-valued analysis conditions on the right-hand side, like the upper semi-continuity. In the forthcoming paper we shall provide sufficient conditions ensuring the existence of weak solutions by using the concept measure of weak noncompactness, the Pettis integration and an appropriate fixed point theorem.

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