mathematics

MDPI

Article

On the Diophantine Equation $z(n) = (2 - 1/k)\,n$ Involving the Order of Appearance in the Fibonacci Sequence

Eva Trojovská

Department of Mathematics, Faculty of Science, University of Hradec Králové, 500 03 Hradec Králové, Czech Republic; eva.trojovska@uhk.cz; Tel.: +42-049-333-2859

Received: 23 December 2019; Accepted: 13 January 2020; Published: 14 January 2020

Abstract: Let $(F_n)_{n\geq 0}$ be the sequence of the Fibonacci numbers. The order (or rank) of appearance $z(n)$ of a positive integer n is defined as the smallest positive integer m such that n divides F_m. In 1975, Sallé proved that $z(n) \leq 2n$, for all positive integers n. In this paper, we shall solve the Diophantine equation $z(n) = (2 - 1/k)n$ for positive integers n and k.

Keywords: diophantine equation; asymptotic; Fibonacci numbers; order (rank) of appearance; p-adic valuation

MSC: primary 11Dxx; 11B39; secondary 11A41; 11Y70

1. Introduction

Undoubtedly one of the most famous sequences of integer numbers is the sequence $(F_n)_{n\geq 0}$ of the Fibonacci numbers given for $n \geq 0$ by $F_{n+2} = F_{n+1} + F_n$, with $F_0 = 0$ and $F_1 = 1$. The Fibonacci numbers have many amazing properties (see [1–4] together with their very extensive annotated bibliography for further references). Many prominent mathematicians have dealt with divisibility properties of the Fibonacci numbers but many questions remain unanswered, e.g., it is an open problem if there exist infinitely many primes in the Fibonacci sequence (we recommend [5,6]). Further, we note that the p-adic order (the exponent of the highest power of a prime number p which divides n is called the p-adic order of n and it is denoted by $\nu_p(n)$) of Fibonacci numbers has been completely characterized by Halton [7] and Lengyel [8] (see some generalizations and applications in [9–14]). The *order (or rank) of appearance* of a positive integer n in the Fibonacci sequence, denoted by $z(n)$, is defined as the smallest natural number m, such that $n \mid F_m$ (sometimes it is called *order of apparition*, or *Fibonacci entry point*), see Table 1.

Table 1. Values of $z(n)$ for $1 \leq n \leq 15$.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	11	12	13	14	15
$z(n)$	1	3	4	6	5	12	8	6	12	15	10	12	7	24	20	12	9	12	18	30

The function $z(n)$ can be implemented in `Mathematica` [15] as

```
z[n_]:=Catch[Do[i;If[Mod[Fibonacci[i],n]==0,Throw[i]],{i,2*n}]].
```

Using Mathematica we can easily get Figure 1, from which it can be seen that the values of $z(n)$ have the upper bounds on the line $y = 2n$ and the lower bounds on the line $y = 0$, with respect to an elementary fact, that $z(F_n) = n$, for $n > 2$ and the well-known fact, that

$$\lim_{n\to\infty} \frac{n}{F_n} = 0\,.$$

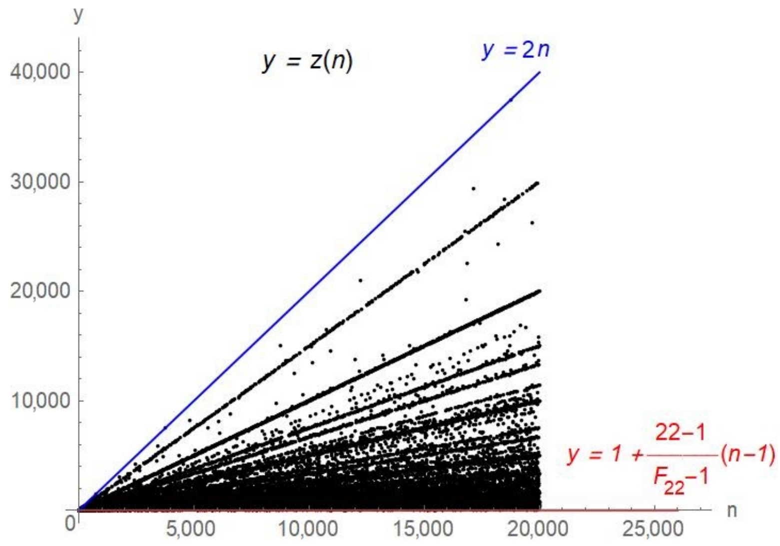

Figure 1. The graph of $z(n)$ for $n \in [1,\ 20{,}000]$, with the upper bounds on the blue line $y = 2n$ and the lower bounds on the red line $y = 1 + (22-1)/(F_{22}-1)(n-1)$.

There are many results about $z(n)$ in the literature. Some advanced properties of $z(n)$ can be found e.g., in [16–19]. Marques [20,21] and Luca and Pomerance [22] investigated a local behavior of the order of appearance in the Fibonacci sequence. Subsequently, considerable efforts were made to find the values of $z(V(n))$, where $V(n)$ is a certain expression containing operations of addition, subtraction, multiplication, as well as powers of Fibonacci and Lucas numbers. Marques [23], Marques and Trojovský [24], and Khaochim and Pongsriiam [25] studied $z(\prod_{i=0}^{k} L_{n+i})$ for some values of k. Similarly, $z(\prod_{i=0}^{k} F_{n+i})$ was found by Marques [26] and Khaochim and Pongsriiam [27] and the values of $z(F_n^k)$ and $z(L_n^k)$ were derived by Marques [28] and Khaochim and Pongsriiam [29]. Trojovský [30,31] found for distinct positive integers $m, n < m$ the values of $z(L_n - L_m)$ and $z(F_m \pm F_n)$, when $m \equiv n \pmod 4$ and $m \equiv n \pmod 2$, respectively. We will also mention a few results regarding the upper bounds for $z(n)$. In 1878, Lucas showed, as an immediate consequence of the Théorème Fondamental of Section XXVI in ([32], p. 300), that $z(n) < \infty$ for all $n \geq 1$. We remark that there is not a closed formula for the $z(n)$, and therefore, Diophantine equations related to $z(n)$ play an important role in its best comprehension. This function gained great interest in 1992, when Sun and Sun [33] proved that to show that all solutions of the Diophantine equation $z(n) = z(n^2)$ are composite numbers, implies Fermat's last theorem. However, it is still known that there are no prime solutions when $n < 3.23 \cdot 10^{15}$ (PrimeGrid Project, May 2017). Recently, deep interest has been shown in investigating some Diophantine equations containing $z(n)$. Independently, Somer and Křížek [34] and Marques [35] showed that all solutions of the Diophantine equation $z(n) = n$ (thus, all fixed points of the function $z(n)$), have the form $n = 5^k$ or $12 \cdot 5^k$, for any integer $k \geq 0$. Lehmer [36] (see Theorem 5.1) proved that all solutions of the equations $z(n) = n + 1$ and $z(n) = n - 1$ are primes. A generalization of these results was studied by Trojovsk y [37], as he considered the Diophantine equation $z(n) = n + \ell$, with $|\ell| \in \{1, 2, \ldots, 9\}$. For instance, it was proved that for $\ell = 2$, the only solution is $n = 4$, and for $\ell = 4$ no solution exists.

Concerning upper bounds for $z(n)$, one can apply the Dirichlet's box principle to the sequence $((F_k, F_{k+1}) \pmod n)_{k \geq 0}$ (sequence of ordered pairs modulo n, so it has at most n^2 distinct terms), to obtain that n must divide F_m, for some $m \leq (n-1)^2 + 1$, in particular, $z(n) \leq (n-1)^2 + 1$ (see [2], Theorem, p. 52). For a prime p there is a better upper bound for $z(p)$, as $z(p) \leq p + 1$.

In 1975, Sallé [38] proved that $z(n) \leq 2n$, for all natural numbers n. The value $2n$ is the sharpest upper bound for $z(n)$, since, e.g., $z(6) = 12$ (Savin [39] showed that $z(p) \mid (p+1)/2$ holds for prime numbers $p \equiv 13, 17 \pmod{20}$, hence $z(p) \leq (p+1)/2$). Actually, proceeding along the same lines as the proof of Theorem 1.1 of [35], one obtains that

$$z(n) = 2n \quad \text{if and only if} \quad n = 6 \cdot 5^k, \text{ for } \ k \geq 0. \tag{1}$$

Thus, the Diophantine equation $z(n) = 2n$ is completely solved. Now, we can think about the related version of this equation, i.e., $z(n) = (2 - 1/k)\, n$. Since $2 - 1/k$ tends to 2 as $k \to \infty$, the following question arises: What other possible solutions could appear?

In this paper, we shall answer completely this question by proving Theorem 1.

Theorem 1. *The only solutions of the Diophantine equation*

$$z(n) = \left(2 - \frac{1}{k}\right) n \tag{2}$$

in positive integers n and k are

$$(k, n) \in \{(1, 5^a), (1, 12 \cdot 5^a), (2, 2 \cdot 5^a), (2, 4 \cdot 5^a)\},$$

for all positive integers a.

It is important to remark that the appearance of the power of 5, in the solutions of Equations (1) and (2), is not a coincidence. Indeed, it comes from the strong relation between the Fibonacci numbers and the number 5. This can be seen in Binet's formula $F_n = (\alpha^n - \beta^n)/\sqrt{5}$ (where $\alpha = (1+\sqrt{5})/2$ and $\beta = -1/\alpha$) which reflects in the powerful fact that the 5-adic valuation of F_n and n are the same (in fact, the prime 5 is the only one with this property, see [8]).

2. Auxiliary Results

As mentioned before, in 1975, Sallé provided the sharpest upper bound for $z(n)$, namely, $z(n) \leq 2n$, where the equality holds if and only if $n = 6 \cdot 5^k$, for all integers $k \geq 0$. However, apart from these cases this upper bound is very weak. For instance, $z(3731) = 280$, thus the smallest upper bound of $z(3731)$ is approximately only 4% of that upper bound $2n = 2 \cdot 3731 = 7462$. In fact, Marques [40] gave sharper upper bounds for $z(n)$ for all positive integers n which are not in the form $6 \cdot 5^k$, where k is any positive integer.

Marques' theorems from [40] will be essential ingredients in our proof. Therefore, we shall present his results as lemmas (in what follows, we denote by $\omega(n)$ and $\nu_2(n)$ the number of distinct prime factors of n and the 2-adic valuation of n, respectively).

Lemma 1. *(Theorem 1.1 of [40]) We have*

(i) $z(2^k) = 3 \cdot 2^{k-2}$ *(for $k \geq 3$),* $z(3^k) = 4 \cdot 3^{k-1}$ *(for $k \geq 1$) and* $z(5^k) = 5^k$ *(for $k \geq 0$).*

(ii) *If $p > 5$ is a prime, then*

$$z(p^k) \leq \left(p - \left(\frac{5}{p}\right)\right) p^{k-1}, \text{ for } k \geq 1,$$

where, as usual, $\left(\frac{a}{q}\right)$ denotes the Legendre symbol of a with respect to a prime $q > 2$.

For the cases when $\omega(n) \geq 2$, Marques [40] proved that

Lemma 2. *(Theorem 1.2 of [40]) Let n be an odd integer with $\omega(n) \geq 2$, then*

$$z(n) \leq 2 \cdot \left(\frac{2}{3}\right)^{\omega(n) - \delta_n} n,$$

where

$$\delta_n = \begin{cases} 0, & \text{if } 5 \nmid n; \\ 1, & \text{if } 5 \mid n. \end{cases}$$

Lemma 3. *(Theorem 1.3 of [40]) Let n be an even integer with $\omega(n) \geq 2$, we have that*

(i) *If $\nu_2(n) \geq 4$, then*

$$z(n) \leq \frac{3}{4} \cdot \left(\frac{2}{3}\right)^{\omega(n)-\delta_n-1} n.$$

(ii) *If $\nu_2(n) = 1$, then*

$$z(n) \leq \begin{cases} 3n/2, & \text{if } \omega(n) = 2 \text{ and } 5 \mid n; \\ 2n, & \text{if } \omega(n) = 2 \text{ and } 5 \nmid n; \\ 3 \cdot (2/3)^{\omega(n)-\delta_n-1} n, & \text{if } \omega(n) > 2. \end{cases}$$

(iii) *If $\nu_2(n) \in \{2, 3\}$, then*

$$z(n) \leq \begin{cases} 3n/2, & \text{if } \omega(n) = 2 \text{ and } 5 \mid n; \\ n, & \text{if } \omega(n) = 2 \text{ and } 5 \nmid n; \\ (2/3)^{\omega(n)-\delta_n-2} n, & \text{if } \omega(n) > 2. \end{cases}$$

Our last tool is a relation between $z(n)$ and $z(p^a)$ for all prime powers dividing n. A proof of this fact can be found in [41].

Lemma 4. *(Theorem 3.3 of [41]) Let $n > 1$ have the prime factorization $n = p_1^{a_1} \cdots p_k^{a_k}$. Then*

$$z(n) = lcm(z(p_1^{a_1}), \ldots, z(p_k^{a_k})),$$

where $lcm()$ denotes the least common multiple.

As usual, from now on we use the well-known notation $[a, b] = \{a, a+1, \ldots, b\}$, for integers $a < b$. Now we are ready to deal with the proof of our main result.

3. The Proof of the Theorem

For $k = 1$, all the solutions are of the form 5^k or $12 \cdot 5^k$ (see [35]). So, we may assume that $k > 1$, hence $2 - 1/k \geq 3/2$ in the rest of the proof. Therefore, in our case, we have that $z(n) \geq 3n/2$. Now, we shall split the proof according to the value of $\omega(n)$.

3.1. The Case $\omega(n) = 1$

In this case, we have that $n = p^a$, for a prime p. For the primes $p = 2, 3$, and 5, respectively, we obtain, by Lemma 1 (i), that

$$\begin{aligned} 3 \cdot 2^{a-1} &\leq z(2^a) = 3 \cdot 2^{a-2}, \\ 3^{a+1}/2 &\leq z(3^a) = 4 \cdot 3^{a-1}, \\ 3 \cdot 5^a/2 &\leq z(5^a) = 5^a. \end{aligned}$$

Clearly, these three inequalities do not hold for any positive integer a. Thus, we suppose that $p > 5$. By Lemma 1 (ii), we get $3p^a/2 \leq z(p^a) \leq (p+1)p^{a-1}$ which arrives in the contradiction that $p \leq 2$. So, we have no solution for a power of all primes.

3.2. The Case in which n is Odd and $\omega(n) \geq 2$

This case does not provide any solution, since, by Lemma 2, we have

$$3n/2 \leq z(n) \leq 2 \cdot (2/3)^{\omega(n)-\delta_n} n \leq 4n/3.$$

3.3. The Case in which n is Even and $\omega(n) \geq 2$

Here, we shall split the proof according to the value of $\nu_2(n)$.

3.3.1. The Case $\nu_2(n) \geq 4$

For this case, we do not have solution, since, by Lemma 3 (i), we obtain

$$3n/2 \leq z(n) \leq (3/4) \cdot (2/3)^0 n = 3n/4.$$

3.3.2. The Case $\nu_2(n) = 1, \omega(n) = 2$ and $\delta_n = 1$

In this case, we have that $n = 2 \cdot 5^a$ which is a solution for $k = 2$. In fact, we can use Lemma 4 to obtain (we remark that we shall use this result many times in this work. So, in order to avoid unnecessary repetition, we shall omit its citation),

$$(2 - 1/k)\, 2 \cdot 5^a = z(2 \cdot 5^a) = \operatorname{lcm}(z(2), z(5^a)) = 3 \cdot 5^a.$$

Thus, $2 - 1/k = 3/2$ and so $k = 2$.

3.3.3. The Case $\nu_2(n) = 1, \omega(n) = 2$ and $\delta_n = 0$

In this case, we have that $n = 2 \cdot p^a$, for a prime $p \neq 2$ or 5. So, we have

$$(2 - 1/k)\, 2p^a = z(2 \cdot p^a) = \operatorname{lcm}(3, z(p^a)).$$

If $p = 3$, then $z(3^a) = 4 \cdot 3^{a-1}$ and then

$$(2 - 1/k)\, 2 \cdot 3^a = z(2 \cdot 3^a) = \operatorname{lcm}(3, 4 \cdot 3^{a-1}) = 4 \cdot 3^{a-1},$$

where we supposed that $a > 1$ (the case $a = 1$ implies that $n = 6$ but as $z(6) = 2 \cdot 6$ we do not have a solution). Hence $k = 3/4 \notin \mathbb{Z}$. Now, we assume that $p > 5$. Observe that $\operatorname{lcm}(a, b)$ is either ab or at most $ab/2$. So, $\operatorname{lcm}(3, z(p^a))$ is either $3z(p^a)$ or at least $3z(p^a)/2$. In the second case, we use the Lemma 3 to arrive at

$$3p^a \leq (2 - 1/k)2p^a \leq 3z(p^a)/2 \leq 3(p+1)p^{a-1}/2$$

and so $2 \leq (p+1)/p < 2$ which is a contradiction. In the case in which $\operatorname{lcm}(3, z(p^a)) = 3z(p^a)$, after a straightforward computation, we obtain that

$$k = \frac{2p^a}{4p^a - 3z(p^a)}.$$

This implies that $4p^a - 3z(p^a)$ divides $2p^a$. Note that all positive divisors of $2p^a$ are p^k or $2p^k$, for any $k \in [0, a]$. Since $p > 5$, we have

$$4p^a - 3z(p^a) \geq 4p^a - 3(p+1)p^{a-1} = p^{a-1}(p-3) > 2p^{a-1},$$

thus p^a must divide $4p^a - 3z(p^a)$ and then p^a divides $z(p^a)$. However, in the proof of the item (ii) of Lemma 1, it was proved the stronger fact that $z(p^a)$ divides $(p - (5/p))p^{a-1}$ (see [40], p. 235). Hence, we have that $p^a \mid z(p^a) \mid (p - (5/p))p^{a-1}$ which yields that p divides $p - (5/p)$ and so $p = 5$ which contradicts our assumption of $p > 5$. So, we do not have solution in this case.

3.3.4. The Case $\nu_2(n) = 1$ and $\omega(n) > 3$

For this, we have

$$3n/2 \leq z(n) \leq 3 \cdot (2/3)^2 n = 4n/3$$

which does not hold.

3.3.5. The Case $\nu_2(n) = 1, \omega(n) = 3$ and $\delta_n = 0$

This case is exactly as the previous one, since all that matters is that $\omega(n) - \delta_n \geq 2$.

3.3.6. The Case $\nu_2(n) = 1, \omega(n) = 3$ and $\delta_n = 1$

We have $n = 2 \cdot 5^a p^b$, where $p \neq 2, 5$ is a prime. Thus

$$\begin{aligned}(2 - 1/k)\, 2 \cdot 5^a p^b &= z(2 \cdot 5^a p^b) = \mathrm{lcm}(3, 5^a, z(p^b)) = \mathrm{lcm}(\mathrm{lcm}(3, 5^a), z(p^b)) \\ &= \mathrm{lcm}(3 \cdot 5^a, z(p^b)),\end{aligned}$$

where we used that $z(5^a) = 5^a$. Since $\mathrm{lcm}(3 \cdot 5^a, z(p^b))$ is either $3 \cdot 5^a z(p^b)$ or at most $3 \cdot 5^a z(p^b)/2$, the proof of the non-existence of solutions for this case follows exactly along the same lines than in Section 3.3.3.

3.3.7. The Case $\nu_2(n) \in \{2, 3\}$

For this case, by Lemma 3 (iii), we do not have solution when $\omega(n) > 2$ (since $(2/3)^{\omega(n) - \delta_n - 2} < 1$). Clearly, the same holds when $\omega(n) = 2$ and $\delta_n = 0$. So, it remains to study the case in which $\omega(n) = 2$ and $\delta_n = 1$. Then $n = 2^t \cdot 5^a$, where $t \in \{2, 3\}$. Thus

$$(2 - 1/k)\, 2^t \cdot 5^a = z(2^t \cdot 5^a) = \mathrm{lcm}(z(2^t), z(5^a)) = \mathrm{lcm}(6, 5^a) = 6 \cdot 5^a,$$

where we used that $z(4) = z(8) = 6$. Now, observe that $(2 - 1/k)\, 2^t \cdot 5^a = 6 \cdot 5^a$ implies in $(2 - 1/k)\, 2^{t-1} = 3$ for $t \in \{2, 3\}$. This holds only for $k = t = 2$ and so we obtain the family of solutions of the form $n = 4 \cdot 5^a$.

This completes the proof of the main theorem. □

4. Conclusions

In this paper, we dealt with the function $z(n)$, which is known as the order (or rank) of appearance of n in the Fibonacci sequence. This function encodes many properties of Fibonacci numbers and it is related to famous problems in mathematics (such as an elementary proof of the Fermat's last theorem). Here, we used sharper upper bounds for $z(n)$ in order to understand its behavior near the extremal case (i.e., near $2n$). More precisely, we solved completely the Diophantine equation $z(n) = (2 - 1/k)\, n$ in positive integers k and n, by expliciting its four (and unique) families of solutions.

Funding: The author was supported by the Project of Specific Research PrF UHK no. 2116/2019, University of Hradec Králové, Czech Republic.

Conflicts of Interest: The author declares no conflict of interest.

References

1. Koshy, T. *Fibonacci and Lucas Numbers with Applications*; Wiley: New York, NY, USA, 2001.
2. Vorobiev, N. N. *Fibonacci Numbers*; Dover Publications: New York, NY, USA, 2013.

3. Knott, R. Fibonacci Numbers and the Golden Section. The Mathematics Department of the University of Surrey, UK. Available online: Http://www.maths.surrey.ac.uk/hosted-sites/R.Knott/Fibonacci/fib.html (accessed on 2 January 2020).
4. The Fibonacci Association, Official Website. Available online: Https://www.mathstat.dal.ca/fibonacci/ (accessed on 2 January 2020).
5. Brillhart, J.; Montgomery, P.L.; Silverman, R.D. Tables of Fibonacci and Lucas Factorizations. *Math. Comput.* **1988**, *50*, 251–260. [CrossRef]
6. Dubner, H.; Keller, W. New Fibonacci and Lucas Primes. *Math. Comput.* **1999**, *68*, 417–427. [CrossRef]
7. Halton, J.H. On the divisibility properties of Fibonacci numbers. *Fibonacci Q.* **1966**, *4*, 217–240.
8. Lengyel, T. The order of the Fibonacci and Lucas numbers. *Fibonacci Q.* **1995**, *33*, 234–239.
9. Kreutz, A.; Lelis, J.; Marques, D.; Silva, E.; Trojovský, P. The p-adic order of the k-Fibonacci and k-Lucas numbers. *p-Adic Numbers Ultrametric Anal. Appl.* **2017**, *9*, 15–21. [CrossRef]
10. Sanna, C. The p-adic valuation of Lucas sequences. *Fibonacci Q.* **2016**, *54*, 118–124.
11. Marques, D.; Trojovský, P. The p-adic order of some Fibonomial Coefficients. *J. Integer Seq.* **2015**, *18*, 15.3.1.
12. Trojovský, P. The p-adic order of some Fibonomial coefficients whose entries are powers of p. *p-Adic Numbers Ultrametric Anal. Appl.* **2017**, *9*, 228–235. [CrossRef]
13. Phunphayap, P.; Pongsriiam, P. Explicit Formulas for the p-adic Valuations of Fibonomial Coefficients. *J. Integer Seq.* **2018**, *21*, 18.3.1.
14. Pongsriiam, P. The order of appearance of factorials in the Fibonacci sequence and certain Diophantine equations. *Period. Math. Hungar.* **2019**, *79*, 141–156. [CrossRef]
15. Wolfram, S. *The Mathematica Book*, 4th ed.; Wolfram Media/Cambridge University Press: Cambridge, UK, 1999.
16. Cubre, P.; Rouse, J. Divisibility properties of the Fibonacci entry point. *Proc. Am. Math. Soc.* **2014**, *142*, 3771–3785. [CrossRef]
17. Chung, C.L. Some Polynomial Sequence Relations. *Mathematics* **2019**, *7*, 750. [CrossRef]
18. Kim, S. The density of the terms in an elliptic divisibility sequence having a fixed G.C.D. with their indices. *J. Number Theory* **2020**, *207*, 22–41. [CrossRef]
19. Leonetti, P.; Sanna, C. On the greatest common divisor of n and the nth Fibonacci number. *Rocky Mt. J. Math.* **2018**, *48*, 1191–1199. [CrossRef]
20. Marques, D. On integer numbers with locally smallest order of appearance in the Fibonacci sequence. *Int. J. Math. Math. Sci.* **2011**, *2011*, 407643. [CrossRef]
21. Marques, D. The order of appearance of integers at most one away from Fibonacci numbers. *Fibonacci Q.* **2012**, *50*, 36–43.
22. Luca, F.; Pomerance, C. On the local behavior of the order of appearance in the Fibonacci sequence. *Int. J. Number Theory* **2014**, *10*, 915–933. [CrossRef]
23. Marques, D. The order of appearance the product of consecutive Lucas numbers. *Fibonacci Q.* **2013**, *51*, 38–43. [CrossRef]
24. Marques, D.; Trojovský, P. The order of appearance of the product of five consecutive Lucas numbers. *Tatra Mt. Math. Publ.* **2014**, *59*, 65–77. [CrossRef]
25. Khaochim, N.; Pongsriiam, P. The general case on the order of appearance of the product of consecutive Lucas numbers. *Acta Math. Univ. Comenian.* **2018**, *59*, 277–289.
26. Marques, D. The order of appearance of product of consecutive Fibonacci numbers. *Fibonacci Q.* **2012**, *50*, 132–139.
27. Khaochim, N.; Pongsriiam, P. On the order of appearance of the product of Fibonacci numbers. *Contrib. Discret. Math.* **2018**, *13*, 45–62.
28. Marques, D. The order of appearance of powers of Fibonacci and Lucas numbers. *Fibonacci Q.* **2012**, *50*, 239–245.
29. Pongsriiam, P. A complete formula for the order of appearance of the powers of Lucas numbers. *Commun. Korean Math. Soc.* **2016**, *31*, 447–450. [CrossRef]
30. Trojovský, P. The order of appearance of the sum and difference between two Fibonacci numbers. *Asian-Eur. J. Math.* **2019**, *12*, 1950046. [CrossRef]
31. Trojovský, P. On the order of appearance of the difference of two Lucas numbers. *Miskolc Math. Notes* **2018**, *19*, 641–648. [CrossRef]

32. Lucas, E. Théorie des fonctions numériques simplement périodiques. *Am. J. Math.* **1878**, *1*, 289–321. [CrossRef]
33. Sun, Z.H.; Sun, Z.W. Fibonacci numbers and Fermat's last theorem. *Acta Arith.* **1992**, *60*, 371–388. [CrossRef]
34. Somer, L.; Křížek, M. Fixed points and upper bounds for the rank of appearance in Lucas sequences. *Fibonacci Q.* **2013**, *51*, 291–306.
35. Marques, D. Fixed points of the order of appearance in the Fibonacci sequence. *Fibonacci Q.* **2012**, *50*, 346–352.
36. Lehmer, D.H. An extended theory of Lucas' functions. *Ann. Math.* **1930**, *31*, 419–448. [CrossRef]
37. Trojovský, P. On Diophantine equations related to order of appearance in Fibonacci sequence. *Mathematics* **2019**, *7*, 1073. [CrossRef]
38. Sallé, H.J.A. A Maximum value for the rank of apparition of integers in recursive sequences. *Fibonacci Q.* **1975**, *13*, 159–161.
39. Savin, D. About Special Elements in Quaternion Algebras Over Finite Fields. *Adv. Appl. Clifford Algebr.* **2017**, *27*, 1801–1813. [CrossRef]
40. Marques, D. Sharper upper bounds for the order of appearance in the Fibonacci sequence. *Fibonacci Q.* **2013**, *51*, 233–238.
41. Renault, M. Properties of the Fibonacci Sequence Under Various Moduli. Master's Thesis, Wake Forest University, Winston-Salem, NC, USA, 1996. Available online: http://webspace.ship.edu/msrenault/fibonacci/FibThesis.pdf (accessed on 2 January 2020).