Geodesic Vector Fields on a Riemannian Manifold

Sharief Deshmukh, Patrik Peska and Nasser Bin Turki

1 Department of Mathematics, College of science, King Saud University, P.O. Box-2455 Riyadh 11451, Saudi Arabia; nassert@ksu.edu.sa
2 Department of Algebra and Geometry, Palacky University, 77146 Olomouc, Czech Republic; peska@seznam.cz
* Correspondence: shariefd@ksu.edu.sa

Received: 29 December 2019; Accepted: 16 January 2020; Published: 19 January 2020

Abstract: A unit geodesic vector field on a Riemannian manifold is a vector field whose integral curves are geodesics, or in other worlds have zero acceleration. A geodesic vector field on a Riemannian manifold is a smooth vector field with acceleration of each of its integral curves is proportional to velocity. In this paper, we show that the presence of a geodesic vector field on a Riemannian manifold influences its geometry. We find characterizations of n-spheres as well as Euclidean spaces using geodesic vector fields.

Keywords: geodesic vector field; eikonal equation; isometric to sphere; isometric to Euclidean space

MSC: 53C20; 53C21; 53C24

1. Introduction

Let \((M, g)\) be an \(n\)-dimensional Riemannian manifold. We call a smooth vector field \(\xi\) on \(M\) geodesic vector field if

\[
\nabla_\xi \xi = \rho \xi,
\]

where \(\nabla\) is the covariant derivative operator with respect to the Riemannian connection on \((M, g)\) and \(\rho : M \to \mathbb{R}\) is a smooth function called the potential function of the geodesic vector field \(\xi\). If the potential function \(\rho = 0\), then \(\xi\) is called a unit geodesic vector field (as in this case the integral curves of \(\xi\) are geodesics). By a non-trivial geodesic vector field, we mean nonzero geodesic vector field for which the potential function \(\rho \neq 0\). Physically, a geodesic vector field has integral curves with an acceleration vector always proportional to the velocity vector. These fields are connected with generalized Fermi coordinates [1]. Geodesic vector fields naturally appear in many situations as seen in the following examples:

1. On Euclidean space \((\mathbb{R}^n, \langle \cdot, \cdot \rangle)\), the position vector field \(\xi = \sum_{i=1}^{n} u^i \frac{\partial}{\partial u^i}\), satisfies \(\nabla_\xi \xi = \xi\), therefore \(\xi\) is a geodesic vector field with potential function \(\rho = 1\).
2. Consider unit hypersphere \(S^n\) in the Euclidean space \((\mathbb{R}^{n+1}, \langle \cdot, \cdot \rangle)\). Then, the restriction of coordinate vector field \(\frac{\partial}{\partial u^1}\) on \(\mathbb{R}^{n+1}\) to \(S^n\) can be expressed as

\[
\frac{\partial}{\partial u^1} = \xi + \rho N,
\]
where $\rho = \langle \frac{\partial}{\partial r}, N \rangle$, $N$ being unit normal to $S^n$ and $\xi$ is vector field on $S^n$, which is the tangential component of $\frac{\partial}{\partial r}$. Then it is easy to see that on $S^n$, we have $\nabla_\xi \xi = \rho \xi$, that is, $\xi$ is a geodesic vector field on $S^n$.

3. Concircular vector fields on Riemannian manifolds have been introduced by A. Fialkow (cf. [2,3]). A vector field $\xi$ on a Riemannian manifold $(M, g)$ is said to be a concircular vector field if $\nabla_X \xi = \rho X$ for any smooth vector field $X$ on $M$, where $\rho$ is a smooth function on $M$. Thus, a concircular vector field $\xi$ satisfies $\nabla_\xi \xi = \rho \xi$, that is, a concircular vector field $\xi$ is a geodesic vector field. It is well known that concircular vector fields play a vital role in the theory of projective and conformal transformations. Moreover, concircular vector fields have applications in general relativity, as for instance trajectories of time-like concircular fields in the de Sitter space determine the world lines of receding or colliding galaxies satisfying the Weyl hypothesis (cf. [4]). Therefore, we could expect that geodesic vector fields also have the scope of applications in general relativity. For example, global questions about the existence of these vector fields were studied in [5–10].

4. Another interesting example comes from Yamabe solitons (cf. [11,12]). Let $(M, g, \xi, \lambda)$ be an $n$-dimensional Yamabe soliton. Then the soliton field $\xi$ satisfies

$$\frac{1}{2} \mathcal{L}_\xi g = (S - \lambda)g,$$

where $\mathcal{L}_\xi g$ is the Lie-derivative of metric $g$, $S$ is the scalar curvature and $\lambda$ is a constant. If the soliton field $\xi$ is a gradient of a smooth function, then $(M, g, \xi, \lambda)$ is called a gradient Yamabe soliton. On gradient Yamabe soliton the soliton field satisfies $\nabla_\xi \xi = (S - \lambda)\xi$, that is, $\xi$ is a geodesic field with potential function $\rho = S - \lambda$.

5. Recall that an Eikonal equation is a nonlinear partial differential equation

$$\|\nabla u\| = \frac{1}{f},$$

where on a non-compact Riemannian manifold $(M, g)$, which is encountered in problems of wave propagation, where $f$ is a positive function (cf. [13,14]). A straightforward observation shows that, above equation gives $\nabla \nabla u \cdot u = -\frac{1}{f^2} \nabla f$, which on choosing $u = \frac{1}{f}$, gives $\nabla \nabla u \cdot u = u \nabla u$, that is, an Eikonal equation gives a non-trivial geodesic vector field $\nabla u$ with potential function $u$. Note that Eikonal equations are also used in tumor invasion margin on Riemannian manifolds of brain fibers (cf. [15]).
the $n$-sphere $S^n(c)$ as well as the Euclidean space $(\mathbb{R}^n, (,))$ using geodesic vector fields (cf. Theorems 1 and 2).

2. Preliminaries

Let $\xi$ be a geodesic vector field on an $n$-dimensional Riemannian manifold $(M, g)$ with potential function $\rho$. We denote by $\alpha$ the smooth 1-form dual to $\xi$. Then we have

$$da(X,Y) = g(\nabla_X \xi, Y) - g(\nabla_Y \xi, X),$$

(2)

$$\mathcal{L}_\xi g (X,Y) = g(\nabla_X \xi, Y) + g(\nabla_Y \xi, X), \quad X, Y \in \mathfrak{X}(M),$$

(3)

where $\nabla$ is the covariant derivative operator with respect the Riemannian connection on $(M, g)$ and $\mathfrak{X}(M)$ is the Lie algebra of smooth vector fields on $M$. Note that the Lie derivative $\mathcal{L}_\xi g$ is symmetric, while the smooth 2-form $da$ is skew-symmetric, which give a symmetric operator $B$ and a skew-symmetric operator $\psi$ on $M$ defined by

$$\mathcal{L}_\xi g (X,Y) = 2g(BX, Y), \quad da(X,Y) = 2g(\psi X, Y).$$

Then using Equations (2) and (3), we conclude

$$\nabla_X \xi = BX + \psi X, \quad X \in \mathfrak{X}(M).$$

(4)

Using the defining Equation (1) of geodesic vector field in Equation (4), we get

$$B\xi + \psi \xi = \rho \xi.$$  

(5)

The curvature tensor field $R$ and the Ricci tensor $\text{Ric}$ of the Riemannian manifold $(M, g)$, are given by

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z,$$

(6)

and

$$\text{Ric}(X,Y) = \sum_{i=1}^n g(R(e_i, X)Y, e_i),$$

(7)

where $\{e_1, \ldots, e_n\}$ is a local orthonormal frame on $M$. The Ricci operator $Q$ of the Riemannian manifold $(M, g)$ is a symmetric operator defined by

$$g(QX, Y) = \text{Ric}(X, Y), \quad X, Y \in \mathfrak{X}(M).$$

The scalar curvature $S$ of the Riemannian manifold is defined by $S = \text{Tr}Q$ the trace of the Ricci operator $Q$. The gradient $\nabla S$ of the scalar curvature satisfies (cf. [30])

$$\frac{1}{2} \nabla S = \sum_{i=1}^n (\nabla Q)(e_i, e_i),$$

(8)

where the covariant derivative

$$(\nabla Q)(X,Y) = \nabla_X QY - Q\nabla_X Y.$$  

Choosing $Y = Z = \xi$ in Equation (6) and using Equations (1) and (4), we conclude

$$R(X,\xi)\xi = X(\rho) \xi + \rho (BX + \psi X) - (\nabla B)(\xi, X) - (\nabla \psi)(\xi, X)$$

$$- B^2 X - \psi^2 X - (B\psi + \psi B)(X).$$

(9)
Taking $X = e_i$ in above equation and the inner product with $e_i$, on summing the resulting equation over an orthonormal frame $\{e_1, \ldots, e_n\}$, we get

$$
Ric(\xi, \xi) = \xi(\rho) + \rho f - \xi(f) - \|B\|^2 + \|\psi\|^2,
$$

(10)

where $f = \text{Tr} B$ the trace of the symmetric operator $B$, we have used $\text{Tr} \psi = 0$ ($\psi$ being skew-symmetric) and the fact that $B\psi + \psi B$ is a skew-symmetric operator and

$$
\|B\|^2 = \sum_{i=1}^n g(Be_i, Be_i), \|\psi\|^2 = \sum_{i=1}^n g(\psi e_i, \psi e_i).
$$

We associate one more smooth function $h : M \to \mathbb{R}$ on a Riemannian manifold $(M, g)$ to geodesic vector field $\xi$, defined by

$$
h = \frac{1}{2} \|\xi\|^2.
$$

(11)

Then, using Equation (4), we get the following expression for the gradient $\nabla h$ of the smooth function $h$,

$$
\nabla h = B\xi - \psi \xi.
$$

(12)

Note that for a smooth function $F : M \to \mathbb{R}$ on a Riemannian manifold $(M, g)$, the Hessian operator $A_F$ and the Laplacian $\Delta F$ are defined by

$$
A_F X = \nabla_X \nabla F, \quad \Delta F = \text{div} \nabla F,
$$

(13)

where

$$
\text{div} X = \sum_{i=1}^n g(\nabla e_i X, e_i).
$$

The Hessian $\text{Hess}(F)$ is defined by

$$
\text{Hess}(F)(X,Y) = g(A_F X, Y), \quad X,Y \in \mathfrak{X}(M).
$$

(14)

3. A Characterization of Euclidean Spaces

In this section, we use a non-trivial geodesic vector field on a connected Riemannian manifold to find a characterization of the Euclidean spaces. We have seen through Example-1 in the introduction that the Euclidean space $(\mathbb{R}^n, \langle , \rangle)$ admits a geodesic vector field $\xi$ with potential function $\rho$ a constant. Recall that a geodesic vector field $\xi$ with potential function $\rho$ is said to be a non-trivial geodesic vector field if $\xi$ is nonzero and $\rho \neq 0$.

**Theorem 1.** Let $(M, g)$ be an $n$-dimensional complete and connected Riemannian manifold. The following two statements are equivalent:

1. There exists a non-trivial geodesic vector field $\xi$ with potential function $\rho$ with the properties that $\text{Tr} L_\xi g$ is constant along the integral curves of $\xi$ and Ricci curvature $Ric(\xi, \xi)$ satisfies

$$
Ric(\xi, \xi) \geq \frac{1}{4} \|d\alpha\|^2 + \frac{1}{4} \left( \text{Tr} L_\xi g \right) \left( 2\rho - \frac{1}{n} \text{Tr} L_\xi g \right) + \xi(\rho).
$$

2. $(M, g)$ is isometric to Euclidean space $(\mathbb{R}^n, \langle , \rangle)$.

**Proof.** Suppose that $\xi$ is a non-trivial geodesic vector field on the connected Riemannian manifold $(M, g)$, such that $\xi(f) = 0$, where $f = \frac{1}{2} \text{Tr} L_\xi g = \text{Tr} B$ and the Ricci curvature $Ric(\xi, \xi)$ satisfies
Now, as $da(X,Y) = 2g(\psi X, Y)$, we get $\frac{1}{4} \|da\|^2 = \|\psi\|^2$ and the above inequality takes the form

$$\text{Ric}(\xi, \xi) \geq \frac{1}{4} \|\psi\|^2 + f \left( \rho - \frac{1}{n} f \right) + \xi(\rho). \quad (15)$$

Using Equation (10) with $\xi(f) = 0$, we get

$$\text{Ric}(\xi, \xi) = \psi \xi + \rho f - \|B\|^2 + \|\psi\|^2,$$

that is,

$$\|B\|^2 - \frac{1}{n} f^2 = \|\psi\|^2 + f \left( \rho - \frac{1}{n} f \right) + \xi(\rho) - \text{Ric}(\xi, \xi). \quad (16)$$

Now, using the inequality (15) in the above equation, we conclude

$$\|B\|^2 - \frac{1}{n} f^2 \leq 0. \quad (17)$$

However, by Schwartz’s inequality, we have $\|B\|^2 \geq \frac{1}{n} f^2$ and the equality holds if and only if $B = \frac{f}{n} I$. Thus, inequality (17), implies

$$B = \frac{f}{n} I. \quad (18)$$

Using Equations (5) and (18), we conclude

$$\psi \xi = \left( \rho - \frac{f}{n} \right) \xi,$$

and taking the inner product with $\xi$ in the above equation and noting that $\psi$ is skew-symmetric, we get

$$\left( \rho - \frac{f}{n} \right) \|\xi\|^2 = 0.$$

As $\xi$ is non-trivial, $\|\xi\| \neq 0$ and consequently, on connected $M$ above two equations give

$$\rho = \frac{f}{n}, \quad \psi \xi = 0. \quad (19)$$

Combining Equations (12), (18) and (19), we conclude $\nabla h = \frac{f}{n} \xi$, which on using Equation (4), gives

$$A_h X = \frac{1}{n} X(f) \xi + \frac{f}{n} \left( \frac{1}{n} X + \psi X \right).$$

Thus, the Hessian $\text{Hess}(h)$ is given by

$$\text{Hess}(h)(X,Y) = \frac{1}{n} X(f) a(Y) + \frac{f^2}{n^2} g(X,Y) + \frac{f}{n} g(\psi X,Y). \quad (20)$$

Now, using the facts that $\text{Hess}(h)$ is symmetric and the operator $\psi$ is skew-symmetric in above equation, we conclude

$$0 = \frac{1}{n} (X(f)a(Y) - Y(f)a(X)) + \frac{2f}{n} g(\psi X,Y),$$

that is,

$$2f \psi X = a(X) \nabla f - X(f) \xi, \quad X \in \mathfrak{X}(M). \quad (21)$$
Taking \( X = \xi \) in above equation and using Equation \((19)\), we get \( \parallel \xi \parallel^2 \nabla f = \xi(f)\xi = 0 \) by the assumption in the statement. Since, \( \xi \) is non-trivial geodesic vector field, the equation \( \parallel \xi \parallel^2 \nabla f = 0 \) on connected \( M \), implies \( \nabla f = 0 \), that is, \( f \) is a constant. Note that the constant \( f \) has to be a nonzero constant, for if \( f = 0 \), then Equation \((19)\) would imply \( \rho = 0 \), which is a contradiction to the fact that \( \xi \) is a non-trivial geodesic vector field. Using this fact that \( f \) is a nonzero constant in Equation \((21)\), we conclude \( \psi = 0 \). Hence, Equation \((20)\), takes the form

\[
\text{Hess}(h) = cg, \tag{22}
\]

where \( c \) is a nonzero constant. Finally, we observe that the smooth function \( h \) is not a constant, for if not, then the Equation \((12)\), would imply \( \xi = 0 \), a contradiction to the fact that \( \xi \) is a non-trivial geodesic vector field. Hence, Equation \((22)\) on a complete and connected Riemannian manifold \((M, g)\) implies that \((M, g)\) is isometric to the Euclidean space \((\mathbb{R}^n, \langle \cdot, \cdot \rangle)\) (cf. [31], Theorem 1, p. 778, [14]).

Conversely, on the Euclidean space \((\mathbb{R}^n, \langle \cdot, \cdot \rangle)\), we have the position vector field

\[
\xi = \sum_{i=1}^n u^i \frac{\partial}{\partial u^i},
\]

which satisfies \( \nabla_X \xi = X, X \in \mathcal{X}(\mathbb{R}^n) \), where \( \nabla \) is the covariant derivative with respect to the Euclidean connection. Then, it follows that \( \xi \) is the non-trivial geodesic vector field with potential function \( \rho = 1 \) and corresponding operators \( B = I \) and \( \psi = 0 \). Thus, \( f = TrB = n \) is a constant and \( Ric(\xi, \xi) = 0 \), that is, we get

\[
Ric(\xi, \xi) = \parallel \psi \parallel^2 + f \left( \rho - \frac{1}{n} f \right) + \xi(\rho),
\]

which meet the requirements in the statement. \( \square \)

4. A Characterization of \( n \)-Spheres

In this section, we use non-trivial geodesic vector field on a compact and connected Riemannian manifold to find a characterization of a \( n \)-sphere \( S^n(c) \). Indeed we prove the following:

**Theorem 2.** Let \((M, g)\) be an \( n \)-dimensional compact and connected Riemannian manifold of positive Ricci curvature and constant scalar curvature. The following two statements are equivalent:

1. There exists a non-trivial geodesic vector field \( \xi \) with potential function \( \rho \) and Ricci curvature \( Ric(\xi, \xi) \) satisfies

\[
\int_M Ric(\xi, \xi) \geq \int_M \left( \frac{n-1}{4n} \left( TrL_\xi g \right)^2 + \frac{1}{4} \parallel da \parallel^2 \right). \tag{23}
\]

2. \((M, g)\) is isometric to \( n \)-sphere \( S^n(c) \).

**Proof.** Let \( \xi \) be a non-trivial geodesic vector field on an \( n \)-dimensional compact and connected Riemannian manifold \((M, g)\) of constant scalar curvature, with potential function \( \rho \) satisfying the condition in the statement. Since, \( f = TrB = \frac{1}{2} TrL_\xi g \) and \( \parallel \psi \parallel^2 = \frac{1}{4} \parallel da \parallel^2 \), the condition in the statement reads

\[
\int_M Ric(\xi, \xi) \geq \int_M \left( \frac{n-1}{n} f^2 + \parallel \psi \parallel^2 \right). \tag{23}
\]

Using Equation \((4)\), we get \( \text{div} \xi = f \) and consequently,

\[
\text{div} f \xi = \xi(f) + f^2 \quad \text{and} \quad \text{div} \rho \xi = \xi(\rho) + \rho f.
\]

Integrating these equations, we conclude

\[
\int_M \xi(f) = - \int_M f^2 \quad \text{and} \quad \int_M (\xi(\rho) + \rho f) = 0. \tag{24}
\]
Now, integrating Equation (10) and using Equation (24), we get
\[
\int_M \left( \text{Ric}(\xi, \xi) + \|B\|^2 - \|\psi\|^2 - f^2 \right) = 0,
\]
which gives
\[
\int_M \left( \|B\|^2 - \frac{1}{n} f^2 \right) = \int_M \left( \frac{n-1}{n} f^2 + \|\psi\|^2 - \text{Ric}(\xi, \xi) \right). \tag{25}
\]
Next, we use the inequality (23) in the above equation, to conclude
\[
\int_M \left( \|B\|^2 - \frac{1}{n} f^2 \right) \leq 0. \tag{26}
\]
However, by Schwartz’s inequality, we have \( \|B\|^2 \geq \frac{1}{n} f^2 \), that is,
\[
\int_M \left( \|B\|^2 - \frac{1}{n} f^2 \right) \geq 0
\]
and combining this inequality with inequality (26), we conclude
\[
\int_M \left( \|B\|^2 - \frac{1}{n} f^2 \right) = 0.
\]
Thus, using Schwartz’s inequality, we get \( \|B\|^2 = \frac{1}{n} f^2 \) and this equality holds if and only if \( B = \frac{1}{n} f I \). Moreover, Equation (25) implies
\[
\int_M \left( \left( \frac{n-1}{n} \right) f^2 + \|\psi\|^2 - \text{Ric}(\xi, \xi) \right) = 0. \tag{27}
\]
Using \( B = \frac{1}{n} f I \), and following the proof of Theorem 1, through Equations (18)–(21), we conclude
\[
2f\psi X = a(X) \nabla f - (f(\xi)) \xi. \tag{28}
\]
Taking \( X = \xi \) in above equation and using \( \psi \xi = 0 \), we have \( \|\xi\|^2 \nabla f = (\xi(f)) \xi \), which on taking the inner product with \( \nabla f \), gives
\[
\|\xi\|^2 \|\nabla f\|^2 = (\xi(f))^2. \tag{29}
\]
Using a local orthonormal frame \( \{e_1, \ldots, e_r\} \) on \( M \), Equation (28), gives
\[
4f^2 g(\psi e_i, \psi e_j) = g(\alpha(e_i) \nabla f - e_i(f) \xi, \alpha(e_j) \nabla f - e_j(f) \xi)
\]
and summing these equations, leads to
\[
4f^2 \|\psi\|^2 = 2\|\xi\|^2 \|\nabla f\|^2 - 2(\xi(f))^2.
\]
Thus, using Equation (29), we conclude \( f^2 \|\psi\|^2 = 0 \). Note that if \( f = 0 \), then Equation (19), gives \( \rho = 0 \), which is contrary to our assumption that \( \xi \) is non-trivial geodesic vector field. Hence, on connected \( M \) equation \( f^2 \|\psi\|^2 = 0 \) implies that \( \psi = 0 \). Now, Equation (5) transforms to
\[
\nabla_X \xi = f \xi, \quad X \in \mathfrak{X}(M), \tag{30}
\]
which on using Equation (6), gives the following expression for the curvature tensor
\[
R(X, Y) \xi = \frac{1}{n} (X(f) Y - Y(f) X), \quad X, Y \in \mathfrak{X}(M).
\]
We use this equation to find

$$Ric(Y, \xi) = -\frac{n-1}{n}Y(f),$$

which gives

$$Q(\xi) = -\frac{n-1}{n} \nabla f. \quad (31)$$

Since, the scalar curvature $S$ is a constant, we find divergence $\text{div}Q(\xi)$ using Equations (8) and (30), a straightforward computation gives $\text{div}Q(\xi) = \frac{1}{n}S$. Inserting this in Equation (31), we conclude

$$\Delta f = -\frac{S}{n-1} f. \quad (32)$$

Now, the Equation (30), gives $\text{div}^{\xi} = f$, that is,

$$\int_M f = 0.$$

If $f$ is a constant, then above equation would imply $f = 0$, which we have seen above, gives a contradiction. Hence, Equation (32) suggests that the non-constant function $f$ is an eigenfunction of the Laplace operator $\Delta$ on compact $M$ with eigenvalue $\frac{n}{n-1}S$, which confirms that the constant $S > 0$. Moreover, Equation (32), implies

$$\frac{1}{2} \Delta f^2 = f\Delta f + \|\nabla f\|^2 = \|\nabla f\|^2 - \frac{S}{n-1} f^2,$$

which, after integration, gives

$$\int_M \|\nabla f\|^2 = \frac{S}{n-1} \int_M f^2. \quad (33)$$

Next, using $\psi = 0$ in Equation (27), we have

$$\int_M Ric(\xi, \xi) = \frac{n-1}{n} \int_M f^2, \quad (34)$$

and taking the inner product with $\nabla f$ in Equation (31), we conclude

$$Ric(\nabla f, \xi) = -\frac{n-1}{n} \|\nabla f\|^2. \quad (35)$$

Recall that the Bochner’s formula states that

$$\int_M \left( Ric(\nabla f, \nabla f) + \|A_f\|^2 - (\Delta f)^2 \right) = 0. \quad (36)$$

We compute

$$Ric \left( \nabla f + \frac{S}{n-1} \xi, \nabla f + \frac{S}{n-1} \xi \right) = Ric(\nabla f, \nabla f) + \frac{2S}{n-1} Ric(\nabla f, \xi) + \frac{S^2}{(n-1)^2} Ric(\xi, \xi),$$

which on integration and the use of Equations (34)–(36), leads to

$$\int_M Ric \left( \nabla f + \frac{S}{n-1} \xi, \nabla f + \frac{S}{n-1} \xi \right) = \int_M \left( -\|A_f\|^2 + (\Delta f)^2 - \frac{2S}{n} \|\nabla f\|^2 + \frac{S^2}{n(n-1)} f^2 \right).$$
Using Equation (33) in above equation, we get
\[
\int_M \text{Ric} \left( \nabla f + \frac{S}{n-1} \xi, \nabla f + \frac{S}{n-1} \xi \right) = \int_M \left( -\left\| A_f \right\|^2 + (\Delta f)^2 - \frac{S^2}{n(n-1)} f^2 \right).
\] (37)

Note that Equation (32), gives
\[
(\Delta f)^2 - \frac{S^2}{n(n-1)} f^2 = \frac{S^2}{n(n-1)} f^2 = \frac{1}{n} (\Delta f)^2.
\]

Inserting this equation in Equation (37), leads to
\[
\int_M \text{Ric} \left( \nabla f + \frac{S}{n-1} \xi, \nabla f + \frac{S}{n-1} \xi \right) = -\int_M \left( \left\| A_f \right\|^2 - \frac{1}{n} (\Delta f)^2 \right).
\]

In this equation, we use the facts that Ric > 0 and the Schwartz' inequality \( \left\| A_f \right\|^2 \geq \frac{1}{n} (\Delta f)^2 \), to conclude
\[
\nabla f = -\frac{S}{n-1} \xi \text{ and } A_f = \frac{\Delta f}{n} I.
\] (38)

Taking the covariant derivative in the first equation of Equation (38) with respect to \( X \in \mathfrak{X}(M) \) and using Equation (30), we get
\[
\nabla_X \nabla f = -cfX, \quad X \in \mathfrak{X}(M),
\] (39)

where \( c \) is a positive constant given by \( S = n(n-1)c \). Note that, we have ruled out above that \( f \) can be a constant. Hence, the non-constant function \( f \) satisfies the Obata's differential Equation (39) (cf. [26]) and consequently, the Riemannian manifold \((M, g)\) is isometric to the sphere \( S^n(c) \).

Conversely, if \((M, g)\) is isometric to \( S^n(c) \), then the Ricci curvature for any smooth vector field \( X \) on \( S^n(c) \) is given by \( \text{Ric}(X, X) = (n-1)c \left\| X \right\|^2 \). We treat \( S^n(c) \) as hypersurface of the Euclidean space \((R^{n+1}, \langle , \rangle)\) with unit normal vector field \( N \) and the shape operator \( A = -\sqrt{c} I \). Now, choosing a nonzero constant vector field \( w \in \mathfrak{X}(R^{n+1}), \) we express its restriction to the sphere \( S^n(c) \) as \( w = \xi + sN \), where \( \xi \) is tangential component of \( w \) to \( S^n(c) \) and \( s = \langle w, N \rangle \) is the smooth function on \( S^n(c) \). Taking covariant derivative with respect to \( X \in \mathfrak{X}(S^n(c)) \) of the equation \( w = \xi + sN \) and using Gauss and Weingarten formulas for the hypersurface, we get
\[
0 = \nabla_X \xi - \sqrt{c}s(X, \xi)N + X(s)N + \sqrt{c}sX.
\]

Equating tangential and normal components in the above equation, we get
\[
\nabla_X \xi = -\sqrt{c}sX, \quad \nabla s = \sqrt{c}\xi.
\] (40)

The first equation in Equation (40) gives \( \nabla_\xi \xi = \rho \xi \), where \( \rho = -\sqrt{c}s \). This proves that \( \xi \) is a geodesic vector field with potential function \( \rho \). Suppose \( \rho = 0 \), this will mean \( s = 0 \) and consequently, the second equation in Equation (40) will imply that \( \xi = 0 \). Thus, \( w = 0 \) on \( S^n(c) \), but as \( w \) is a constant vector field, we get \( w = 0 \) on \( R^{n+1} \), contrary to our assumption that \( w \) is a nonzero constant vector field. Hence, \( \rho \neq 0 \). Similarly, we can show that \( \xi \) is a nonzero vector field. Hence, \( \xi \) is a non-trivial geodesic vector field on \( S^n(c) \). Next, by second equation in the Equation (40), we have \( c \left\| \xi \right\|^2 = \left\| \nabla s \right\|^2 \), and that
\[
\int_{S^n(c)} \text{Ric}(\xi, \xi) = (n-1)c \int_{S^n(c)} \left\| \xi \right\|^2 = (n-1) \int_{S^n(c)} \left\| \nabla s \right\|^2.
\] (41)
Also, by the first equation in (40), for the geodesic vector field $\xi$, the operators $B$ and $\psi$ are $B = -\sqrt{cs}I$ and $\psi = 0$, and that $f = TrB = -n\sqrt{cs}$. Moreover, using Equation (40), we find $\text{div}\xi = -n\sqrt{cs}$ and $\Delta s = -ncs$. Thus, we get

$$\int_{S^n(c)} \|\nabla s\|^2 = nc \int_{S^n(c)} s^2 = \frac{1}{n} \int_{S^n(c)} f^2.$$

(42)

Finally, using Equations (41) and (42), we conclude

$$\int_{S^n(c)} \text{Ric}(\xi, \xi) = \frac{n-1}{n} \int_{S^n(c)} f^2 = \int_{S^n(c)} \left(\frac{n-1}{n} f^2 + \|\psi\|^2\right),$$

which is the equation in (23), that is, all the requirements in the statement are met. □

Author Contributions: Conceptualization, S.D., P.P. and N.B.T.; methodology, S.D. and P.P.; software, S.D. and N.B.T.; formal analysis, S.D. and P.P.; investigation, S.D. and P.P.; resources, S.D.; data curation, S.D.; writing—original draft preparation, S.D. and P.P.; writing—review and editing, S.D. and N.B.T.; visualization, N.B.T.; supervision, S.D. and P.P.; project administration, N.B.T.; funding acquisition, N.B.T. All authors have read and agreed to the published version of the manuscript.

Funding: The authors extend their appreciations to the Deanship of Scientific Research King Saud University for funding this work through research group no (RG-1440-142).

Conflicts of Interest: The authors declare no conflict of interest.

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