Fractional Supersymmetric Hermite Polynomials

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Abstract: We provide a realization of fractional supersymmetry quantum mechanics of order r , where the Hamiltonian and the supercharges involve the fractional Dunkl transform as a Klein type operator. We construct several classes of functions satisfying certain orthogonality relations. These functions can be expressed in terms of the associated Laguerre orthogonal polynomials and have shown that their zeros are the eigenvalues of the Hermitian supercharge. We call them the supersymmetric generalized Hermite polynomials.

Keywords: orthogonal polynomials; difference-differential operator; supersymmetry

1. Introduction

Supersymmetry relates bosons and fermions on the basis of $\mathbb{Z}_2$ -graded superalgebras [1,2], where the fermionic set is realized in terms of matrices of finite dimension or in terms of Grassmann variables [3]. The supersymmetric quantum mechanics (SUSYQM), introduced by Witten [2], may be generated by three operators Q_- , Q_+ and H satisfying

$$Q_{\pm}^2 = 0, \quad [Q_{\pm}, H] = 0, \quad \{Q_-, Q_+\} = H. \quad (1)$$

Superalgebra (1) corresponds to the case $N = 2$ supersymmetry. The usual construction of Witten's supersymmetric quantum mechanics with the superalgebra (1) is performed by introduction of fermion degrees of freedom (realized in a matrix form, or in terms of Grassmann variables) which commute with bosonic degrees of freedom. Another realization of supersymmetric quantum mechanics, called minimally bosonized supersymmetric quantum [1,4,5], is built by taking the supercharge as the following Dunkl-type operator:

$$Q = \partial_x R + v(x),$$

where $v(x)$ is a superpotential.

The fractional supersymmetric quantum mechanics of order r (FSUYQM) are an extension of the ordinary supersymmetric quantum mechanics for which the $\mathbb{Z}_2$ -graded superalgebras are replaced by a $\mathbb{Z}_r$ -graded superalgebras [3,6,7]. The framework of the fractional supersymmetric quantum mechanics has been shown to be quite fruitful. Amongst many works, we may quote the deformed Heisenberg algebra introduced in connection with parafermionic and parabosonic systems [3,4], the C_λ -extended oscillator algebra developed in the framework of parasupersymmetric quantum mechanics [8], and the generalized Weyl–Heisenberg algebra W_k related to $\mathbb{Z}_k$ -graded supersymmetric quantum mechanics [3].

Note that the construction of fractional supersymmetric quantum mechanics without employment of fermions and parafermions degrees of freedom was started in [4,9,10]. In particular, the idea of realization of fractional supersymmetry in the form as it was presented in [3,8] was initially proposed in [4] and also in [9]. In this work, we develop a fractional supersymmetric quantum of order r without parafermionic degrees of freedom. We essentially use a difference-differential operators generated from a special case of the well known fractional Dunkl transform. We then investigate the characteristics of the (r) -scheme.

The paper is organized as follows. In Section 2, we discuss some of basic properties of the fractional Dunkl transform and we define the generalized Klein operator. In Section 3, we present a realization of the fractional supersymmetric quantum mechanics and we construct a basis involving the generalized Hermite functions that diagonalize the Hamiltonian. In Section 4, we define the associated generalized Hermite polynomials and we provide its weight function and we show that the eigenvalues of the supercharge are the zeros of the associated generalized Hermite polynomials.

2. Preliminaries

Recall that the fractional Dunkl transform on the real line, introduced in [11,12], is both an extension of the fractional Hankel transform and the Fourier transform. For $0 < |\alpha| < \pi$, the fractional Dunkl transform is defined by:

$$\mathcal{F}_\nu^\alpha f(t) = \frac{e^{i(\nu+1/2)(\tilde{\alpha}\pi/2-\alpha)}}{(2|\sin(\alpha)|)^{\nu+1/2}\Gamma(\nu+1/2)} \int_{\mathbb{R}} e^{-i\frac{t^2+x^2}{2\tan\alpha}} \mathcal{E}_\nu\left(\frac{itx}{\sin\alpha}\right) f(x) |x|^{2\nu} dx,$$

where

$$\tilde{\alpha} = \operatorname{sgn}(\sin(\alpha))$$

and

$$\begin{aligned} \mathcal{E}_\nu(x) &:= \mathcal{J}_{\nu-1/2}(ix) + \frac{x}{2\nu+1} \mathcal{J}_{\nu+1/2}(ix), \\ \mathcal{J}_\nu(x) &:= \Gamma(\nu+1) (2/x)^\nu J_\nu(x). \end{aligned}$$

Notice that $J_\nu(x)$ is the standard Bessel function ([13] Ch. 10) and $\Gamma(x)$ is the Gamma function. It is well known that, for $\nu > 0$, the function $\mathcal{E}_\nu(\lambda x)$ is the unique analytic solution of the following system that can be found in [14]:

$$\begin{cases} Y_\nu \mathcal{E}_\nu(\lambda x) = i\lambda \mathcal{E}_\nu(\lambda x), \\ \mathcal{E}_\nu(0) = 1, \end{cases} \quad (2)$$

where Y_ν is the Dunkl operator related to root system A_1 (see ([14] Definition 4.4.2)), which is a differential-difference operator, depending on a parameter $\nu \in \mathbb{R}$ and acting on $C^\infty(\mathbb{R})$ as:

$$Y_\nu := \frac{d}{dx} + \frac{\nu}{x}(1-R), \quad (3)$$

where R is the Klein operator :

$$(Rf)(x) = f(-x). \quad (4)$$

The operator Y_ν is also related by a simple similarity transformation to the Yang–Dunkl operator used in Refs. [1,4,10]. The corresponding Dunkl harmonic oscillator and the annihilation and creation operators take the forms [15]

$$H_\nu = -\frac{1}{2}Y_\nu^2 + \frac{1}{2}x^2 = -\frac{1}{2}\frac{d^2}{dx^2} - \frac{\nu}{x}\frac{d}{dx} + \frac{\nu}{2x^2}(1-R) + \frac{1}{2}x^2, \quad (5)$$

$$A_- = \frac{1}{\sqrt{2}}(Y_\nu + x), \quad A_+ = \frac{1}{\sqrt{2}}(-Y_\nu + x). \quad (6)$$

They satisfy the (anti)commutation relations

$$[A_-, A_+] = 1 + 2\nu R, \quad R^2 = 1, \quad \{A_\pm, R\} = 0, \quad [1, A_\pm] = [1, R] = 0. \quad (7)$$

The generators 1 , $A_\pm$, R , and relations (7) give us a realization of the R -deformed Heisenberg algebra [1,10]. In [9,13], the authors show that the R -deformed algebra is intimately related to parabosons, parafermions [13] and to the $osp(1|2)$ $osp(2|2)$ superalgebras.

From now, we assume that $\nu > 0$. The adjoint Y_ν^* of the Dunkl operators Y_ν with domain $\mathcal{S}(\mathbb{R})$ (the space $\mathcal{S}(\mathbb{R})$ being dense in $L^2(\mathbb{R}, |x|^{2\nu} dx)$) is $-Y_\nu$ and therefore the operator H_ν is self-adjoint, its spectrum is discrete, and the wave functions corresponding to the well-known eigenvalues

$$\lambda_n = n + \nu + \frac{1}{2}, \quad n = 0, 1, 2, \dots \quad (8)$$

are given by

$$\psi_n^{(\nu)}(x) = \gamma_n^{-1/2} e^{-x^2/2} H_n^{(\nu)}(x), \quad (9)$$

where

$$\gamma_n = 2^{2n} \Gamma\left(\left[\frac{n}{2}\right] + 1\right) \Gamma\left(\left[\frac{n+1}{2}\right] + \nu + \frac{1}{2}\right), \quad n = 0, 1, 2, \dots$$

$[x]$ denotes the greatest integer function and $H_n^{(\nu)}(x)$ is the generalized Hermite polynomial introduced by Szegő [15–17] and obtained from Laguerre polynomial $L_n^{(\nu)}(x)$ as follows:

$$\begin{cases} H_{2n}^{(\nu)}(x) = (-1)^n 2^{2n} n! L_n^{(\nu-\frac{1}{2})}(x^2), \\ H_{2n+1}^{(\nu)}(x) = (-1)^n 2^{2n+1} n! x L_n^{(\nu+\frac{1}{2})}(x^2). \end{cases}$$

It is well known that for $\nu > 0$, these polynomials satisfy the orthogonality relations :

$$\int_{\mathbb{R}} H_n^{(\nu)}(x) H_m^{(\nu)}(x) |x|^{2\nu} e^{-x^2} dx = \gamma_n \delta_{nm}. \quad (10)$$

We define the generalized Klein operator K as a special case of the fractional Dunkl transform $\mathcal{F}_\nu^\alpha$ corresponding to $\alpha = \frac{2\pi}{r}$. That is,

$$K = \mathcal{F}_\nu^{\frac{2\pi}{r}}. \quad (11)$$

It is well known that the wave functions $\psi_n^{(\nu)}(x)$ form an orthonormal basis of $L^2(\mathbb{R}, |x|^{2\nu} dx)$ and are also eigenfunctions of the Fourier–Dunkl transform [11,12,15]. In particular, the generalized Klein operator K acts on the wave functions $\psi_n^{(\nu)}(x)$ as:

$$K\psi_n(x) = \varepsilon_r^n \psi_n^{(\nu)}(x), \quad \varepsilon_r = e^{\frac{2i\pi}{r}}.$$

Let us consider the $\mathbb{Z}_r$ -grading structure on the space $L^2(\mathbb{R}, |x|^{2\nu} dx)$ as

$$L^2(\mathbb{R}, |x|^{2\nu} dx) = \bigoplus_{j=0}^{r-1} L_j^2(\mathbb{R}, |x|^{2\nu} dx), \quad (12)$$

where $L_j^2(\mathbb{R}, |x|^{2\nu} dx)$ is a linear subspace of $L^2(\mathbb{R}, |x|^{2\nu} dx)$ generated by the generalized wave functions

$$\{\psi_{nr+j}^{(\nu)}(x) : n = 0, 1, 2, \dots\}.$$

For $j = 0, 1, \dots, r-1$, we denote by Π_j , the orthogonal projection from $L^2(\mathbb{R}, |x|^{2\nu} dx)$ onto its subspace $L_j^2(\mathbb{R}, |x|^{2\nu} dx)$. The action of Π_j on $L^2(\mathbb{R}, |x|^{2\nu} dx)$ can be taken to be

$$\Pi_k \psi_{nr+j}^{(\nu)}(x) = \delta_{kj} \psi_{nr+j}^{(\nu)}(x).$$

It is clear that they form a system of resolution of the identity:

$$\Pi_0 + \Pi_1 + \dots + \Pi_{r-1} = 1, \quad \Pi_i \Pi_j = \delta_{ij} \Pi_i, \quad \Pi_j^* = \Pi_j. \quad (13)$$

Note that the orthogonal projection Π_j is related to the Klein operator K by

$$\Pi_j = \frac{1}{r} \sum_{l=0}^{r-1} \varepsilon_r^{-lj} K^l.$$

3. Fractional Supersymmetric Dunkl Harmonic Oscillator

In this section, we shall present a construction of the fractional supersymmetric quantum mechanics of order r ($r = 2, 3, \dots$) by using the generalized Klein's operator defined in Equation (11). Following Khare [6,7], a fractional supersymmetric quantum mechanics model of arbitrary order r can be developed by generalizing the fundamental Equations (1) to the forms

$$\begin{aligned} Q_{\pm}^r &= 0, \quad [H, Q_{\pm}] = 0, \quad Q_-^{\dagger} = Q_+, \\ Q_-^{r-2} H &= Q_-^{r-1} Q_+ + Q_-^{r-2} Q_+ Q_- + \dots + Q_- Q_+ Q_-^{r-2} + Q_+ Q_-^{r-1}. \end{aligned}$$

We introduce the supercharges Q_- and Q_+ as :

$$Q_- = \frac{1}{\sqrt{2}}(Y_{\nu} + x)(1 - \Pi_0), \quad Q_+ = \frac{1}{\sqrt{2}}(1 - \Pi_0)(-Y_{\nu} + x) \quad (14)$$

and the fractional supersymmetric Dunkl harmonic oscillator $\mathcal{H}_{\nu}$ by

$$\mathcal{H}_{\nu} = -(r-1)\frac{1}{2}Y_{\nu}^2 + (r-1)\frac{1}{2}x^2 - \sum_{k=0}^{r-1} \Theta_k \Pi_{r-k-1}, \quad (15)$$

where

$$\Theta_k = \frac{(r-1)(r-2k)}{2} + 2\nu \left[\frac{2r + (-1)^k - 1}{4} \right] R, \quad k = 0, \dots, r-1, \quad (16)$$

and recall that $[\cdot]$ denotes the greatest integer function. Obviously, the operators $Q_{\pm}$ and $\mathcal{H}_{\nu}$ with common domain $\mathcal{S}(\mathbb{R})$ are densely defined in the Hilbert space $L^2(\mathbb{R}, |x|^{2\nu} dx)$ and have the Hermitian conjugation relations

$$\mathcal{H}_{\nu}^* = \mathcal{H}_{\nu}, \quad Q_-^* = Q_+. \quad (17)$$

Furthermore, they satisfy the intertwining relations valid for $s = 0, \dots, r-1$:

$$\Pi_s Q_- = Q_- \Pi_{s+1}, \quad Q_+ \Pi_s = \Pi_{s+1} Q_+, \quad \mathcal{H}_\nu \Pi_s = \Pi_s \mathcal{H}_\nu. \quad (18)$$

Proposition 1. The supercharges $Q_\pm$ are nilpotent operators of order r .

Proof. By making use of the following relations

$$Y_\nu \Pi_s = \Pi_{s-1} Y_\nu, \quad x \Pi_s = \Pi_{s+1} x, \quad (19)$$

we can easily show by induction that

$$Q_-^k = \begin{cases} A_-^k (1 - \sum_{s=0}^{k-1} \Pi_s), & \text{if } 1 \leq k \leq r-1, \\ 0, & \text{if } k = r. \end{cases} \quad (20)$$

Since $Q_+ = Q_-^*$, we also have $Q_+^r = 0$. $\square$

The first main result is

Theorem 1. The Hermitian operators Q_- , Q_+ and $\mathcal{H}_\nu$ defined in Equations (14) and (15) satisfy the commutation relations:

$$\begin{aligned} (i) \quad & Q_\pm^r = 0, \quad [\mathcal{H}_\nu, Q_\pm] = 0, \quad Q_-^\dagger = Q_+, \\ (ii) \quad & Q_-^{r-2} \mathcal{H}_\nu = Q_-^{r-1} Q_+ + Q_-^{r-2} Q_+ Q_- + \dots + Q_- Q_+ Q_-^{r-2} + Q_+ Q_-^{r-1}. \end{aligned}$$

Proof. From the commutation relation (7), we can show by induction that

$$A_+ A_-^k = A_-^k A_+ - \vartheta_k A_-^{k-1}, \quad k \geq 1, \quad (21)$$

where

$$\vartheta_k = \begin{cases} k, & \text{if } k \text{ is even,} \\ k + 2\nu R, & \text{if } k \text{ is odd.} \end{cases} \quad (22)$$

Combining this with Equation (20), we obtain, for $k = 1, \dots, r-2$:

$$\begin{aligned} Q_+ Q_-^{r-1} &= A_-^{r-2} (A_- A_+ - \vartheta_{r-1}) \Pi_{r-1}, \\ Q_-^{r-1} Q_+ &= A_-^{r-2} A_- A_+ \Pi_{r-2}, \\ Q_-^{r-1-k} Q_+ Q_-^k &= A_-^{r-2} (A_- A_+ - \vartheta_k) (\Pi_{r-2} + \Pi_{r-1}). \end{aligned}$$

Additionally, a straightforward computation shows that

$$\sum_{k=1}^{r-1} \vartheta_k = \frac{r(r-1)}{2} + 2\nu \left[\frac{r}{2} \right] R.$$

Thus, we get

$$\begin{aligned} \sum_{k=0}^{r-1} Q_-^{r-1-k} Q_+ Q_-^k &= A_-^{r-2} [(r-1) A_- A_+ (\Pi_{r-2} + \Pi_{r-1}) - (\sum_{k=1}^{r-2} \vartheta_k) \Pi_{r-2} - (\sum_{k=1}^{r-1} \vartheta_k) \Pi_{r-1}] \\ &= Q_-^{r-2} [(r-1) A_- A_+ - \Theta_1 \Pi_{r-2} - \Theta_0 \Pi_{r-1}]. \end{aligned} \quad (23)$$

From Equation (13), we easily see that

$$(\Pi_{r-2} + \Pi_{r-1}) \sum_{k=2}^{r-1} \Theta_k \Pi_{r-k-1} = 0,$$

and combining with Equation (23), we get

$$\sum_{k=0}^{r-1} Q_-^{r-1-k} Q_+ Q_-^k = Q_-^{r-2} \mathcal{H}_v.$$

It remains to prove that $[\mathcal{H}_v, Q_-] = [\mathcal{H}_v, Q_+] = 0$. Observe that for $k = 0, \dots, r-1$, we have

$$r-1 = \left\lfloor \frac{2r + (-1)^k - 1}{4} \right\rfloor + \left\lfloor \frac{2r + (-1)^{k+1} - 1}{4} \right\rfloor,$$

and then, for $k = 0, \dots, r-2$, we have

$$\Theta_k - (r-1)(1-2\nu R) = \Theta_{k+1}, \quad (24)$$

which leads to

$$\begin{aligned} Q_- \mathcal{H}_v &= \left\{ (r-1)A_- A_+ + 1 - 2\nu R - \sum_{k=0}^{r-2} \Theta_k \Pi_{r-k-2} \right\} A_- (1 - \Pi_0) \\ &= \left\{ (r-1)A_- A_+ - \sum_{k=0}^{r-2} \Theta_{k+1} \Pi_{r-k-2} \right\} (1 - \Pi_{r-1}) A_- \\ &= \mathcal{H}_v Q_-. \end{aligned}$$

Finally, we have obtained $[\mathcal{H}_v, Q_-] = 0$, and since the operator $\mathcal{H}_v$ is self-adjoint and $Q_+ = Q_-^*$, we conclude that $[\mathcal{H}_v, Q_+] = 0$. $\square$

Proposition 2. For even integer r , the fractional supersymmetric Dunkl harmonic oscillator $\mathcal{H}_v$ has $r/2$ -fold degenerate spectrum and acts on the wave functions $\psi_n^{(v)}(x)$ as:

$$\mathcal{H}_v \psi_{nr+s}^{(v)}(x) = \lambda_{nr} \psi_{nr+s}^{(v)}(x), \quad s = 0, 1, \dots, r-1, \quad n = 0, 1, 2, \dots$$

where

$$\lambda_{nr} = (r-1)(nr + \nu + \frac{r+1}{2}) + (-1)^s \nu r, \quad s = 0, \dots, r-1.$$

Proof. From ([15] [formulas (3.7.1) and (3.7.2)]), the creation and annihilation operators A_+ and A_- act on the wave functions $\psi_{nr+s}^{(v)}$ as:

$$\begin{aligned} A_- \psi_{nr+s}^{(v)} &= \sqrt{nr + s + \nu(1 - (-1)^s)} \psi_{nr+s-1}^{(v)}, \\ A_+ \psi_{nr+s}^{(v)} &= \sqrt{nr + s + 1 + \nu(1 - (-1)^{s+1})} \psi_{nr+s+1}^{(v)}. \end{aligned}$$

Then, the supercharges Q_- and Q_+ take the value

$$Q_- \psi_{nr+s}^{(v)} = \sqrt{(nr + s + \nu(1 - (-1)^s))/2} \psi_{nr+s-1}^{(v)}, \quad s = 1, \dots, r-1, \quad (25)$$

$$Q_+ \psi_{nr+s}^{(v)} = \sqrt{(nr + s + 1 + \nu(1 - (-1)^{s+1}))/2} \psi_{nr+s+1}^{(v)}, \quad s = 0, \dots, r-2, \quad (26)$$

$$Q_- \psi_{nr}^{(v)} = 0, \quad Q_+ \psi_{(n+1)r-1}^{(v)} = 0. \quad (27)$$

A straightforward computation shows that

$$\mathcal{H}_\nu \psi_{nr+s}^\nu = \lambda_{nr} \psi_{nr+s}^\nu, \quad s = 0, \dots, r-1,$$

where $\lambda_{nr} = (r-1)(nr + \nu + \frac{r+1}{2}) + (-1)^s \nu r$. $\square$

4. Supersymmetric Generalized Hermite Polynomials

4.1. Associated Generalized Hermite Polynomials

Starting from the following recurrence relations for the generalized Hermite polynomials $\{H_n^{(\nu)}(x)\}$,

$$\begin{aligned} H_{n+1}^{(\nu)}(x) &= 2xH_n^{(\nu)}(x) - 2(n + \nu(1 - (-1)^n))H_{n-1}^{(\nu)}(x) \\ H_0^{(\nu)}(x) &= 1, \quad H_1^{(\nu)}(x) = 2x, \end{aligned} \quad (28)$$

given in [15–17], one defines, for each real number c , the system of polynomials $H_n^{(\nu)}(x, c)$ by the recurrence relation:

$$H_{n+1}^{(\nu)}(x, c) = 2xH_n^{(\nu)}(x, c) - 2(n + c + \nu(1 - (-1)^n))H_{n-1}^{(\nu)}(x, c), \quad (29)$$

with initial conditions

$$H_0^{(\nu)}(x, c) = 1, \quad H_1^{(\nu)}(x, c) = 2x. \quad (30)$$

Now, assume that

$$c > 0, \quad c + 2\nu > -1. \quad (31)$$

By Favard's theorem [16], it follows that the family of polynomials $\{H_n^{(\nu)}(x, c)\}$ satisfying the recurrence relation (29) and the initial condition (30), is orthogonal with respect to some positive measure on the real line. We shall refer to the polynomials $\{H_n^{(\nu)}(x, c)\}$ as the associated generalized Hermite polynomials. As shown in ([18] Theorem 5.6.1)(see also [19–21]), there are two different systems of associated Laguerre polynomials denoted by $L_n^{(\nu)}(x, c)$ and $\mathcal{L}_n^{(\nu)}(x, c)$. They satisfy the recurrence relations:

$$(2n + 2c + \nu + 1 - x)L_n^{(\nu)}(x, c) = (n + c + 1)L_{n+1}^{(\nu)}(x, c) + (n + c + \nu)L_{n-1}^{(\nu)}(x, c), \quad (32)$$

$$L_0^{(\nu)}(x, c) = 1, \quad L_1^{(\nu)}(x, c) = \frac{2c + \nu + 1 - x}{c + 1} \quad (33)$$

and

$$(2n + 2c + \nu + 1 - x)\mathcal{L}_n^{(\nu)}(x, c) = (n + c + 1)\mathcal{L}_{n+1}^{(\nu)}(x, c) + (n + c + \nu)\mathcal{L}_{n-1}^{(\nu)}(x, c), \quad (34)$$

$$\mathcal{L}_0^{(\nu)}(x, c) = 1, \quad \mathcal{L}_1^{(\nu)}(x, c) = \frac{c + \nu + 1 - x}{c + 1}. \quad (35)$$

Recall the Tricomi function $\Psi(a, c; x)$ given by

$$\Psi(a, c; x) = \frac{1}{\Gamma(a)} \int_0^\infty e^{-xt} t^{a-1} (1+t)^{c-a-1} dt, \quad \Re(a), \Re(x) > 0.$$

By [18], the polynomials $L_n^{(\nu)}(x, c)$ and $\mathcal{L}_n^{(\nu)}(x, c)$ satisfy the orthogonality relations

$$\int_0^\infty L_n^{(\nu)}(x, c) L_m^{(\nu)}(x, c) x^\nu e^{-x} \frac{|\Psi(c, 1 - \nu; x e^{-i\pi})|^{-2}}{\Gamma(c+1)\Gamma(\nu+c+1)} dx = \frac{(\nu+c+1)_n}{(c+1)_n} \delta_{nm}, \quad (36)$$

$$\int_0^\infty \mathcal{L}_n^{(\nu)}(x, c) \mathcal{L}_m^{(\nu)}(x, c) x^\nu e^{-x} \frac{|\Psi(c, -\nu; x e^{-i\pi})|^{-2}}{\Gamma(c+1)\Gamma(\nu+c+1)} dx = \frac{(\nu+c+1)_n}{(c+1)_n} \delta_{nm}, \quad (37)$$

when one of the following conditions is satisfied:

$$\nu + c > -1, \quad c \geq 0 \quad \text{or} \quad \nu + c \geq -1, \quad c \geq -1.$$

The monic polynomial version of $H_n^{(\nu)}(x, c)$ is given by

$$\mathcal{H}_n^{(\nu)}(x, c) = 2^{-n} H_n^{(\nu)}(x, c), \quad n = 0, 1, \dots,$$

and satisfies

$$\begin{aligned} \mathcal{H}_{n+1}^{(\nu)}(x, c) &= x \mathcal{H}_n^{(\nu)}(x, c) - \frac{1}{2} (n + c + \nu(1 - (-1)^n)) \mathcal{H}_{n-1}^{(\nu)}(x, c), \\ \mathcal{H}_{-1}^{(\nu)}(x, c) &= 0, \quad \mathcal{H}_0^{(\nu)}(x, c) = 1. \end{aligned} \quad (38)$$

It is easy to see that the polynomial $(-1)^n \mathcal{H}_n^{(\nu)}(-x, c)$ also satisfies (38). Thus,

$$\mathcal{H}_n^{(\nu)}(-x, c) = (-1)^n \mathcal{H}_n^{(\nu)}(x, c).$$

Thus, by induction, we write them in the form

$$\mathcal{H}_{2n}^{(\nu)}(x, c) = S_n(x^2) \quad \text{and} \quad \mathcal{H}_{2n+1}^{(\nu)}(x, c) = x Q_n(x^2), \quad (39)$$

where $S_n(x)$, $Q_n(x)$ are monic polynomials of degree n .

Theorem 2. Let $c > 0$ and $\nu > -c/2$. The associated generalized Hermite polynomials $H_n^{(\nu)}(x, c)$, defined in (29), have the explicit form:

$$\begin{aligned} H_{2n}^{(\nu)}(x, c) &= (-1)^n 2^{2n} (1 + c/2)_n \mathcal{L}_n^{(\nu-1/2)}(x^2, c/2), \\ H_{2n+1}^{(\nu)}(x, c) &= (-1)^n 2^{2n+1} (1 + c/2)_n x \mathcal{L}_n^{(\nu+1/2)}(x^2, c/2), \end{aligned}$$

and the orthogonality relations

$$\int_{\mathbb{R}} H_n^{(\nu)}(x, c) H_m^{(\nu)}(x, c) |x|^{2\nu} e^{-x^2} \frac{|\Psi(c/2, 1/2 - \nu; x^2 e^{-i\pi})|^{-2}}{\Gamma(1 + c/2)\Gamma(\nu + c/2 + 1/2)} dx = \zeta_n \delta_{nm}, \quad (40)$$

where

$$\zeta_n = \begin{cases} 2^{4k} (1 + c/2)_k (\nu + c/2 + 1/2)_k, & \text{if } n = 2k, \\ 2^{4k+2} (1 + c/2)_k (\nu + c/2 + 3/2)_k, & \text{if } n = 2k + 1. \end{cases}$$

Proof. It is directly verified that the polynomials $S_n(x)$, $Q_n(x)$ given in (39) are orthogonal as they satisfy the recurrence relations

$$\begin{aligned} S_{n+1}(x) &= (x - (2n + c + \nu + 1/2))S_n(x) - (n + c/2) \\ &\quad \times (n + c/2 - 1/2 + \nu)S_{n-1}(x), \\ S_{-1}(x) &= 0, \quad S_0(x) = 1, \end{aligned}$$

and

$$\begin{aligned} Q_{n+1}(x) &= (x - (2n + c + 3/2 + \nu))Q_n(x) - (n + c/2) \\ &\quad \times (n + (1 + c)/2 + \nu)Q_{n-1}(x) \\ Q_{-1}(x) &= 0, \quad Q_0(x) = 1. \end{aligned}$$

From Equation (32), we see that the polynomials $S_n(x)$ satisfy the same recurrence relation as $(-1)^n(1 + c/2)_n \mathcal{L}_n^{(\nu-1/2)}(x, c/2)$, so that

$$S_n(x) = (-1)^n(1 + c/2)_n \mathcal{L}_n^{(\nu-1/2)}(x, c/2). \quad (41)$$

A similar analysis shows that

$$Q_n(x) = (-1)^n(1 + c/2)_n L_n^{(\nu+1/2)}(x, c/2). \quad (42)$$

In view of Equations (41) and (42), the explicit form of the associated generalized Hermite polynomials is given by

$$H_{2n}^{(\nu)}(x, c) = (-1)^n 2^{2n} (1 + c/2)_n \mathcal{L}_n^{(\nu-1/2)}(x^2, c/2), \quad (43)$$

$$H_{2n+1}^{(\nu)}(x, c) = (-1)^n 2^{2n+1} (1 + c/2)_n x L_n^{(\nu+1/2)}(x^2, c/2). \quad (44)$$

From Equations (36) and (37), we deduce that the system $\mathcal{H}_n^\nu(x, c)$ satisfies the orthogonality relations

$$\int_{\mathbb{R}} H_n^{(\nu)}(x, c) H_m^{(\nu)}(x, c) |x|^{2\nu} e^{-x^2} \frac{|\Psi(c/2, 1/2 - \nu; x^2 e^{-i\pi})|^{-2}}{\Gamma(1 + c/2)\Gamma(\nu + c/2 + 1/2)} = \zeta_n \delta_{nm}, \quad (45)$$

with

$$\zeta_n = \begin{cases} 2^{4k} (1 + c/2)_k (\nu + c/2 + 1/2)_k, & \text{if } n = 2k, \\ 2^{4k+2} (1 + c/2)_k (\nu + c/2 + 3/2)_k, & \text{if } n = 2k + 1. \end{cases}$$

□

4.2. Supersymmetric Generalized Hermite Polynomials

In the sequel, we assume that r is an even integer and we consider the Hermitian supercharge operator Q , defined on $\mathcal{S}(\mathbb{R})$, by

$$Q = \frac{1}{\sqrt{2}} Y_\nu (\Pi_{r-1} - \Pi_0) + \frac{x}{\sqrt{2}} (2 - \Pi_0 - \Pi_{r-1}).$$

From Equation (14), we have

$$Q = \frac{1}{\sqrt{2}} (Q_- + Q_+),$$

so it has a self-adjoint extension on $L^2(\mathbb{R}, |x|^{2\nu} dx)$. Furthermore, it acts on the basis ψ_n^ν as

$$\begin{aligned} Q\psi_{nr+s}^\nu &= a_s^{(n)}\psi_{nr+s-1}^\nu + a_{s+1}^{(n)}\psi_{nr+s+1}^\nu, \quad s = 1, \dots, r-1, \\ Q\psi_{nr}^\nu &= a_1^{(n)}\psi_{nr+1}^\nu, \quad Q\psi_{(n+1)r-1}^\nu = a_{r-1}^{(n)}\psi_{(n+1)r-1}^\nu, \end{aligned} \quad (46)$$

where

$$a_s^{(n)} := \sqrt{(nr + s + \nu(1 - (-1)^s))/2}, \quad s = 1, \dots, r-1.$$

On the other hand, by (46), we see that the operator Q leaves invariant the finite dimensional subspace of $L^2(\mathbb{R}, |x|^{2\nu} dx)$ generated by ψ_{nr+s}^ν , $s = 0, 1, \dots, r-1$. Hence, Q can be represented in this basis by the following $r \times r$ tridiagonal Jacobi matrix $A_r^{(n)}$

$$A_r^{(n)} = \begin{pmatrix} 0 & a_1^{(n)} & 0 & & \\ a_1^{(n)} & 0 & a_2^{(n)} & 0 & \\ 0 & a_2^{(n)} & 0 & a_3^{(n)} & \ddots \\ & \ddots & \ddots & \ddots & \ddots & 0 \\ & & \ddots & a_{r-2}^{(n)} & 0 & a_{r-1}^{(n)} \\ & & & 0 & a_{r-1}^{(n)} & 0 \end{pmatrix}.$$

It is well known that, if the coefficients of the subdiagonal of some Jacobi Matrix are different from zero, then all the eigenvalues of this matrix are real and nondegenerate [16]. We introduce the normalized eigenvectors ϕ_s of the supercharge Q

$$Q\phi_s = x_s\phi_s, \quad s = 0, \dots, r-1 \quad (47)$$

that can be expanded in the basis ψ_{nr+k} , $k = 0, 1, \dots, r-1$, as

$$\phi_s = \sum_{k=0}^{r-1} \sqrt{w_s} p_k(x_s) \psi_{nr+k}, \quad (48)$$

where the coefficients p_k obey the three-term recurrence relation [22]

$$\begin{aligned} a_k^{(n)} p_{k-1}(x) + a_{k+1}^{(n)} p_{k+1}(x) &= x p_k(x), \\ p_{-1}(x) &= 0, \quad p_0(x_s) = 1, \end{aligned}$$

Hence, they become orthogonal polynomials. We denote by $P_k(x)$, the monic orthogonal polynomial related to $p_k(x)$ by

$$P_k(x) = h_k p_k(x), \quad (49)$$

where

$$h_k = a_k^{(n)} \dots a_1^{(n)} \quad (50)$$

and satisfying

$$\begin{aligned} xP_k(x) &= P_{k+1}(x) + \frac{1}{2}(k + nr + \nu(1 - (-1)^k)) P_{k-1}(x), \quad k = 0, \dots, r-1, \\ P_{-1}(x) &= 0, \quad P_0(x) = 1. \end{aligned} \quad (51)$$

From the three terms recurrence relations (51), the polynomials $P_k(x)$ can be identified with the associated generalized Hermite polynomial $\mathcal{H}_k^{(v)}(x, c)$, namely,

$$P_k(x) = \mathcal{H}_k^{(v)}(x, nr).$$

It is well known from the theory of orthogonal polynomials that the eigenvalues of the Jacobi matrix $A_r^{(n)}$ coincide with the roots of the characteristic polynomial $\mathcal{H}_r^{(v)}(x, nr)$ [16,22]. The weights w_s defined in (56) are given by the following formula

$$w_s = \frac{h_r^2}{\mathcal{H}_{r-1}^{(v)}(x_s, nr)(\mathcal{H}_r^{(v)})'(x_s, nr)}, \quad (52)$$

where $(\mathcal{H}_r^{(v)})'(x, nr)$ denotes the derivative of $\mathcal{H}_r^{(v)}(x, nr)$, h_r is defined in Equation (50) and $x_{nr,1} > \dots > x_{nr,r}$ are the zeros of $\mathcal{H}_r^{(v)}(x, nr)$. For more detail, we refer to [16]. Then, it turns out that

$$\phi_s = \sum_{k=0}^{r-1} u_{ks}^{(n)} \psi_{nr+k}, \quad (53)$$

where

$$u_{ks}^{(n)} = \frac{h_r}{h_k} \frac{\mathcal{H}_k^{(v)}(x_s, nr)}{(\mathcal{H}_{r-1}^{(v)}(x_s, nr)(\mathcal{H}_r^{(v)})'(x_s, nr))^{1/2}}, \quad 0 \leq s, k \leq r-1. \quad (54)$$

Since both bases $\{\psi_{nr+k}, k = 0, \dots, r-1\}$ and $\{\phi_s, s = 0, \dots, r-1\}$ are orthonormal and all the coefficients are real, then the matrix $(u_{ks}^{(n)})$ is orthogonal and hence the system $\{\mathcal{H}_k^{(v)}(x)\}$ becomes orthogonal polynomials:

$$\sum_{s=0}^{r-1} w_s \mathcal{H}_k^{(v)}(x_s) \mathcal{H}_{k'}^{(v)}(x_s) = \delta_{kk'} / h_k^2. \quad (55)$$

We call supersymmetric generalized Hermite polynomials the orthogonal polynomials, denoted by $\mathbb{H}_N^{(r,v)}(x)$, extracted from the orthogonal function ϕ_s :

$$\mathbb{H}_N^{(r,v)}(x) = \sum_{k=0}^{r-1} H_k^{(v)}(x_s, nr) H_{nr+k}^{(v)}(x), \quad N = nr + s, \quad (56)$$

and we obtain the following:

Theorem 3. The supersymmetric generalized Hermite polynomials $\mathbb{H}_N^{(r,v)}(x)$ satisfy the orthogonality relations

$$\int_{-\infty}^{\infty} \mathbb{H}_N^{(r,v)}(x) \mathbb{H}_{N'}^{(r,v)}(x) |x|^{2v} e^{-x^2} dx = \varrho_N \delta_{NN'}, \quad (57)$$

where $\varrho_N = \gamma_{nr} / w_s$ for $s = 0, \dots, r-1$ and $N = nr + s$.

Proof. From Equations (10) and (56), we obtain

$$\begin{aligned} \int_{-\infty}^{\infty} \mathbb{H}_{nr+s}^{(r,v)}(x) \mathbb{H}_{nr+s'}^{(r,v)}(x) |x|^{2v} e^{-x^2} dx &= \delta_{nn'} \sum_{k=0}^{r-1} H_k^{(v)}(x_s, nr) H_k^{(v)}(x_{s'}, nr) \gamma_{nr+k} \\ &= \delta_{nn'} \gamma_{nr} \sum_{k=0}^{r-1} h_k^2 \mathcal{H}_k^{(v)}(x_s, nr) \mathcal{H}_k^{(v)}(x_{s'}, nr) \end{aligned}$$

and, from ([18] Theorem 2.11.2), we obtain the dual orthogonality relation for $\{\mathcal{H}_k^{(\nu)}(x)\}$:

$$\sum_{k=0}^{r-1} \mathcal{H}_k^{(\nu)}(x_s) \mathcal{H}_k^{(\nu)}(x_{s'}) \frac{[\frac{nr+k}{2}]! \Gamma(\frac{nr+k+1}{2} + \nu + \frac{1}{2})}{[\frac{nr}{2}]! \Gamma(\frac{nr+1}{2} + \nu + \frac{1}{2})} = \delta_{ss'} / w_s \quad (58)$$

and, finally,

$$\int_{-\infty}^{\infty} \mathbb{H}_{nr+s}^{(r,\nu)}(x) \mathbb{H}_{n'r+s'}^{(r,\nu)}(x) |x|^{2\nu} e^{-x^2} dx = \delta_{nn'} \delta_{ss'} \gamma_{nr} / w_s.$$

□

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