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On a New Half-Discrete Hilbert-Type Inequality Involving the Variable Upper Limit Integral and Partial Sums

Jianquan Liao ¹, Shanhe Wu ^{2,*} and Bicheng Yang ¹

¹ Department of Mathematics, Guangdong University of Education, Guangzhou 510303, Guangdong, China; liaojianquan@gdei.edu.cn (J.L.); bcyang@gdei.edu.cn (B.Y.)

² Department of Mathematics, Longyan University, Longyan 364012, Fujian, China

* Correspondence: shanhewu@163.com

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Abstract: In this paper we establish a new half-discrete Hilbert-type inequality involving the variable upper limit integral and partial sums. As applications, an inequality obtained from the special case of the half-discrete Hilbert-type inequality is further investigated; moreover, the equivalent conditions of the best possible constant factor related to several parameters are proved.

Keywords: weight coefficient; Euler–Maclaurin summation formula; half-discrete Hilbert-type inequality; partial sum; variable upper limit integral

1. Introduction

If $p > 1, \frac{1}{p} + \frac{1}{q} = 1, a_m, b_n \geq 0, 0 < \sum_{m=1}^{\infty} a_m^p < \infty$ and $0 < \sum_{n=1}^{\infty} b_n^q < \infty$, then the famous Hardy–Hilbert’s inequality with the best possible constant factor $\frac{\pi}{\sin(\pi/p)}$ reads as follows ([1], Theorem 315):

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m b_n}{m+n} < \frac{\pi}{\sin(\pi/p)} \left(\sum_{m=1}^{\infty} a_m^p \right)^{\frac{1}{p}} \left(\sum_{n=1}^{\infty} b_n^q \right)^{\frac{1}{q}} \quad (1)$$

In [2], an extension of (1) was established by introducing parameters $\lambda_i \in (0, 2] (i = 1, 2)$, $\lambda_1 + \lambda_2 = \lambda \in (0, 4]$, that is

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m b_n}{(m+n)^{\lambda}} < B(\lambda_1, \lambda_2) \left[\sum_{m=1}^{\infty} m^{p(1-\lambda_1)-1} a_m^p \right]^{\frac{1}{p}} \left[\sum_{n=1}^{\infty} n^{q(1-\lambda_2)-1} b_n^q \right]^{\frac{1}{q}} \quad (2)$$

where the constant factor $B(\lambda_1, \lambda_2)$ is the best possible, and

$$B(u, v) = \int_0^{\infty} \frac{t^{u-1}}{(1+t)^{u+v}} dt \quad (u, v > 0) \quad (3)$$

is the beta function.

In [3], by applying inequality (2) and the Abel’s summation by parts formula, Adiyasuren et al. gave a new inequality with the kernel $\frac{1}{(m+n)^{\lambda}}$ involving partial sums. The Hardy–Hilbert’s inequality (1) and its integral analogues play an important role in analysis and its applications ([4–15]).

In 1934, a half-discrete Hilbert-type inequality was given as follows ([1], Theorem 351): Assuming that $K(t)$ ($t > 0$) is a decreasing function, and

$$p > 1, \frac{1}{p} + \frac{1}{q} = 1, 0 < \phi(s) = \int_0^\infty K(t)t^{s-1}dt < \infty,$$

$a_n \geq 0$ such that $0 < \sum_{n=1}^\infty a_n^p < \infty$, then we have the following inequality:

$$\int_0^\infty x^{p-2} \left(\sum_{n=1}^\infty K(nx)a_n \right)^p dx < \phi^p\left(\frac{1}{q}\right) \sum_{n=1}^\infty a_n^p. \quad (4)$$

Rassias, Yang, and Krnić et al. presented some extensions of inequality (4) in [16–20].

Hong and Wen [21] showed the equivalent statements of the extensions of (1) with the best possible constant factor related to several parameters. Some similar results relating to the extensions of inequalities (2) and (4) were given in [22–27].

Recently, Yang and Wu et al. [28,29] gave a reverse half-discrete Hardy–Hilbert’s inequality and an extended Hardy–Hilbert’s inequality. For these inequalities, the equivalent statements of the best possible constant factor related to several parameters were also discussed therein.

Following the way of [2,4,21], the aim of this paper is to establish a new half-discrete Hilbert-type inequality involving the variable upper limit integral and partial sums via the kernel $\frac{1}{(x+n^\alpha)^\lambda}$. With regard to the obtained inequality, the equivalent conditions of the best possible constant factor related to several parameters are also proved.

2. Some Lemmas

In what follows, we suppose that $p > 1, \frac{1}{p} + \frac{1}{q} = 1, \lambda \in (0, 4], \alpha \in (0, 1], \lambda_1 \in (0, \lambda + 1), \lambda_2 \in (0, \frac{2}{\alpha} - 1] \cap (0, \lambda + 1)$,

$$k_\lambda(\lambda_i) := B(\lambda_i, \lambda - \lambda_i) \quad (i = 1, 2),$$

where the beta function $B(u, v)$ ($u, v > 0$) is defined by (3). For a nonnegative real function f and a sequence of nonnegative real numbers $\{a_n\}_{n=1}^\infty$, we define the following variable upper limit integral

$$F(x) := \int_0^x f(t)dt \quad (x \geq 0)$$

and the partial sums

$$A_n := \sum_{k=1}^n a_k \quad (n \in \mathbb{N} = \{1, 2, \dots\}),$$

satisfying $F(x) = o(e^{tx}), A_n = o(e^{tn^\alpha})$ ($t > 0; x, n \rightarrow \infty$) such that

$$0 < \int_0^\infty x^{-p(\frac{\lambda-\lambda_2}{p} + \frac{\lambda_1}{q})-1} F^p(x) < \infty \text{ and } 0 < \sum_{n=1}^\infty n^{q[1-\alpha(1+\frac{\lambda-\lambda_1}{q} + \frac{\lambda_2}{p})]-1} A_n^q < \infty. \quad (5)$$

Lemma 1. (i) ([5], (2.2.3)) If $g(t)$ is a positive real function and $g \in C^3(0, \infty)$ such that $(-1)^i \frac{d^i}{dt^i} g(t) > 0, t \in [m, \infty)$ ($m \in \mathbb{N}$) with $g^{(i)}(\infty) = 0$ ($i = 0, 1, 2, 3$) and $P_i(t), B_i$ ($i \in \mathbb{N}$) are the Bernoulli functions and the Bernoulli numbers of i -order, then we have

$$\int_m^\infty P_{2q-1}(t)g(t)dt = -\varepsilon_q \frac{B_{2q}}{2q} g(m) \quad (0 < \varepsilon_q < 1; q = 1, 2, \dots). \quad (6)$$

In particular, for $q = 1$, in view of $B_2 = \frac{1}{6}$, we have

$$-\frac{1}{12}g(m) < \int_m^\infty P_1(t)g(t)dt < 0; \quad (7)$$

for $q = 2$, in view of $B_4 = -\frac{1}{30}$, we have

$$0 < \int_m^\infty P_3(t)g(t)dt < \frac{1}{120}g(m). \quad (8)$$

(ii) ([5], (2.2.16)) If $h(t)$ is a positive real function, $h \in C^3[m, \infty)$, $h^{(i)}(\infty) = 0$ ($i = 0, 1, 2, 3$), then the Euler–Maclaurin summation formulas hold:

$$\sum_{k=m}^\infty h(k) = \int_m^\infty h(t)dt + \frac{1}{2}h(m) + \int_m^\infty P_1(t)h'(t)dt, \quad (9)$$

$$\int_m^\infty P_1(t)h'(t)dt = -\frac{1}{12}h'(m) + \frac{1}{6}\int_m^\infty P_3(t)h'''(t)dt, \quad (10)$$

where $P_i(t)$ ($i = 1, 3$) are the Bernoulli functions.

Lemma 2. For $\alpha \in (0, 1]$, $s \in (0, 6]$, $s_2 \in (0, \frac{2}{\alpha}] \cap (0, s)$, $k_s(s_2) = B(s_2, s - s_2)$, let us define the following weight coefficient:

$$\omega_\alpha(s_2, x) := \alpha \cdot x^{s-s_2} \sum_{n=1}^\infty \frac{1}{(x+n^\alpha)^s} n^{\alpha s_2-1} (x \in \mathbb{R}_+ = (0, \infty)) \quad (11)$$

Then the following inequalities hold:

$$0 < k_s(s_2)(1 - O(\frac{1}{x^{s_2}})) < \omega_\alpha(s_2, x) < k_s(s_2), \quad (12)$$

where $O(\frac{1}{x^{s_2}}) := \frac{1}{k_s(s_2)} \int_0^{\frac{1}{x}} \frac{u^{s_2-1}}{(1+u)^s} du > 0$.

Proof. For fixed $x > 0$, we define the real function $g_x(t)$ as follows:

$$g_x(t) := \frac{\alpha t^{\alpha s_2-1}}{(x+t^\alpha)^s} (t > 0).$$

By using (9), we have

$$\begin{aligned} \sum_{n=1}^\infty g_x(n) &= \int_1^\infty g_x(t)dt + \frac{1}{2}g_x(1) + \int_1^\infty P_1(t)g'_x(t)dt \\ &= \int_0^\infty g_x(t)dt - h(x), \end{aligned}$$

where

$$h(x) := \int_0^1 g_x(t)dt - \frac{1}{2}g_x(1) - \int_1^\infty P_1(t)g'_x(t)dt.$$

We obtain $-\frac{1}{2}g_x(1) = \frac{-\alpha}{2(x+1)^s}$. Integrating by parts, it follows that

$$\begin{aligned} \int_0^1 g_x(t)dt &= \alpha \int_0^1 \frac{t^{\alpha s_2-1}}{(x+t^\alpha)^s} dt \stackrel{u=t^\alpha}{=} \int_0^1 \frac{u^{\frac{s_2-1}{s}}}{(x+u)^s} du \\ &= \frac{1}{s_2} \int_0^1 \frac{du^{\frac{s_2}{s}}}{(x+u)^s} = \frac{1}{s_2} \frac{u^{\frac{s_2}{s}}}{(x+u)^s} \Big|_0^1 + \frac{s}{s_2} \int_0^1 \frac{u^{\frac{s_2}{s}}}{(x+u)^{s+1}} du \\ &= \frac{1}{s_2} \frac{1}{(x+1)^s} + \frac{s}{s_2(s_2+1)} \int_0^1 \frac{du^{\frac{s_2+1}{s}}}{(x+u)^{s+1}} \\ &> \frac{1}{s_2} \frac{1}{(x+1)^s} + \frac{s}{s_2(s_2+1)} \left[\frac{u^{\frac{s_2+1}{s}}}{(x+u)^{s+1}} \right]_0^1 + \frac{s(s+1)}{s_2(s_2+1)(x+1)^{s+2}} \int_0^1 u^{s_2+1} du \\ &= \frac{1}{s_2} \frac{1}{(x+1)^s} + \frac{\lambda}{s_2(s_2+1)} \frac{1}{(x+1)^{s+1}} + \frac{s(s+1)}{s_2(s_2+1)(s_2+2)} \frac{1}{(x+1)^{s+2}}, \end{aligned}$$

and

$$\begin{aligned} -g'_x(t) &= -\frac{\alpha(\alpha s_2-1)t^{\alpha s_2-2}}{(x+t^\alpha)^s} + \frac{\alpha^2 s t^{\alpha+\alpha s_2-2}}{(x+t^\alpha)^{s+1}} \\ &= -\frac{\alpha(\alpha s_2-1)t^{\alpha s_2-2}}{(x+t^\alpha)^s} + \frac{\alpha^2 s(x+t^\alpha-x)t^{\alpha s_2-2}}{(x+t^\alpha)^{s+1}} \\ &= \frac{\alpha(\alpha s-\alpha s_2+1)t^{\alpha s_2-2}}{(x+t^\alpha)^s} - \frac{\alpha^2 s x t^{\alpha s_2-2}}{(x+t^\alpha)^{s+1}}. \end{aligned}$$

For $0 < s_2 \leq \frac{2}{\alpha}, 0 < \alpha \leq 1, s_2 < s \leq 6$, we have

$$(-1)^i \frac{\partial^i}{\partial t^i} \left[\frac{t^{\alpha s_2-2}}{(x+t^\alpha)^s} \right] > 0, (-1)^i \frac{\partial^i}{\partial t^i} \left[\frac{t^{\alpha s_2-2}}{(x+t^\alpha)^{s+1}} \right] > 0 (i = 0, 1, 2, 3).$$

By (7)–(10), we obtain

$$\begin{aligned} &\alpha(\alpha s - \alpha s_2 + 1) \int_1^\infty P_1(t) \frac{t^{\alpha s_2-2}}{(x+t^\alpha)^s} dt > -\frac{\alpha(\alpha s - \alpha s_2 + 1)}{12(x+1)^s}, \\ &- \alpha^2 x s \int_1^\infty P_1(t) \frac{t^{\alpha s_2-2}}{(x+t^\alpha)^{s+1}} dt \\ &= \frac{\alpha^2 x s}{12(x+1)^{s+1}} - \frac{\alpha^2 x s}{6} \int_1^\infty P_3(t) \left[\frac{t^{\alpha s_2-2}}{(x+t^\alpha)^{s+1}} \right]'' dt \\ &> \frac{\alpha^2 x s}{12(x+1)^{s+1}} - \frac{\alpha^2 x s}{720} \left[\frac{t^{\alpha s_2-2}}{(x+t^\alpha)^{s+1}} \right]''_{t=1} \\ &> \frac{\alpha^2(x+1-1)s}{12(x+1)^{s+1}} - \frac{\alpha^2(x+1)s}{720} \left[\frac{(s+1)(s+2)\alpha^2}{(x+1)^{s+3}} + \frac{\alpha(s+1)(5-\alpha-2\alpha s_2)}{(x+1)^{s+2}} + \frac{(2-\alpha s_2)(3-\alpha s_2)}{(x+1)^{s+1}} \right] \\ &= \frac{\alpha^2 s}{12(x+1)^s} - \frac{\alpha^2 s}{12(x+1)^{s+1}} \\ &\quad - \frac{\alpha^2 s}{720} \left[\frac{(s+1)(s+2)\alpha^2}{(x+1)^{s+2}} + \frac{\alpha(s+1)(5-\alpha-2\alpha s_2)}{(x+1)^{s+1}} + \frac{(2-\alpha s_2)(3-\alpha s_2)}{(x+1)^s} \right]. \end{aligned}$$

Then we have

$$\begin{aligned} h(x) &> \frac{1}{(x+1)^s} h_1 + \frac{\lambda}{(x+1)^{s+1}} h_2 + \frac{s(s+1)}{(x+1)^{s+2}} h_3, \\ h_1 &:= \frac{1}{s_2} - \frac{\alpha}{2} - \frac{\alpha - \alpha^2 s_2}{12} - \frac{\alpha^2 s(2 - \alpha s_2)(3 - \alpha s_2)}{720}, \\ h_2 &:= \frac{1}{s_2(s_2+1)} - \frac{\alpha^2}{12} - \frac{\alpha^3(s+1)(5-\alpha-2\alpha s_2)}{720} \end{aligned}$$

and

$$h_3 := \frac{1}{s_2(s_2+1)(s_2+2)} - \frac{\alpha^4(s+2)}{720}.$$

We obtain

$$h_1 \geq \frac{1}{s_2} - \frac{\alpha}{2} - \frac{\alpha - \alpha^2 s_2}{12} - \frac{\alpha^2(2 - \alpha s_2)(3 - \alpha s_2)}{720} = \frac{g(s_2)}{720s_2},$$

where for $\sigma \in (0, \frac{2}{\alpha}]$,

$$g(\sigma) := 720 - (420\alpha + 6s\alpha^2)\sigma + (60\alpha^2 + 5s\alpha^3)\sigma^2 - s\alpha^4\sigma^3.$$

Namely, g was previously defined as a function. We obtain that for $\alpha \in (0, 1], s \in (0, 6]$,

$$\begin{aligned} g'(\sigma) &= -(420\alpha + 6s\alpha^2) + 2(60\alpha^2 + 5s\alpha^3)\sigma - 3\alpha^4\sigma^2 \\ &\leq -420\alpha - 6s\alpha^2 + 2(60\alpha^2 + 5s\alpha^3)\frac{2}{\alpha} \\ &= (14s\alpha - 180)\alpha < 0, \end{aligned}$$

and then it follows that

$$h_1 \geq \frac{g(s_2)}{720s_2} \geq \frac{g(2/\alpha)}{720s_2} = \frac{1}{6s_2} > 0.$$

We obtain that for $s_2 \in (0, \frac{2}{\alpha}]$,

$$h_2 > \frac{\alpha^2}{6} - \frac{\alpha^2}{12} - \frac{5(s+1)\alpha^2}{720} = (\frac{1}{12} - \frac{s+1}{140})\alpha^2 > 0, \text{ and}$$

$$h_3 \geq (\frac{1}{24} - \frac{s+2}{720})\alpha^3 > 0 (0 < s \leq 6).$$

Hence, we have $h(x) > 0$, and then setting $t = x^{1/\alpha}u^{1/\alpha}$, it follows that

$$\begin{aligned} \omega(s_2, x) &= x^{s-s_2} \sum_{n=1}^{\infty} g_x(n) < x^{s-s_2} \int_0^{\infty} g_x(t) dt \\ &= \alpha x^{s-s_2} \int_0^{\infty} \frac{t^{\alpha s_2-1}}{(x+t^\alpha)^s} dt = \int_0^{\infty} \frac{u^{s_2-1}}{(1+u)^s} du \\ &= B(s_2, s-s_2) = k_s(s_2). \end{aligned}$$

On the other hand, by (9), we have

$$\begin{aligned} \sum_{n=1}^{\infty} g_x(n) &= \int_1^{\infty} g_x(t) dt + \frac{1}{2}g_x(1) + \int_1^{\infty} P_1(t)g'_x(t) dt \\ &= \int_1^{\infty} g_x(t) dt + H(x), \end{aligned}$$

where

$$H(x) := \frac{1}{2}g_x(1) + \int_1^{\infty} P_1(t)g'_x(t) dt.$$

We have obtained $\frac{1}{2}g_x(1) = \frac{\alpha}{2(x+1)^s}$ and

$$g'_x(t) = -\frac{\alpha(\alpha s - \alpha s_2 + 1)t^{\alpha s_2-2}}{(x+t^\alpha)^s} + \frac{\alpha^2 s x t^{\alpha s_2-2}}{(x+t^\alpha)^{s+1}}.$$

For $s_2 \in (0, \frac{2}{\alpha}] \cap (0, s), 0 < s \leq 6$, by (7), we have

$$\begin{aligned} &-\alpha(\alpha s - \alpha s_2 + 1) \int_1^{\infty} P_1(t) \frac{t^{\alpha s_2-2}}{(x+t^\alpha)^s} dt > 0 \text{ and} \\ &\alpha^2 s \int_1^{\infty} P_1(t) \frac{t^{\alpha s_2-2}}{(x+t^\alpha)^{s+1}} dt > -\frac{\alpha^2 x s}{12(x+1)^{s+1}} > -\frac{\alpha^2 s}{12(x+1)^s}. \end{aligned}$$

Hence, it follows that

$$H(m) > \frac{\alpha}{2(x+1)^s} - \frac{\alpha^2 s}{12(x+1)^s} \geq \frac{\alpha}{2(x+1)^s} - \frac{6\alpha}{12(x+1)^s} = 0,$$

and then we obtain

$$\begin{aligned} \omega(\lambda_2, x) &= x^{s-s_2} \sum_{n=1}^{\infty} g_x(n) > x^{s-s_2} \int_1^{\infty} g_x(t) dt \\ &= x^{s-s_2} \int_0^{\infty} g_x(t) dt - x^{s-s_2} \int_0^1 g_x(t) dt \\ &= k_s(s_2) [1 - \frac{1}{k_s(s_2)} \int_0^{\frac{1}{x}} \frac{u^{s_2-1}}{(1+u)^s} du] > 0, \end{aligned}$$

where we set $O(\frac{1}{x^2}) = \frac{1}{k_s(s_2)} \int_0^{\frac{1}{x}} \frac{u^{s_2-1}}{(1+u)^s} du$ satisfying

$$0 < \int_0^{\frac{1}{x}} \frac{u^{s_2-1}}{(1+u)^s} du < \int_0^{\frac{1}{x}} u^{s_2-1} du = \frac{1}{s_2 x^{s_2}}.$$

Therefore, inequalities (12) follow. Thus, Lemma 2 is proved. $\square$

Lemma 3. If $s \in (0, 6]$, $s_1 \in (0, s)$, $s_2 \in (0, \frac{2}{\alpha}] \cap (0, s)$, then the following extended half-discrete Hardy–Hilbert’s inequality holds:

$$I = \int_0^\infty \sum_{n=1}^\infty \frac{a_n f(x)}{(x+n^\alpha)^s} dx \leq (\frac{1}{\alpha} k_s(s_2))^{\frac{1}{p}} (k_s(s_1))^{\frac{1}{q}} \cdot \left\{ \int_0^\infty x^{p[1-(\frac{s-s_2}{p} + \frac{s_1}{q})]-1} f^p(x) dx \right\}^{\frac{1}{p}} \left\{ \sum_{n=1}^\infty n^{q[1-\alpha(\frac{s_2}{p} + \frac{s-s_1}{q})]-1} a_n^q \right\}^{\frac{1}{q}} \quad (13)$$

Proof. Setting $u = x/n^\alpha$, we obtain the following weight coefficient:

$$\omega_\alpha(s_1, n) := n^{\alpha(s-s_1)} \int_0^\infty \frac{x^{s_1-1}}{(x+n^\alpha)^s} dx = \int_0^\infty \frac{u^{s_1-1}}{(u+1)^s} du = k_s(s_1) \quad (n \in \mathbb{N}). \quad (14)$$

By Hölder’s inequality ([30]), we obtain

$$\begin{aligned} I &= \int_0^\infty \sum_{n=1}^\infty \frac{1}{(x+n^\alpha)^s} \left[\frac{x^{(1-s_1)/q} (\alpha n^{\beta-1})^{1/p}}{n^{\alpha(1-s_2)/p}} f(x) \right] \left[\frac{n^{\alpha(1-s_2)/p}}{x^{(1-s_1)/q} (\alpha n^{\alpha-1})^{1/p}} a_n \right] dx \\ &\leq \left[\sum_{n=1}^\infty \int_0^\infty \frac{\alpha}{(x+n^\alpha)^s} \frac{x^{(1-s_1)(p-1)} n^{\alpha-1}}{n^{\alpha(1-s_2)}} f^p(x) dx \right]^{\frac{1}{p}} \\ &\quad \cdot \left[\int_0^\infty \sum_{n=1}^\infty \frac{1}{(x+n^\alpha)^s} \frac{n^{\alpha(1-s_2)(q-1)} a_n^q}{x^{1-s_1} (\alpha n^{\alpha-1})^{q-1}} dx \right]^{\frac{1}{q}} \\ &= \frac{1}{\alpha^{1/p}} \left\{ \int_0^\infty \omega_\alpha(s_2, x) x^{p[1-(\frac{s-s_2}{p} + \frac{s_1}{q})]-1} f^p(x) dx \right\}^{\frac{1}{p}} \\ &\quad \cdot \left\{ \sum_{n=1}^\infty \omega_\alpha(s_1, n) n^{q[1-\alpha(\frac{s-s_1}{q} + \frac{s_2}{p})]-1} a_n^q \right\}^{\frac{1}{q}}. \end{aligned}$$

Thus, by (12) and (14), we get inequality (13). The proof of Lemma 3 is complete. $\square$

Remark 1. For $s = \lambda + 2 \in (2, 6]$, $\lambda \in (0, 4]$, $s_1 = \lambda_1 + 1 \in (1, s)$, $\lambda_1 \in (0, \lambda + 1)$,

$$s_2 = \lambda_2 + 1 \in (1, \frac{2}{\alpha}], \lambda_2 \in (0, \frac{2}{\alpha} - 1] \cap (0, \lambda + 1),$$

by replacing in (13), $f(x)$ and a_n respectively with $F(x)$ and A_n , in view of (5), we have

$$\begin{aligned} \int_0^\infty \sum_{n=1}^\infty \frac{A_n}{(x+n^\alpha)^{\lambda+2}} F(x) dx &< (\frac{1}{\alpha} k_{\lambda+2}(\lambda_2 + 1))^{\frac{1}{p}} (k_{\lambda+2}(\lambda_1 + 1))^{\frac{1}{q}} \\ &\cdot \left[\int_0^\infty x^{-p(\frac{\lambda-\lambda_2}{p} + \frac{\lambda_1}{q})-1} F^p(x) dx \right]^{\frac{1}{p}} \left\{ \sum_{n=1}^\infty n^{q[1-\alpha(1+\frac{\lambda_2}{p} + \frac{\lambda-\lambda_1}{q})]-1} A_n^q \right\}^{\frac{1}{q}}. \end{aligned} \quad (15)$$

Lemma 4. For $t > 0$, we have

$$\int_0^\infty e^{-tx} f(x) dx = t \int_0^\infty e^{-tx} F(x) dx \quad (16)$$

and

$$\sum_{n=1}^\infty e^{-tn^\alpha} a_n \leq t \sum_{n=1}^\infty e^{-tn^\alpha} A_n. \quad (17)$$

Proof. Integrating by parts, in view of $F(x) = o(e^{tx}) (t > 0; x \rightarrow \infty)$, it follows that

$$\begin{aligned} \int_0^\infty e^{-tx} f(x) dx &= \int_0^\infty e^{-tx} dF(x) = e^{-tx} F(x) \Big|_0^\infty - \int_0^\infty F(x) d e^{-tx} \\ &= \lim_{x \rightarrow \infty} e^{-tx} F(x) + t \int_0^\infty e^{-tx} F(x) dx = t \int_0^\infty e^{-tx} F(x) dx, \end{aligned}$$

and then we obtain inequality (16).

In view of $A_n e^{-tn^\alpha} = o(1) (n \rightarrow \infty)$, by Abel's summation by parts formula, we obtain

$$\begin{aligned} \sum_{n=1}^\infty e^{-tn^\alpha} a_n &= \lim_{n \rightarrow \infty} A_n e^{-tn^\alpha} + \sum_{n=1}^\infty A_n [e^{-tn^\alpha} - e^{-t(n+1)^\alpha}] \\ &= \sum_{n=1}^\infty A_n [e^{-tn^\alpha} - e^{-t(n+1)^\alpha}]. \end{aligned}$$

Since $1 - e^{-t} < t (t > 0)$ and for $\alpha \in (0, 1]$,

$$(n+1)^\alpha - n^\alpha - 1 = \alpha(n + \theta_n)^{\alpha-1} - 1 \leq 0 (\theta_n \in (0, 1)),$$

it follows that $e^{t[(n+1)^\alpha - n^\alpha - 1]} \leq 1 (t > 0)$, which implies $e^{-t(n+1)^\alpha} \geq e^{-t(n^\alpha + 1)}$. Thus, we obtain

$$\begin{aligned} \sum_{n=1}^\infty e^{-tn^\alpha} a_n &\leq \sum_{n=1}^\infty A_n [e^{-tn^\alpha} - e^{-t(n^\alpha + 1)}] \\ &= (1 - e^{-t}) \sum_{n=1}^\infty A_n e^{-tn^\alpha} \leq t \sum_{n=1}^\infty A_n e^{-tn^\alpha}, \end{aligned}$$

Hence, the inequality (17) is derived. This completes the proof of Lemma 4. $\square$

3. Main Results

Theorem 1. The following half-discrete Hilbert-type inequality holds true:

$$\begin{aligned} I := \int_0^\infty \sum_{n=1}^\infty \frac{a_n}{(x+n^\alpha)^\lambda} f(x) dx &< \frac{\Gamma(\lambda+2)}{\Gamma(\lambda)} \left(\frac{1}{\alpha} k_{\lambda+2} (\lambda_2 + 1) \right)^{\frac{1}{p}} (k_{\lambda+2} (\lambda_1 + 1))^{\frac{1}{q}} \\ &\cdot \left[\int_0^\infty x^{-p(\frac{\lambda-\lambda_2}{p} + \frac{\lambda_1}{q})-1} F^p(x) dx \right]^{\frac{1}{p}} \left\{ \sum_{n=1}^\infty n^{q[1-\alpha(1+\frac{\lambda-\lambda_1}{q} + \frac{\lambda_2}{p})]-1} A_n^q \right\}^{\frac{1}{q}}, \end{aligned} \quad (18)$$

where $\Gamma(\cdot)$ is the gamma function.

In particular, for $\lambda_1 + \lambda_2 = \lambda$, which implies that $k_\lambda(\lambda_1) = B(\lambda_1, \lambda_2)$ and

$$\begin{aligned} \frac{\lambda-\lambda_2}{p} + \frac{\lambda_1}{q} &= \frac{\lambda_1}{p} + \frac{\lambda_1}{q} = \lambda_1, \quad \frac{\lambda-\lambda_1}{q} + \frac{\lambda_2}{p} = \frac{\lambda_2}{q} + \frac{\lambda_2}{p} = \lambda_2, \\ 0 &< \int_0^\infty x^{-p\lambda_1-1} F^p(x) dx < \infty, \quad 0 < \sum_{n=1}^\infty n^{q[1-\alpha(\lambda_2+1)]-1} A_n^q < \infty, \end{aligned}$$

and hence, the following inequality holds:

$$\begin{aligned} I &= \int_0^\infty \sum_{n=1}^\infty \frac{a_n}{(x+n^\alpha)^\lambda} f(x) dx < \frac{\lambda_1 \lambda_2}{\alpha^{1/p}} k_\lambda(\lambda_1) \\ &\cdot \left(\int_0^\infty x^{-p\lambda_1-1} F^p(x) dx \right)^{\frac{1}{p}} \left\{ \sum_{n=1}^\infty n^{q[1-\alpha(1+\lambda_2)]-1} A_n^q \right\}^{\frac{1}{q}}. \end{aligned} \quad (19)$$

Proof. Since

$$\frac{1}{(x+n^\alpha)^\lambda} = \frac{1}{\Gamma(\lambda)} \int_0^\infty t^{\lambda-1} e^{-(x+n^\alpha)t} dt,$$

it follows that

$$\begin{aligned}
 I &= \frac{1}{\Gamma(\lambda)} \int_0^\infty \sum_{n=1}^\infty a_n f(x) \int_0^\infty t^{\lambda-1} e^{-(x+n^\alpha)t} dt dx \\
 &= \frac{1}{\Gamma(\lambda)} \int_0^\infty t^{\lambda-1} \int_0^\infty e^{-xt} f(x) dx \sum_{n=1}^\infty e^{-n^\alpha t} a_n dt \\
 &\leq \frac{1}{\Gamma(\lambda)} \int_0^\infty t^{\lambda+1} \int_0^\infty e^{-xt} F(x) dx \sum_{n=1}^\infty e^{-n^\alpha t} A_n dt \\
 &= \frac{1}{\Gamma(\lambda)} \int_0^\infty \sum_{n=1}^\infty A_n F(x) \int_0^\infty t^{\lambda+1} e^{-(x+n^\alpha)t} dt dx \\
 &= \frac{\Gamma(\lambda+2)}{\Gamma(\lambda)} \int_0^\infty \sum_{n=1}^\infty \frac{A_n}{(x+n^\alpha)^{\lambda+2}} F(x) dx.
 \end{aligned}$$

By inequality (15), we obtain inequality (18). Thus, Theorem 1 is proved. $\square$

Remark 2. Putting $\alpha = 1, \lambda_1 \in (0, \lambda + 1), \lambda_2 \in (0, 1] \cap (0, \lambda + 1)$ in (18), we have the following half-discrete Hilbert-type inequality:

$$\begin{aligned}
 \int_0^\infty \sum_{n=1}^\infty \frac{a_n}{(x+n)^\lambda} f(x) dx &< \frac{\Gamma(\lambda+2)}{\Gamma(\lambda)} (k_{\lambda+2}(\lambda_2 + 1))^{\frac{1}{p}} (k_{\lambda+2}(\lambda_1 + 1))^{\frac{1}{q}} \\
 &\cdot \left[\int_0^\infty x^{-p(\frac{\lambda-\lambda_2}{p} + \frac{\lambda_1}{q})-1} F^p(x) dx \right]^{\frac{1}{p}} \left[\sum_{n=1}^\infty n^{-q(\frac{\lambda-\lambda_1}{q} + \frac{\lambda_2}{p})-1} A_n^q \right]^{\frac{1}{q}}.
 \end{aligned} \quad (20)$$

Theorem 2. For $\lambda_1 \in (0, \lambda), \lambda_2 \in (0, 1] \cap (0, \lambda)$, the constant factor

$$\frac{\Gamma(\lambda+2)}{\Gamma(\lambda)} (k_{\lambda+2}(\lambda_2 + 1))^{\frac{1}{p}} (k_{\lambda+2}(\lambda_1 + 1))^{\frac{1}{q}}$$

in (20) is the best possible if and only if $\lambda_1 + \lambda_2 = \lambda (\in (0, 4])$.

Proof. Firstly, we shall prove that under the condition $\lambda_1 + \lambda_2 = \lambda (\in (0, 4])$, the constant factor given in (20) is the best possible.

If $\lambda_1 + \lambda_2 = \lambda (\in (0, 4]) (\lambda_1 \in (0, \lambda), \lambda_2 \in (0, 1] \cap (0, \lambda))$, then we obtain

$$\begin{aligned}
 k_{\lambda+2}(\lambda_2 + 1) &= \int_0^\infty \frac{u^{(\lambda_2+1)-1}}{(1+u)^{\lambda+2}} du = B(\lambda_2 + 1, (\lambda + 2) - (\lambda_2 + 1)) \\
 &= B(\lambda_2 + 1, \lambda_1 + 1) = B(\lambda_1 + 1, (\lambda + 2) - (\lambda_1 + 1)) \\
 &= k_{\lambda+2}(\lambda_1 + 1) = \frac{\Gamma(\lambda_1+1)\Gamma(\lambda_2+1)}{\Gamma(\lambda+2)} = \frac{\lambda_1 \lambda_2 \Gamma(\lambda_1) \Gamma(\lambda_2)}{\Gamma(\lambda+2)},
 \end{aligned}$$

$$\begin{aligned}
 &\frac{\Gamma(\lambda+2)}{\Gamma(\lambda)} (k_{\lambda+2}(\lambda_2 + 1))^{\frac{1}{p}} (k_{\lambda+2}(\lambda_1 + 1))^{\frac{1}{q}} \\
 &= \frac{\Gamma(\lambda+2)}{\Gamma(\lambda)} k_{\lambda+2}(\lambda_1 + 1) = \frac{\Gamma(\lambda+2)}{\Gamma(\lambda)} \lambda_1 \lambda_2 \frac{\Gamma(\lambda_1) \Gamma(\lambda_2)}{\Gamma(\lambda+2)} \\
 &= \lambda_1 \lambda_2 \frac{\Gamma(\lambda_1) \Gamma(\lambda_2)}{\Gamma(\lambda)} = \lambda_1 \lambda_2 B(\lambda_1, \lambda_2),
 \end{aligned}$$

and $\frac{\lambda-\lambda_2}{p} + \frac{\lambda_1}{q} = \lambda_1, \frac{\lambda-\lambda_1}{q} + \frac{\lambda_2}{p} = \lambda_2$. Thus, we can reduce (20) to the following:

$$\begin{aligned}
 &\int_0^\infty \sum_{n=1}^\infty \frac{a_n}{(x+n)^\lambda} f(x) dx \\
 &< \lambda_1 \lambda_2 B(\lambda_1, \lambda_2) \left(\int_0^\infty x^{-p\lambda_1-1} F^p(x) dx \right)^{\frac{1}{p}} \left(\sum_{n=1}^\infty n^{-q\lambda_2-1} A_n^q \right)^{\frac{1}{q}}.
 \end{aligned} \quad (21)$$

For any $0 < \varepsilon < \min\{p\lambda_1, q\lambda_2\}$, we set the following function and sequence:

$$\tilde{f}(x) := \begin{cases} 0, & 0 < x < 1, \\ x^{\lambda_1 - \frac{\varepsilon}{p} - 1}, & x \geq 1 \end{cases}, \quad \tilde{a}_n := n^{\lambda_2 - \frac{\varepsilon}{q} - 1} \quad (n \in \mathbb{N}).$$

Then it follows that

$$\begin{aligned}\widetilde{F}(x) &:= \int_0^x \widetilde{f}(t)dt \leq \begin{cases} 0, 0 < x < 1 \\ \frac{1}{\lambda_1 - \frac{\varepsilon}{p}} x^{\lambda_1 - \frac{\varepsilon}{p}}, x \geq 1 \end{cases}, \\ \widetilde{A}_n &:= \sum_{k=1}^n \widetilde{a}_k = \sum_{k=1}^n k^{\lambda_2 - \frac{\varepsilon}{q} - 1} < \int_0^n t^{\lambda_2 - \frac{\varepsilon}{q} - 1} dt = \frac{1}{\lambda_2 - \frac{\varepsilon}{q}} n^{\lambda_2 - \frac{\varepsilon}{q}} (n \in \mathbb{N}).\end{aligned}$$

If there exists a positive constant

$$M \leq \lambda_1 \lambda_2 B(\lambda_1, \lambda_2)$$

such that (21) is valid when replacing $\lambda_1 \lambda_2 B(\lambda_1, \lambda_2)$ by M , then in particular, by substitution of $f(x) = \widetilde{f}(x)$, $a_n = \widetilde{a}_n$, $F(x) = \widetilde{F}(x)$ and $A_n = \widetilde{A}_n$ in (21), we have

$$\widetilde{I} := \int_0^\infty \sum_{n=1}^\infty \frac{\widetilde{a}_n \widetilde{f}(x)}{(x+n)^\lambda} dx < M \left(\int_0^\infty x^{-p\lambda_1-1} \widetilde{F}^p(x) dx \right)^{\frac{1}{p}} \left(\sum_{n=1}^\infty n^{-q\lambda_2-1} \widetilde{A}_n^q \right)^{\frac{1}{q}}. \quad (22)$$

By (22) and the decreasingness property of series, we obtain

$$\begin{aligned}\widetilde{I} &< M \frac{1}{\lambda_1 - \frac{\varepsilon}{p}} \left(\int_1^\infty x^{-p\lambda_1-1} x^{p\lambda_1 - \varepsilon} dx \right)^{\frac{1}{p}} \frac{1}{\lambda_2 - \frac{\varepsilon}{q}} \left(\sum_{n=1}^\infty n^{-q\lambda_2-1} n^{q\lambda_2 - \varepsilon} \right)^{\frac{1}{q}} \\ &= M \left(\frac{1}{\lambda_1 - \frac{\varepsilon}{p}} \right) \left(\frac{1}{\lambda_2 - \frac{\varepsilon}{q}} \right) \left(\int_1^\infty x^{-\varepsilon-1} dx \right)^{\frac{1}{p}} \left(1 + \sum_{n=2}^\infty n^{-\varepsilon-1} \right)^{\frac{1}{q}} \\ &< M \left(\frac{1}{\lambda_1 - \frac{\varepsilon}{p}} \right) \left(\frac{1}{\lambda_2 - \frac{\varepsilon}{q}} \right) \left(\int_1^\infty x^{-\varepsilon-1} dx \right)^{\frac{1}{p}} \left(1 + \int_1^\infty y^{-\varepsilon-1} dy \right)^{\frac{1}{q}} \\ &= \frac{M}{\varepsilon} \left(\frac{1}{\lambda_1 - \frac{\varepsilon}{p}} \right) \left(\frac{1}{\lambda_2 - \frac{\varepsilon}{q}} \right) (\varepsilon + 1)^{\frac{1}{q}}.\end{aligned}$$

By (12) (for $\alpha = 1$), setting $\widetilde{\lambda}_2 := \lambda_2 - \frac{\varepsilon}{q} \in (0, 1) \cap (0, \lambda)$ ($0 < \widetilde{\lambda}_1 := \lambda_2 + \frac{\varepsilon}{q} = \lambda - \widetilde{\lambda}_2 < \lambda$), we find

$$\begin{aligned}\widetilde{I} &= \int_1^\infty \left[x^{\lambda_1 - \frac{\varepsilon}{q}} \sum_{n=1}^\infty \frac{1}{(x+n)^\lambda} n^{(\lambda_2 - \frac{\varepsilon}{q}) - 1} \right] x^{-\varepsilon-1} dx \\ &= \int_1^\infty \omega_1(\widetilde{\lambda}_2, x) x^{-\varepsilon-1} dx > B(\widetilde{\lambda}_1, \widetilde{\lambda}_2) \int_1^\infty (1 - O(x^{-\widetilde{\lambda}_2})) x^{-\varepsilon-1} dx \\ &= B(\widetilde{\lambda}_1, \widetilde{\lambda}_2) \left(\int_1^\infty x^{-\varepsilon-1} dx - \int_1^\infty O(x^{-\lambda_2 - \frac{\varepsilon}{p} - 1}) dx \right) \\ &= \frac{1}{\varepsilon} B\left(\lambda_1 + \frac{\varepsilon}{q}, \lambda_2 - \frac{\varepsilon}{q}\right) (1 - \varepsilon O(1)).\end{aligned}$$

Then, in virtue of the above results, we have

$$B\left(\lambda_1 + \frac{\varepsilon}{q}, \lambda_2 - \frac{\varepsilon}{q}\right) (1 - \varepsilon O(1)) < \varepsilon \widetilde{I} < M \left(\frac{1}{\lambda_1 - \frac{\varepsilon}{p}} \right) \left(\frac{1}{\lambda_2 - \frac{\varepsilon}{q}} \right) (\varepsilon + 1)^{\frac{1}{q}}.$$

For $\varepsilon \rightarrow 0^+$, in view of the continuity of the beta function, we obtain

$$\lambda_1 \lambda_2 B(\lambda_1, \lambda_2) \leq M$$

Hence, $M = \lambda_1 \lambda_2 B(\lambda_1, \lambda_2)$ is the best possible constant factor of (21).

Secondly, we need to prove that if the constant factor given in (20) is the best possible, then $\lambda_1 + \lambda_2 = \lambda \in (0, 4]$.

Setting $\hat{\lambda}_1 := \frac{\lambda-\lambda_2}{p} + \frac{\lambda_1}{q}$, $\hat{\lambda}_2 := \frac{\lambda-\lambda_1}{q} + \frac{\lambda_2}{p}$, we reduce (20) to the following:

$$\int_0^\infty \sum_{n=1}^\infty \frac{a_n}{(x+n)^\lambda} f(x) dx < \frac{\Gamma(\lambda+2)}{\Gamma(\lambda)} (k_{\lambda+2}(\lambda_2+1))^{\frac{1}{p}} (k_{\lambda+2}(\lambda_1+1))^{\frac{1}{q}} \cdot \left(\int_0^\infty x^{-p\hat{\lambda}_1-1} F^p(x) dx \right)^{\frac{1}{p}} \left(\sum_{n=1}^\infty n^{-q\hat{\lambda}_2-1} A_n^q \right)^{\frac{1}{q}} \quad (23)$$

We obtain

$$\begin{aligned} \hat{\lambda}_1 + \hat{\lambda}_2 &= \frac{\lambda-\lambda_2}{p} + \frac{\lambda_1}{q} + \frac{\lambda-\lambda_1}{q} + \frac{\lambda_2}{p} = \lambda, \\ 0 < \hat{\lambda}_1, \hat{\lambda}_2 &< \frac{\lambda}{p} + \frac{\lambda}{q} = \lambda, \end{aligned}$$

and then we have

$$\hat{\lambda}_1 \hat{\lambda}_2 k_\lambda(\hat{\lambda}_1) = \hat{\lambda}_1 \hat{\lambda}_2 B(\hat{\lambda}_1, \hat{\lambda}_2) \in \mathbb{R}_+ = (0, \infty)$$

If the constant factor

$$\frac{\Gamma(\lambda+2)}{\Gamma(\lambda)} (k_{\lambda+2}(\lambda_2+1))^{\frac{1}{p}} (k_{\lambda+2}(\lambda_1+1))^{\frac{1}{q}}$$

in (20) (or (23)) is the best possible, then by (21), the unified best possible constant factor must be $\hat{\lambda}_1 \hat{\lambda}_2 B(\hat{\lambda}_1, \hat{\lambda}_2) \in \mathbb{R}_+$, namely,

$$\begin{aligned} &\frac{\Gamma(\lambda+2)}{\Gamma(\lambda)} (k_{\lambda+2}(\lambda_2+1))^{\frac{1}{p}} (k_{\lambda+2}(\lambda_1+1))^{\frac{1}{q}} \\ &= \hat{\lambda}_1 \hat{\lambda}_2 B(\hat{\lambda}_1, \hat{\lambda}_2) = \frac{\Gamma(\lambda+2)}{\Gamma(\lambda)} k_{\lambda+2}(\hat{\lambda}_1+1). \end{aligned}$$

It follows that

$$k_{\lambda+2}(\hat{\lambda}_1+1) = (k_{\lambda+2}(\lambda_2+1))^{\frac{1}{p}} (k_{\lambda+2}(\lambda_1+1))^{\frac{1}{q}}$$

By Hölder's inequality with weight, we obtain

$$\begin{aligned} k_{\lambda+2}(\hat{\lambda}_1+1) &= k_{\lambda+2} \left(\frac{\lambda-\lambda_2}{p} + \frac{\lambda_1}{q} + 1 \right) \\ &= \int_0^\infty \frac{1}{(1+u)^{\lambda+2}} u^{\frac{\lambda-\lambda_2}{p} + \frac{\lambda_1}{q}} du = \int_0^\infty \frac{1}{(1+u)^{\lambda+2}} \left(u^{\frac{\lambda-\lambda_2}{p}} \right) \left(u^{\frac{\lambda_1}{q}} \right) du \\ &\leq \left[\int_0^\infty \frac{1}{(1+u)^{\lambda+2}} u^{\lambda-\lambda_2} du \right]^{\frac{1}{p}} \left[\int_0^\infty \frac{1}{(1+u)^{\lambda+2}} u^{\lambda_1} du \right]^{\frac{1}{q}} \\ &= \left[\int_0^\infty \frac{1}{(1+v)^{\lambda+2}} v^{(\lambda_2+1)-1} dv \right]^{\frac{1}{p}} \left[\int_0^\infty \frac{1}{(1+u)^{\lambda+2}} u^{(\lambda_1+1)-1} du \right]^{\frac{1}{q}} \\ &= (k_{\lambda+2}(\lambda_2+1))^{\frac{1}{p}} (k_{\lambda+2}(\lambda_1+1))^{\frac{1}{q}}. \end{aligned} \quad (24)$$

We observe that (24) keeps the form of equality if and only if there exist constants A and B such that they are not all zero and ([30])

$$Au^{\lambda-\lambda_2} = Bu^{\lambda_1} a.e. \text{ in } \mathbb{R}_+.$$

Assuming that $A \neq 0$, we have

$$u^{\lambda-\lambda_2-\lambda_1} = \frac{B}{A} a.e. \text{ in } \mathbb{R}_+,$$

and then one has $\lambda - \lambda_2 - \lambda_1 = 0$, namely, $\lambda_1 + \lambda_2 = \lambda (\in (0, 4])$. This completes the proof of Theorem 2. $\square$

Remark 3. (i) For $\lambda = 1, \lambda_1 = \frac{1}{r}, \lambda_2 = \frac{1}{s} (r > 1, \frac{1}{r} + \frac{1}{s} = 1)$ in (21), we have the following half-discrete Hilbert-type inequality with the best possible constant factor $\frac{\pi}{rs \sin(\pi/r)}$:

$$\int_0^\infty \sum_{n=1}^\infty \frac{a_n f(x)}{x+n} dx < \frac{\pi}{rs \sin(\pi/r)} \left(\int_0^\infty x^{-\frac{p}{r}-1} F^p(x) dx \right)^{\frac{1}{p}} \left(\sum_{n=1}^\infty n^{-\frac{q}{s}-1} A_n^q \right)^{\frac{1}{q}}. \quad (25)$$

In particular, for $r = p, s = q$, we get

$$\int_0^\infty \sum_{n=1}^\infty \frac{a_n f(x)}{x+n} dx < \frac{\pi}{pq \sin(\pi/p)} \left(\int_0^\infty x^{-2} F^p(x) dx \right)^{\frac{1}{p}} \left(\sum_{n=1}^\infty n^{-2} A_n^q \right)^{\frac{1}{q}}; \quad (26)$$

for $r = q, s = p$, we obtain

$$\int_0^\infty \sum_{n=1}^\infty \frac{a_n f(x)}{x+n} dx < \frac{\pi}{pq \sin(\pi/p)} \left(\int_0^\infty x^{-p} F^p(x) dx \right)^{\frac{1}{p}} \left(\sum_{n=1}^\infty n^{-q} A_n^q \right)^{\frac{1}{q}}. \quad (27)$$

(ii) For $\lambda = 2, \lambda_1 = \lambda_2 = 1$ in (21), we have the following half-discrete Hilbert-type inequality with the best possible constant factor 1:

$$\int_0^\infty \sum_{n=1}^\infty \frac{a_n}{(x+n)^2} f(x) dx < \left(\int_0^\infty \frac{F^p(x)}{x^{p+1}} dx \right)^{\frac{1}{p}} \left(\sum_{n=1}^\infty \frac{A_n^q}{n^{q+1}} \right)^{\frac{1}{q}} \quad (28)$$

4. Conclusions

In this paper, by means of the weight coefficients, the idea of introduced parameters, the Euler–Maclaurin summation formula and Abel’s summation by parts formula, a new half-discrete Hilbert-type inequality involving the variable upper limit integral and partial sums is given in Theorem 1. As an application, an inequality obtained from the special case of the half-discrete Hilbert-type inequality is investigated in Theorem 2, we obtained the equivalent conditions of the best possible constant factor related to several parameters. The lemmas and theorems proved in this paper reveal some new and interesting properties of this type of inequalities.

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