mathematics

MDPI

Article

Unicyclic Graphs Whose Completely Regular Endomorphisms form a Monoid

Rui Cu and Hailong Hou *

School of Mathematics and Statistics, Henan University of Science and Technology, Luoyang 471023, China; gurui@haust.edu.cn

* Correspondence: hailonghou@haust.edu.cn

Received: 19 January 2020; Accepted: 6 February 2020; Published: 13 February 2020

check for updates

Abstract: In this paper, completely regular endomorphisms of unicyclic graphs are explored. Let G be a unicyclic graph and let $cEnd(G)$ be the set of all completely regular endomorphisms of G. The necessary and sufficient conditions under which $cEnd(G)$ forms a monoid are given. It is shown that $cEnd(G)$ forms a submonoid of $End(G)$ if and only if G is an odd cycle or $G = G(n, m)$ for some odd $n \geq 3$ and integer $m \geq 1$.

Keywords: endomorphism; monoid; completely regular; unicyclic graph

MSC: 05C25; 20M20

1. Introduction

The endomorphism monoids of graphs allow to establish a natural connection between graph theory and algebraic theory of semigroups. They have valuable applications (cf. [1]), many of which are related to automata theory (cf. [2,3]). In recent years, more and more scholors have paid attention to the endomorphism monoids of graphs and a large number of interesting results concerning combinatorial properties of graphs and algebraic properties of their endomorphism monoids have been obtained (see [4–6] and their references). In [7], endomorphisms and weak endomorphisms of a finite undirected path were characterized, the ranks of its endomorphism monoids and weak endomorphism monoids were determined. In [8], we studied regular endomorphisms of trees and determined the trees whose regular endomorphisms form a monoid. In [9], quasi-strong endomorphisms of a join of split graphs were explored, the conditions under which quasi-strong endomorphisms of the join of split graphs form a monoid were given. In [10], the endomorphism monoid of $\overline{P_n}$ was explored. It was shown that $End(\overline{P_n})$ is orthodox. In [11], Wilkeit determined endomorphism regular bipartite graphs. In [12], Hou and Luo constructed four classes of new endomorphism regular graphs by means of the join and the lexicographic product of two graphs with certain conditions. In particular, the join of connected bipartite graphs with regular endomorphism monoids were determined. The endomorphism regularity and endomorphism complete regularity of split graphs were studied separately by Li in [13,14]. Unicyclic graphs is a class of famous graphs; its endomorphism regularity was studied in [15].

An element a of a monoid S is said to be completely regular if there exists $x \in S$ such that $a = axa$ and $xa = ax$. Let $End(G)$ be the endomorphism monoid of a graph G and $f \in End(G)$. Then f is called a completely regular endomorphism of G if it is a completely regular element of $End(G)$. The set of all completely regular endomorphisms of G is denoted by $cEnd(X)$. In general, the composition of two completely regular elements of a monoid S is not completely regular. In [16], Hou and Gu posed the question: Under what conditions does the set $cEnd(G)$ form a monoid? However, it seems quite hard to obtain a complete solution to this question. Therefore a natural strategy for dealing with this question is to discover various kinds of conditions for various classes of graphs. In this paper, we shall give necessary and sufficient conditions under which completely regular endomorphisms of unicyclic graphs form a monoid. The main result of this paper will establish the relations between the combinatorial structure of unicyclic graphs and completely regular submonoids of its endomorphism monoids. The research of this scheme will enrich the contents of graph theory and algebraic theory of semigroups. We present a new method for characterizing completely regular endomorphisms of unicyclic graphs, which may be applied to characterizing completely regular endomorphisms of other classes of graphs.

2. Preliminary Concepts

The graph $G = (V(G), E(G))$ considered here is finite, undirected and simple. If x_1 and x_2 are adjacent in G, denote the edge connecting x_1 and x_2 by $\{x_1, x_2\}$. For $v \in V(G)$, set $N(v) = \{x \in V(G) | \{x, v\} \in E(G)\}$. Denote the cardinality of $N(v)$ by $d(v)$ and we call it the degree of v in G. A connected graph G is called a unicyclic graph if the number of its edges equal to the number of its vertices. Clearly, a unicyclic graph has a unique cycle.

Let G be a graph. A mapping f from $V(G)$ to itself is called an endomorphism if $\{x_1, x_2\} \in E(G)$ implies $\{f(x_1), f(x_2)\} \in E(G)$. An endomorphism f is called an automorphism if and only if it is bijective. The set of all endomorphisms and automorphisms of G are written as $End(G)$ and $Aut(G)$, respectively. Let $f \in End(G)$. The endomorphic image of G under f is denoted by I_f. Obviously, I_f is a subgraph of G with $V(I_f) = f(V(G))$ and $\{f(a), f(b)\} \in E(I_f)$ if and only if there exist $s \in f^{-1}(f(a))$ and $t \in f^{-1}(f(b))$ such that $\{s, t\} \in E(G)$. An endomorphism f of G is called a retraction if and only if f is an idempotent in $End(G)$. A subgraph H of G is called a retract if and only if there exists a retraction f from G to H with $H = I_f$. Denote the set of all idempotents of $End(G)$ by $Idpt(G)$.

We refer the reader to [2,17–19] for all the concepts not defined here. The following two lemmas quoted from the references will be used later.

Lemma 1 ([20] Theorem 2.7)**.** *Let G be a graph and let $f \in End(G)$. Then f is completely regular if and only if $f|_{I_f} \in Aut(I_f)$.*

Lemma 2 ([21] Corollary 2.3)**.** *Let G be a graph and H be a retract of G. If $cEnd(H)$ does not form a monoid, then $cEnd(G)$ does not form a monoid.*

3. Unicyclic Graphs Whose Completely Regular Endomorphisms form a Monoid

In this section, completely regular endomorphisms of unicyclic graphs are explored. We give necessary and sufficient conditions for a unicyclic graph G under which $cEnd(G)$ forms a monoid. Firstly, we consider the unicyclic graph with a unique even cycle.

Lemma 3. *Let G be a unicyclic graph with a unique even cycle C_n. If there exists $x \in V(G)$ such that $d(x) \geq 3$, then $cEnd(G)$ does not form a monoid.*

Proof. Let G be a unicyclic graph with a unique even cycle C_n. Clearly, G is bipartite. If there exists $x \in V(G)$ such that $d(x) \geq 3$, then S_3 (see Figure 1) is a retract of G. By Lemma 2, we only need to show that the completely regular endomorphisms of S_3 do not form a monoid. Let

$$f = \begin{pmatrix} y_0 & y_1 & y_2 & y_3 \\ y_0 & y_2 & y_1 & y_3 \end{pmatrix}$$

and

$$g = \begin{pmatrix} y_0 & y_1 & y_2 & y_3 \\ y_0 & y_1 & y_3 & y_3 \end{pmatrix}.$$

Then $f \in Aut(S_3), g \in Idpt(S_3)$. So $f, g \in cEnd(S_3)$. Now

$$fg = \begin{pmatrix} y_0 & y_1 & y_2 & y_3 \\ y_0 & y_2 & y_3 & y_3 \end{pmatrix}.$$

It is routine to check that $fg \in End(S_3)$ and $y_2, y_3 \in V(I_{fg})$. However, $fg(y_2) = fg(y_3) = y_3$. By Lemma 1, $fg \notin cEnd(S_3)$. Hence $cEnd(S_3)$ does not form a monoid. $\square$

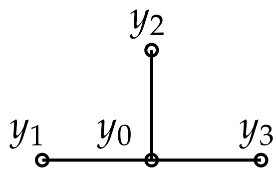

Figure 1. Graph S_3.

Lemma 4. *Let $G = C_{2n}$ with $n \geq 3$. Then $cEnd(G)$ does not form a monoid.*

Proof. Let G be an even cycle C_{2n} with $n \geq 3$. As $diam(G) \geq 3$, P_3 (see Figure 2) is a retract of G. By Lemma 2, we hope to show that all completely regular endomorphisms of P_3 do not form a monoid. Let

$$f = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ x_1 & x_2 & x_3 & x_2 \end{pmatrix}$$

and

$$g = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ x_4 & x_3 & x_2 & x_1 \end{pmatrix}.$$

Then $f \in Idpt(P_3), g \in Aut(P_3)$. Thus $f, g \in cEnd(P3)$. Now

$$fg = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ x_2 & x_3 & x_2 & x_1 \end{pmatrix}.$$

It is not difficult to verify that $fg \in End(P_3)$ and $x_1, x_3 \in V(I_{fg})$. However, $fg(x_1) = fg(x_3) = x_2$. By Lemma 1, $fg \notin cEnd(P_3)$. Hence $cEnd(P_3)$ does not form a monoid. $\square$

Figure 2. Graph P_3.

Lemma 5. *$cEnd(C_4)$ does not form a monoid.*

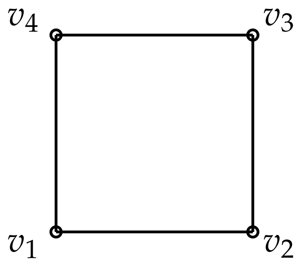

Figure 3. Graph C_4.

Proof. Let

$$f = \begin{pmatrix} v_1 & v_2 & v_3 & v_4 \\ v_4 & v_1 & v_2 & v_3 \end{pmatrix}$$

and

$$g = \begin{pmatrix} v_1 & v_2 & v_3 & v_4 \\ v_3 & v_2 & v_3 & v_4 \end{pmatrix}.$$

Then $f \in Aut(C_4)$, $g \in Idpt(C_4)$. So $f, g \in cEnd(C_4)$. Now

$$fg = \begin{pmatrix} v_1 & v_2 & v_3 & v_4 \\ v_2 & v_1 & v_2 & v_3 \end{pmatrix}.$$

It is not difficult to verify that $fg \in End(C_4)$ and $v_1, v_3 \in V(I_{fg})$. However, $fg(v_1) = fg(v_3) = v_2$. By Lemma 1, $fg \notin cEnd(C_4)$. Hence $cEnd(C_4)$ does not form a monoid. □

Lemma 6. *Let G be a unicyclic graph with a unique even cycle C_n. Then $cEnd(G)$ does not form a monoid.*

Proof. This follows directly from Lemmas 3–5. □

Secondly, we start to look for conditions for a unicyclic graph G with a unique odd cycle, under which all completely regular endomorphisms of G form a monoid.

Lemma 7. *Let G be a unicyclic graph with a unique odd cycle C_n. If there exist two vertices $u, v \in C_n$ such that $d(u) \geq 3$ and $d(v) \geq 3$, then $cEnd(G)$ does not form a monoid.*

Proof. Let G be a unicyclic graph with a unique odd cycle C_n. If there exist two vertices $u_1, v_1 \in C_n$ such that $d(u_1) \geq 3$ and $d(v_1) \geq 3$, then G_1 (see Figure 4) is a retract of G. By Lemma 2, we only need to show that $cEnd(G_1)$ does not form a monoid. Define a mapping f from $V(G_1)$ to itself by

$$f(x) = \begin{cases} v_n, & x = v_{11}, \\ x, & Otherwise. \end{cases}$$

Let

$$g = \begin{pmatrix} v_1 & v_2 & \cdots & v_{i-1} & v_i & v_{i+1} & \cdots & v_n & v_{11} & v_{i1} \\ v_i & v_{i-1} & \cdots & v_2 & v_1 & v_n & \cdots & v_{i+1} & v_{i1} & v_{11} \end{pmatrix}.$$

Then $f \in Idpt(G_1)$ and $g \in cEnd(G_1)$. Now

$$fg = \begin{pmatrix} v_1 & v_2 & \cdots & v_{i-1} & v_i & v_{i+1} & \cdots & v_n & v_{11} & v_{i1} \\ v_i & v_{i-1} & \cdots & v_2 & v_1 & v_n & \cdots & v_{i+1} & v_{i1} & v_n \end{pmatrix}.$$

It is easy to see that $fg \in End(G_1)$ and $v_{i1}, v_{i+1} \in V(I_{fg})$. However, $fg(v_{i1}) = fg(v_{i+1}) = v_n$. By Lemma 1, $fg \notin cEnd(G_1)$. Hence $cEnd(G_1)$ does not form a monoid. □

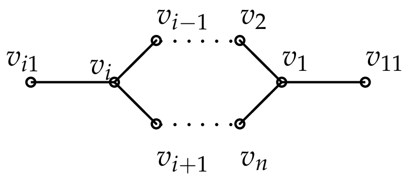

Figure 4. Graph G_1.

Lemma 8. *Let G be a unicyclic graph with a unique odd cycle C_n. If there exists $v \in V(C_n)$ such that $d(v) \geq 4$, then $cEnd(G)$ does not form a monoid.*

Proof. Let G be a unicyclic graphs with a unique odd cycle C_n. If there exists $v \in V(G)$ such that $d(v) \geq 4$, then G_2 (see Figure 5) is a retract of G. By Lemma 2, we only need to show that $cEnd(G_2)$ does not form a monoid. Let

$$f = \begin{pmatrix} v_1 & v_2 & \cdots & v_n & v_{11} & v_{12} \\ v_1 & v_2 & \cdots & v_n & v_{12} & v_{12} \end{pmatrix}$$

and

$$g = \begin{pmatrix} v_1 & v_2 & \cdots & v_n & v_{11} & v_{12} \\ v_1 & v_2 & \cdots & v_n & v_{11} & v_2 \end{pmatrix}.$$

Then $f, g \in Idpt(G_2)$. Hence $f, g \in cEnd(G_2)$. Now

$$fg = \begin{pmatrix} v_1 & v_2 & \cdots & v_n & v_{11} & v_{12} \\ v_1 & v_2 & \cdots & v_n & v_{12} & v_2 \end{pmatrix}.$$

It is not hard to see that $fg \in End(G_2)$ and $v_2, v_{12} \in V(I_{fg})$. However, $fg(v_2) = fg(v_{12}) = v_2$. By Lemma 1, $fg \notin cEnd(G_2)$. Hence $cEnd(G_2)$ does not form a monoid. □

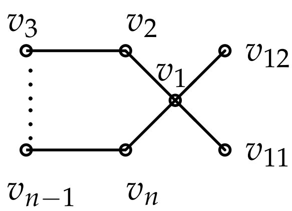

Figure 5. Graph G_2.

Lemma 9. *Let G be a unicyclic graph with a unique odd cycle C_n. If there exists $u \in V(G) \setminus V(C_n)$ such that $d(u) \geq 3$, then $cEnd(G)$ does not form a monoid.*

Proof. As there exists $u \in G \setminus C_n$ such that $d(u) \geq 3$, G_3 (see Figure 6) is a retract of G. By Lemma 2, we hope to show that $cEnd(G_3)$ does not form a monoid. Let

$$f = \begin{pmatrix} v_1 & v_2 & \cdots & v_n & v_{11} & v_{12} & \cdots & v_{1i-1} & v_{1i} & v_{1i+1} & v_{1i+2} \\ v_1 & v_2 & \cdots & v_n & v_{11} & v_{12} & \cdots & v_{1i-1} & v_{1i} & v_{1i+2} & v_{1i+2} \end{pmatrix}$$

and

$$g = \begin{pmatrix} v_1 & v_2 & \cdots & v_n & v_{11} & v_{12} & \cdots & v_{1i-1} & v_{1i} & v_{1i+1} & v_{1i+2} \\ v_1 & v_2 & \cdots & v_n & v_{11} & v_{12} & \cdots & v_{1i-1} & v_{1i} & v_{1i+1} & v_{1i-1} \end{pmatrix}.$$

Then $f, g \in Idpt(G_3)$. Hence $f, g \in cEnd(G_3)$. Now

$$fg = \begin{pmatrix} v_1 & v_2 & \cdots & v_n & v_{11} & v_{12} & \cdots & v_{1i-1} & v_{1i} & v_{1i+1} & v_{1i+2} \\ v_1 & v_2 & \cdots & v_n & v_{11} & v_{12} & \cdots & v_{1i-1} & v_{1i} & v_{1i+2} & v_{1i-1} \end{pmatrix}.$$

It is not difficult to see that $fg \in End(G_3)$ and $v_{1i-1}, v_{1i+2} \in V(I_{fg})$. However, $fg(v_{1i-1}) = fg(v_{1i+2}) = v_{1i-1}$. By Lemma 1, $fg \notin cEnd(G_3)$. Hence $cEnd(G_3)$ does not form a monoid. □

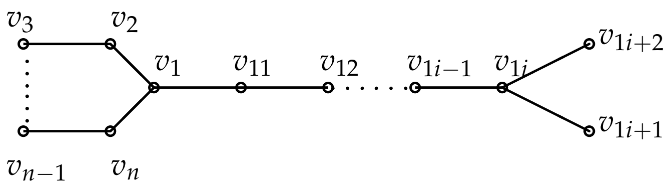

Figure 6. Graph G_3.

Lastly, we investigate the completely regular endomorphisms of unicyclic graph $G(n, m)$ (see Figure 7), where $n \geq 3$ is an odd, $m \geq 1$ is an integer.

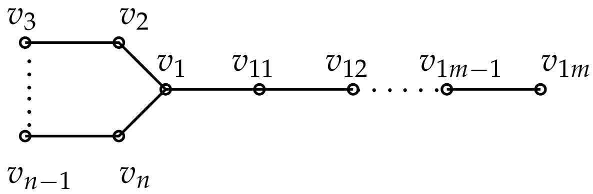

Figure 7. Graph $G(n, m)$.

Lemma 10. *Let $G = G(n, m)$ and $f \in End(G)$. Then $I_f = C_n$ or $I_f = G(n, i)$ for some positive integer i with $1 \leq i \leq m$.*

Proof. As the endomorphism image of a connected graph is connected, I_f is connected. Note that any endomorphism image of an odd cycle contains an odd cycle and C_n is the unique odd cycle in G. Hence $f(C_n) = C_n$. This means that $C_n \in I_f$. Hence $I_f = C_n$ or $I_f = G(n, i)$ for some positive integer i with $1 \leq i \leq m$. □

Lemma 11. *Let $G = G(n, i)$ for some positive integer i and $f \in Aut(G)$. Then $f(x) = x$ for any $x \in \{v_1, v_{11}, v_{12}, \cdots, v_{1i}\}$.*

Proof. As $d(v_1) = 3$, $d(f(v_1)) = 3$. Note that v_1 is the only vertex in G with degree 3. Hence $f(v_1) = v_1$. Recall that $f(C_n) = C_n$. This means $f|_{C_n} \in Aut(C_n)$. Note that $\{v_1, v_{11}\} \in E(G)$. Then $\{f(v_1), f(v_{11})\} = \{v_1, f(v_{11})\} \in E(G)$. Thus $f(v_{11}) \in N(v_1)$. As $N(v_1) = \{v_{11}, v_2, v_n\}$, $f(v_{11}) \in \{v_{11}, v_2, v_n\}$. Since $v_2, v_n \in f(C_n)$, $f(v_{11}) = v_{11}$. A similar argument will show that $f(v_{1s}) = v_{1s}$ for any $s = 2, 3, \cdots, i$. □

Lemma 12. *Let $G = G(n, m)$. Then $f \in End(G)$ is completely regular if and only if*

(1) $I_f = C_n$, or
(2) $I_f = G(n, i)$ for some positive integer i with $1 \leq i \leq m$. In this case, $f|_{C_n} \in Aut(C_n)$ and $f(x) = x$ for any $x \in \{v_1, v_{11}, v_{12}, \cdots, v_{1i}\}$.

Proof. Necessity. Let $f \in cEnd(G)$. By Lemmas 10, $I_f = C_n$ or $I_f = G(n,i)$ for some positive integer i with $1 \leq i \leq m$. As f is completely regular, by Lemma 1, $f|_{I_f} \in Aut(I_f)$. If $I_f = G(n,i)$, then $f(x) = x$ for any $x \in \{v_1, v_{11}, v_{12}, \cdots, v_{1i}\}$ by Lemma 11.

Sufficiency. As any endomorphism image of an odd cycle contains an odd cycle and C_n is the unique odd cycle of G, $f(C_n) = C_n$. If $I_f = C_n$, then $f(I_f) = f(C_n) = C_n$, i.e. $f|_{I_f} \in Aut(I_f)$. Hence f is completely regular. If $I_f = G(n,i)$, then $f|_{C_n} \in Aut(C_n)$ and $f(x) = x$ for any $x \in \{v_1, v_{11}, v_{12}, \cdots, v_{1i}\}$. It is easy to check that $f|_{I_f} \in Aut(I_f)$. By Lemma 1, f is completely regular. □

Lemma 13. *Let $G = G(n,m)$. Then $cEnd(G)$ forms a monoid.*

Proof. Let f and g be two completely regular endomorphisms of G. We hope to show that $fg \in cEnd(G)$. By Lemma 10, $I_f = C_n$ or $I_f = G(n,i)$ for some positive integer i with $1 \leq i \leq m$, and $I_g = C_n$ or $I_g = G(n,j)$ for some positive integer j with $1 \leq j \leq m$. There are two cases:

Case 1. $I_f = C_n$ or $I_g = C_n$. Then $I_{fg} = C_n$. Thus $fg|_{I_{fg}} \in Aut(I_{fg})$. By Lemma 1, fg is completely regular.

Case 2. $I_f = G(n,i)$ $(1 \leq i \leq m)$ and $I_g = G(n,j$ $(1 \leq j \leq m)$. Without loss of generality, we suppose $j \leq i$. By Lemma 12, f, g have the following form:

$$f = \begin{pmatrix} v_1 & v_2 & \cdots & v_n & v_{11} & v_{12} & \cdots & v_{1i} & v_{1i+1} & \cdots & v_{1m} \\ v_1 & v_2^{'} & \cdots & v_n^{'} & v_{11} & v_{12} & \cdots & v_{1i} & v_{1i+1}^{'} & \cdots & v_{1m}^{'} \end{pmatrix},$$

where $v_2^{'}, \cdots, v_n^{'} \in f(C_n \setminus \{v_1\})$, $\{v_2^{'}, \cdots, v_n^{'}\} = \{v_2, \cdots, v_n\}$ and $v_{1s}^{'}$ $(i+1 \leq s \leq m)$ is the image of v_{1s}. As $I_f = G(n,i)$, $v_{1s}^{'} \in \{v_1, v_2, \cdots, v_n, v_{11}, v_{12}, \cdots, v_{1i}\}$ for $s = i+1, \cdots, m$.

$$g = \begin{pmatrix} v_1 & v_2 & \cdots & v_n & v_{11} & v_{12} & \cdots & v_{1j} & v_{1j+1} & \cdots & v_{1m} \\ v_1 & v_2^{''} & \cdots & v_n^{''} & v_{11} & v_{12} & \cdots & v_{1j} & v_{1j+1}^{''} & \cdots & v_{1m}^{''} \end{pmatrix},$$

where $v_2^{''}, \cdots, v_n^{''} \in f(C_n \setminus \{v_1\})$, $\{v_2^{''}, \cdots, v_n^{''}\} = \{v_2, \cdots, v_n\}$ and $v_{1t}^{''}$ $(j+1 \leq t \leq m)$ is the image of v_{1t}, as $I_g = G(n,j)$, $v_{1t}^{''} \in \{v_1, v_2, \cdots, v_n, v_{11}, v_{12}, \cdots, v_{1j}\}$ for $t = j+1, \cdots, m$.

Then

$$fg = \begin{pmatrix} v_1 & v_2 & \cdots & v_n & v_{11} & v_{12} & \cdots & v_{1j} & v_{1j+1} & \cdots & v_{1i} & v_{1i+1} & v_{1m} \\ v_1 & \widetilde{v_2} & \cdots & \widetilde{v_n} & v_{11} & v_{12} & \cdots & v_{1j} & \widetilde{v_{1j+1}} & \cdots & \widetilde{v_{1i}} & \widetilde{v_{1i+1}} & \widetilde{v_{1m}} \end{pmatrix}.$$

where $\widetilde{v_2}, \cdots, \widetilde{v_n} \in f(C_n \setminus \{v_1\})$, $\{\widetilde{v_2}, \cdots, \widetilde{v_n}\} = \{v_2, \cdots, v_n\}$ and $\widetilde{v_{1k}} \in \{v_1, v_2, \cdots, v_n, v_{11}, v_{12}, \cdots, v_{1j}\}$ for $k = j+1, \cdots, m$.

It is trivial to see that $I_{fg} = G(n,j)$ and $fg(x) = x$ for any $x \in \{v_1, v_{11}, v_{12}, \cdots, v_{1j}\}$. Since $f|_{C_n} \in Aut(C_n)$ and $g|_{C_n} \in Aut(C_n)$, $fg|_{C_n} \in Aut(C_n)$. By Lemma 12, $fg \in cEnd(G)$. □

Now we give the main result in this paper.

Theorem 1. *Let G be a unicyclic graph. Then $cEnd(G)$ forms a monoid if and only if*

(1) *G is an odd cycle, or*
(2) *$G = G(n,m)$, where $n \geq 3$ is an odd and $m \geq 1$ is an integer.*

Proof. Necessity. This follows from Lemmas 6–9.

Sufficiency. If G is an odd cycle, then $End(G) = Aut(G)$. Hence $End(G) = cEnd(G)$ and so $cEnd(G)$ forms a monoid. If $G = G(n,m)$, then $cEnd(G)$ forms a monoid by Lemma 13. □

Author Contributions: Create and conceptualize ideas, H.H. and R.G.; writing-original draft preparation, H.H.; writing-review and editing, H.H. and R.G. All authors have read and agreed to the published version of the manuscript.

Funding: This research was partially supported by the National Natural Science Foundation of China (No.11301151) and the Innovation Team Funding of Henan University of Science and Technology (NO.2015XTD010).

Acknowledgments: The authors want to express their gratitude to the referees for their helpful suggestions and comments. The corresponding author Hailong Hou would like to thank Chris Godsil for discussions during his study in University of Waterloo from November 2016 to November 2017.

Conflicts of Interest: The authors declare no conflicts of interest.

References

1. Kelarev, A.V.; Ryan, J.; Yearwood, J. Cayley graphs as classifiers for data mining: The influence of asymmetries. *Discret. Math.* **2009**, *309*, 5360–5369.
2. Kelarev, A.V. *Graph Algebras and Automata*; Marcel Dekker: New York, NY, USA, 2003.
3. Kelarev, A.V.; Praeger, C.E. Labelled Cayley graphs and minimal automata. *Aust. J. Combin.* **2004**, *30*, 95–101.
4. Böttcher, M.; Knauer, U. Endomorphism spectra of graphs. *Discret. Math.* **1992**, *109*, 45–57.
5. Gu, R.; Hou, H. End-regular and End-orthodox generalized lexicographic products of bipartite graphs. *Open Math.* **2016**, *14*, 229–236.
6. Kelarev, A.V.; Praeger, C.E. On transitive Cayley graphs of groups and semigroups. *Eur. J. Combin.* **2003**, *24*, 59–72.
7. Dimitrova, I.; Fernandes, V.H.; Koppitz, J.; Quinteiro, T.M. Ranks of Monoids of Endomorphisms of a Finite Undirected Path. *Bull. Malays. Math. Sci. Soc.* Available online: https://link.springer.com/article/10.1007/s40840-019-00762-4 (accessed on 19 April 2019).
8. Gu, R.; Hou, H. Trees whose regular endomorphisms form a monoid. *Discret. Math.* **2020**, *343*, 111655.
9. Hou, H.; Gu, R.; Shang, Y. The join of split graphs whose quasi-strong endomorphisms form a monoid. *Bull. Aust. Math. Soc.* **2015**, *91*, 1–10.
10. Hou, H.; Luo, Y.; Cheng, Z. The Endomorphism monoid of $\overline{P_n}$. *Eur. J. Combin.* **2008**, *29*, 1173–1185.
11. Wilkeit, E. Graphs with regular endomorphism monoid. *Arch. Math.* **1996**, *66*, 344–352.
12. Hou, H.; Luo, Y. Graphs whose endomorphism monoids are regular. *Discret. Math.* **2008**, *308*, 3888–3896.
13. Li, W.; Chen, J. Endomorphism-regularity of split graphs. *Eur. J. Combin.* **2001**, *22*, 207–216.
14. Li, W. Split graphs with completely regular endomorphism monoids. *J. Math. Res. Exposition.* **2006**, *26*, 253–263.
15. Ma, X.; Wong, D.; Zhou, J. Unicyclic graphs with regular endomorphism monoids. *Discret. Math. Algorithms Appl.* **2016**, *8*, 1650020.
16. Hou, H.; Gu, R. Split graphs whose completely regular endomorphisms form a monoid. *ARS Combin.* **2016**, *127*, 79–88.
17. Kelarev, A.V. *Ring Constructions and Applications*; World Scientific:River Edge, NJ, USA, 2002.
18. Knauer, U. *Algebraic Graph Theory: Morphisms, Monoids and Matrices*; De Gruyter: Boston, MA, USA, 2011.
19. Rashmanlou, H.; Pal, M.; Raut, S.; Mofidnakhaei, F.; Sarkar, B. Novel concepts in intuitionistic fuzzy graphs with application. *J. Intell. Fuzzy Syst.* **2019**, *37*, 3743–3749.
20. Li, W. On completely regular endomorphisms of a graph. *J. Math. (PRC)* **1997**, *17*, 1–7.
21. Gu, R.; Hou, H. Graphs whose completely regular endomorphisms form a monoid. *Ital. J. Pure Appl. Math.* **2020**, to appear.

© 2020 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (http://creativecommons.org/licenses/by/4.0/).