mathematics

MDPI

Article

Characterizing Complete Fuzzy Metric Spaces Via Fixed Point Results

Salvador Romaguera and Pedro Tirado *

Instituto Universitario de Matemática Pura y Aplicada-IUMPA, Universitat Politècnica de València, 46022 Valencia, Spain; sromague@mat.upv.es

* Correspondence: pedtipe@mat.upv.es.com

Received: 29 January 2020; Accepted: 11 February 2020; Published: 18 February 2020

Abstract: With the help of C-contractions having a fixed point, we obtain a characterization of complete fuzzy metric spaces, in the sense of Kramosil and Michalek, that extends the classical theorem of H. Hu (see "*Am. Math. Month.* 1967, 74, 436–437") that a metric space is complete if and only if any Banach contraction on any of its closed subsets has a fixed point. We apply our main result to deduce that a well-known fixed point theorem due to D. Mihet (see "*Fixed Point Theory* 2005, 6, 71–78") also allows us to characterize the fuzzy metric completeness.

Keywords: fuzzy metric space; complete; fixed point; hicks contraction

MSC: 54H25; 54A40; 54E50; 54E70.t

1. Introduction

The problem of characterizing complete metric spaces by means of fixed point properties has its origin, in great part, in a work of Connell published in 1959, where the author constructed an example of a non-complete metric space for which every Banach contraction has fixed point [1] (second part of Example 4). In connection to Connell's example, Hu proved in [2] that a metric space is complete if and only if any Banach contraction on closed subsets thereof has a fixed point. Later, Subrahmanyam [3] and Kirk [4], respectively proved that both the famous Kannan fixed point theorem [5] and the famous Caristi fixed point theorem [6] characterize the metric completeness. Remarkable is also the contribution of Suzuki and Takahashi who gave in [7] (Theorem 4) a necessary and sufficient condition for a metric space to be complete by means of weakly contractive mappings having a fixed point. In a paper published in 2008, Suzuki [8] obtained a nice and elegant characterization of the metric completeness with the help of a weak form of the Banach contraction principle, and, very recently, it was proved in [9] that an important fixed point theorem by Samet, Vetro and Vetro [10] (Theorem 2.2) is also able of characterizing complete metric spaces.

In contrast to the situation described in the preceding paragraph, the natural question of characterizing complete fuzzy metric spaces via fixed point properties has received little attention. Indeed, despite the large backlog of published fixed point theorems, only appear in the literature a few efforts, with positive partial results, to obtain a suitable version of Caristi's theorem that allows us to characterize the completeness of fuzzy metric spaces [11,12].

This paper deals with giving an impulse to the study of characterizing complete fuzzy metric spaces by means of fixed point results. In this direction, our main result extends to the framework of fuzzy metric spaces, in the sense of Kramosil and Michalek [13], Hu's characterization cited above. In fact, we will show that Hu's theorem can be recovered from our main result, and, as an application, we will deduce that a well-known fixed point theorem of Mihet [14] (Theorem 2.2) also allows us to obtain a characterization of fuzzy metric completeness. Since the main ingredient in our approach is the celebrated Hicks fixed point theorem [15,16], we conclude the paper showing that another fundamental and distinguished the fuzzy version of the Banach contraction principle due to Sehgal and Bharucha-Reid [17] is not suitable to characterize complete fuzzy metric spaces in this setting.

2. Background

In order to help the reader, we start this section by collecting some concepts and properties which will be useful in the rest of the paper.

The sets of real numbers and positive integer numbers will be denoted by $\mathbb{R}$ and $\mathbb{N}$, respectively. Our basic reference for general topology is [18].

A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is said to be a continuous t-norm if the following conditions hold: (i) The pair $([0,1], *)$ is an Abelian semigroup with neutral element 1. (ii) $*$ is continuous on $[0,1] \times [0,1]$; (iii) $u * w \leq v * w$ if $u \leq v$, where $u, v, w \in [0,1]$.

The books [19,20] provide excellent references in the study of continuous t-norms. In particular, the following are basic but crucial examples of continuous t-norms.

(i) Minimum $\wedge$ given by $u \wedge v = \min\{u, v\}$.
(ii) Product $*_P$ given by $u *_P v = uv$.
(iii) The Łukasiewicz t-norm $*_L$ given by $u *_L v = \max\{u + v - 1, 0\}$.

We will also consider continuous t-norms of H-type (see e.g., [20] (Chapter 1, Section 1.6). Recall that $\wedge$ is of H-type but $*_P$ and $*_L$ not.

It is well known that $*_L \leq *_P \leq \wedge$. In fact, $* \leq \wedge$ for any continuous t-norm $*$.

Definition 1. *(Kramosil and Michalek [13]). A fuzzy metric on a set $\mathcal{X}$ is a pair $(\mathcal{M}, *)$ such that $*$ is a continuous t-norm and $\mathcal{M}$ is a function from $\mathcal{X} \times \mathcal{X} \times [0, \infty)$ to $[0,1]$ such that for all $\xi, \eta, \theta \in \mathcal{X}$:*

(km1) $\mathcal{M}(\xi, \eta, 0) = 0$;
(km2) $\xi = \eta$ if and only if $\mathcal{M}(\xi, \eta, t) = 1$ for all $t > 0$;
(km3) $\mathcal{M}(\xi, \eta, t) = \mathcal{M}(\eta, \xi, t)$;
(km4) $\mathcal{M}(\xi, \theta, t + s) \geq \mathcal{M}(\xi, \eta, t) * \mathcal{M}(\eta, \theta, s)$ for all $t, s \geq 0$;
(km5) $\mathcal{M}(\xi, \eta, \cdot) : [0, \infty) \to [0,1]$ is left continuous.

If $(\mathcal{M}, *)$ is a fuzzy metric on a set $\mathcal{X}$, the triple $(\mathcal{X}, \mathcal{M}, *)$ is said to be a fuzzy metric space.

If $(\mathcal{M}, *)$ is a fuzzy metric on a set $\mathcal{X}$, the family

$$\{B_{\mathcal{M}}(\xi, \varepsilon, t) : \xi \in \mathcal{X}, \varepsilon \in (0,1), t > 0\},$$

where $B_{\mathcal{M}}(\xi, \varepsilon, t) = \{\eta \in \mathcal{X} : \mathcal{M}(\xi, \eta, t) > 1 - \varepsilon\}$ for all $\varepsilon \in (0,1)$, $t > 0$, is a base of open sets for a metrizable topology $\tau_{\mathcal{M}}$ on $\mathcal{X}$, namely, the topology induced by $(\mathcal{M}, *)$.

A sequence $(\xi_p)_{p \in \mathbb{N}}$ in a fuzzy metric space $(\mathcal{X}, \mathcal{M}, *)$ is called a Cauchy sequence if for any $\varepsilon \in (0,1)$ and $t > 0$ there exists a $p_0 \in \mathbb{N}$ such that $\mathcal{M}(\xi_p, \xi_q, t) > 1 - \varepsilon$ for all $p, q \geq p_0$.

A fuzzy metric space $(\mathcal{X}, \mathcal{M}, *)$ is said to be complete if every Cauchy sequence $(\xi_p)_{p \in \mathbb{N}}$ converges with respect to the topology $\tau_{\mathcal{M}}$.

The following is an easy but very useful example of a fuzzy metric space.

Example 1. *Let $(\mathcal{X}, \sigma)$ be a metric space. Define a function $\mathcal{M}_{01}^{\sigma} : \mathcal{X} \times \mathcal{X} \times [0, \infty) \to [0,1]$ as $\mathcal{M}_{01}^{\sigma}(\xi, \eta, t) = 1$ if $\sigma(\xi, \eta) < t$, and $\mathcal{M}_{01}^{\sigma}(\xi, \eta, t) = 0$ if $\sigma(\xi, \eta) \geq t$. Then, $(\mathcal{M}_{01}^{\sigma}, *)$ is a fuzzy metric on $\mathcal{X}$ for any continuous t-norm $*$. Moreover, the topologies induced by σ and $(\mathcal{M}_{01}^{\sigma}, *)$ coincide, and we also have that $(\mathcal{X}, \mathcal{M}_{01}^{\sigma}, *)$ is complete if and only if $(\mathcal{X}, \sigma)$ is complete.*

The concepts and results from Hicks, Radu, and Miheţ, cited in the sequel was originally established by these authors in the slightly more general framework of Menger spaces.

Hicks introduced in [15] the following notion, under the name of a C-contraction, in the study of the fixed point theory for Menger spaces and fuzzy metric spaces.

Definition 2. *([15]). Let $(\mathcal{X}, \mathcal{M}, *)$ be a fuzzy metric space. We say that a mapping $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ is a Hicks contraction on $\mathcal{X}$ (with constant c) if it satisfies the following condition:*

There exists $c \in (0,1)$ such that for any $\xi, \eta \in \mathcal{X}$ and $t > 0$,

$$\mathcal{M}(\xi, \eta, t) > 1 - t \Rightarrow \mathcal{M}(\mathcal{T}\xi, \mathcal{T}\eta, ct) > 1 - ct.$$

In fact, Hicks proved that if $(\mathcal{X}, \mathcal{M}, \wedge)$ is a complete fuzzy metric space, then every Hicks contraction on $\mathcal{X}$ has a unique fixed point.

In [16], Radu proved the following improvement of Hicks' fixed point theorem, which can be also deduced as an immediate consequence of [21] (Theorem 2.2).

Theorem 1. *Let $(\mathcal{X}, \mathcal{M}, *)$ be a complete fuzzy metric space. Then, every Hicks contraction on $\mathcal{X}$ has a unique fixed point.*

Later, Miheţ's introduced in [14] the notion of a weak-Hicks contraction and obtained, among other results, a fixed point theorem for this class of contractions that strictly contains the class of Hicks contractions [14] (Theorem 2.2, Corollary 2.2.1 and Proposition 2.1).

Miheţ's approach motivates the following notion.

Definition 3. *Let $(\mathcal{X}, \mathcal{M}, *)$ be a fuzzy metric space. We say that a mapping $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ is a Miheţ contraction on $\mathcal{X}$ if it satisfies the following two conditions:*

(mi1) there exists $\xi_0 \in \mathcal{X}$ such that $\mathcal{M}(\xi_0, \mathcal{T}\xi_0, 1) > 0$;

(mi2) there exists $c \in (0,1)$ such that for any $\xi, \eta \in \mathcal{X}$ and $t \in (0,1)$,

$$\mathcal{M}(\xi, \eta, t) > 1 - t \Rightarrow \mathcal{M}(\mathcal{T}\xi, \mathcal{T}\eta, ct) > 1 - ct.$$

From [14] (Theorem 2.2) and its proof we deduce the following restatement of [14] (Corollary 2.2.1).

Theorem 2. *(Miheţ [14]) Let $(\mathcal{X}, \mathcal{M}, *)$ be a complete fuzzy metric space such that $*_L \leq *$ or $*$ is of H-type. Then, every Miheţ contraction on $\mathcal{X}$ has a fixed point.*

According to [14] (Definition 2.2), a mapping $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ satisfying condition (mi2) in Definition 3 is said to be a weak-Hicks contraction. It is well-known [14] (Lemma 2.1) that every weak-Hicks contraction is a continuous mapping. Obviously, every Hicks contraction is a weak Hicks contraction and hence a continuous mapping. In fact, we have the following better conclusion.

Proposition 1. *Every Hicks contraction is a Miheţ contraction.*

Proof of Proposition 1. Let $\mathcal{T}$ be a Hicks contraction (with constant c) on a fuzzy metric space $(\mathcal{X}, \mathcal{M}, *)$. It suffices to show that $\mathcal{T}$ satisfies condition (mi1) in Definition 3. Indeed, suppose that $\mathcal{M}(\xi, \mathcal{T}\xi, 1) = 0$ for all $\xi \in \mathcal{X}$. Since $c \in (0,1)$ we deduce that $1 - 1/c < 0$, so $\mathcal{M}(\xi, \mathcal{T}\xi, 1/c) > 1 - 1/c$. Putting $\eta = \mathcal{T}\xi$ we deduce that $\mathcal{M}(\eta, \mathcal{T}\eta, 1) > 1 - 1 = 0$, a contradiction. □

3. Results and Examples

As we recalled in Section 1, the Banach contraction principle does not characterize the metric completeness. The following example (based on the example given in [7]) shows that, similarly to the metric case, there exist non-complete fuzzy metric spaces for which every Hicks contraction has fixed points.

Example 2. *(compare [7], Example on pages 377-378). Let*

$$\mathcal{X} := \bigcup_{p\in\mathbb{N}} \{(\delta, \frac{\delta}{p}) : \delta \in (0,1]\} \cup \{(0,0)\},$$

and let σ be the restriction of the Euclidean metric on $\mathbb{R}^2$ to $\mathcal{X}$. Clearly $(\mathcal{X}, \sigma)$ is not complete and thus the fuzzy metric space $(\xi, \mathcal{M}_{01}^{\sigma}, \wedge)$ is not complete (see Example 1). However, every continuous mapping $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ has at least a fixed point, as Suzuki and Takahashi proved in [7]. Therefore, every Hicks contraction on $\mathcal{X}$ has at least a fixed point because Hicks contractions are continuous mappings.

Nevertheless, we can obtain a characterization of fuzzy metric completeness similar to the one given by Hu for metric spaces with the help of Hicks contractions having a fixed point. To this end, the notion of a semi-metric and an important theorem due to Radu (Theorem 3 below) will be fundamental to our approach.

A semi-metric (compare e.g., [22]) for a topological space $(\mathcal{X}, \tau)$ is a function $\sigma : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ satisfying the following conditions for every $\xi, \eta \in \mathcal{X}$ and $\mathcal{A}$ a subset of $\mathcal{X}$:

(sm1) $\sigma(\xi, \eta) = 0 \Leftrightarrow \xi = \eta$;

(sm2) $\sigma(\xi, \eta) = \sigma(\eta, \xi)$;

(sm3) $\xi \in \overline{\mathcal{A}} \Leftrightarrow \inf\{\sigma(\xi, \alpha) : \alpha \in \mathcal{A}\} = 0$. (As usual, $\overline{\mathcal{A}}$ denotes the closure of $\mathcal{A}$ in $(\mathcal{X}, \tau)$.)

As in the metric case, if σ is a semi-metric for a topological space $(\mathcal{X}, \tau)$, a sequence $(\xi_p)_{p\in\mathbb{N}}$ in $\mathcal{X}$ is called a Cauchy sequence if for each $\varepsilon > 0$ there exists $p_0 \in \mathbb{N}$ such that $\sigma(\xi_p, \xi_q) < \varepsilon$ for all $p, q \geq p_0$.

Theorem 3. *(Radu [23] Proposition 2.1.1) Let $(\mathcal{X}, \mathcal{M}, *)$ be a fuzzy metric space. For each $\xi, \eta \in \mathcal{X}$ put*

$$\sigma_{\mathcal{M}}(\xi, \eta) = \sup\{t \geq 0 : \mathcal{M}(\xi, \eta, t) \leq 1 - t\}.$$

Then $\sigma_{\mathcal{M}}$ is a semi-metric for $(\mathcal{X}, \tau_{\mathcal{M}})$, satisfying $\sigma_{\mathcal{M}} \leq 1$ and

$$\sigma_{\mathcal{M}}(\xi, \eta) < t \Longleftrightarrow \mathcal{M}(\xi, \eta, t) > 1 - t,$$

for all $t > 0$.Therefore, a sequence in $\mathcal{X}$ is a Cauchy sequence for $\sigma_{\mathcal{M}}$ if and only if it is a Cauchy sequence in $(\mathcal{X}, \mathcal{M}, *)$.

Moreover, if $*_L \leq *$, then $\sigma_{\mathcal{M}}$ is a metric on X whose induced topology coincides with $\tau_{\mathcal{M}}$.

Our main result is the following.

Theorem 4. *A fuzzy metric space $(\mathcal{X}, \mathcal{M}, *)$ is complete if and only if every Hicks contraction on any closed subset of $(\mathcal{X}, \mathcal{M}, *)$ has a fixed point.*

Proof of Theorem 4. Suppose that $(\mathcal{X}, \mathcal{M}, *)$ is a complete fuzzy metric space. Let $\mathcal{A}$ be a closed subset of $(\mathcal{X}, \mathcal{M}, *)$. Then $(\mathcal{A}, \mathcal{M}_{|\mathcal{A}}, *)$ is a complete fuzzy metric space, where by $\mathcal{M}_{|\mathcal{A}}$ we denote the restriction of $\mathcal{M}$ to $\mathcal{A} \times \mathcal{A} \times [0, \infty)$. Therefore, every Hicks contraction on $\mathcal{A}$ has a (unique) fixed point by Theorem 1.

Conversely, suppose that $(\mathcal{X}, \mathcal{M}, *)$ is not complete. Then, we can find a Cauchy sequence $(\xi_p)_{p\in\mathbb{N}}$ in $\mathcal{X}$, with $\xi_p \neq \xi_q$ for $p \neq q$, which does not converge for $\tau_{\mathcal{M}}$. Therefore, the set $\mathcal{A} = \{\xi_p : p \in \mathbb{N}\}$ is closed and also are closed all sets of the form $\mathcal{A} \backslash \{\xi_p\}$, $p \in \mathbb{N}$. So, by condition (sm3) in the definition of a semi-metric, we have that for each $p \in \mathbb{N}$,

$$\inf\{\sigma_{\mathcal{M}}(\xi_p, \xi_r) : r \in \mathbb{N},\ r \neq p\} > 0.$$

Since, by Theorem 3, $(\xi_p)_{p\in\mathbb{N}}$ is also a Cauchy sequence for the semi-metric $\sigma_{\mathcal{M}}$, there is a strictly increasing function $\varphi : \mathbb{N} \to \mathbb{N}$ such that $\varphi(p) > p$ for all $p \in \mathbb{N}$ and

$$\sigma_{\mathcal{M}}(\xi_n, \xi_m) < \frac{1}{2} \inf\{\sigma_{\mathcal{M}}(\xi_p, \xi_r) : r \in \mathbb{N},\ r \neq p\},$$

for all $n, m \in \mathbb{N}$ with $n, m \geq \varphi(p)$.

Now let $\mathcal{T} : \mathcal{A} \to \mathcal{A}$ be the mapping given by $\mathcal{T}\xi_p = \xi_{\varphi(p)}$ for all $\xi_p \in \mathcal{A}$. Since $\varphi(p) > p$, $\xi_p \neq \xi_{\varphi(p)}$, so $\mathcal{T}$ has no fixed points. We shall show that, nevertheless, $\mathcal{T}$ is a Hicks contraction on the fuzzy metric space $(\mathcal{A}, \mathcal{M}_{|\mathcal{A}}, *)$, with constant $1/2$.

Indeed, let $\mathcal{M}(\xi_p, \xi_q, t) > 1 - t$, with $p, q \in \mathbb{N}$ and $t > 0$. By Theorem 3, $\sigma_{\mathcal{M}}(\xi_p, \xi_q) < t$. Assume, without loss of generality, that $q > p$. Then $\varphi(q) > \varphi(p)$, and hence

$$\begin{aligned} \sigma_{\mathcal{M}}(\mathcal{T}\xi_p, \mathcal{T}\xi_q) &= \sigma_{\mathcal{M}}(\xi_{\varphi(p)}, \xi_{\varphi(q)}) < \frac{1}{2} \inf\{\sigma_{\mathcal{M}}(\xi_p, \xi_r) : r \in \mathbb{N},\ r \neq p\} \\ &\leq \frac{1}{2}\sigma_{\mathcal{M}}(\xi_p, \xi_q) < \frac{t}{2}. \end{aligned}$$

Consequently $\mathcal{M}(\mathcal{T}\xi_p, \mathcal{T}\xi_q, t/2) > 1 - t/2$ by Theorem 3, which implies that $\mathcal{T}$ is a Hicks contraction on $(\mathcal{A}, \mathcal{M}_{|\mathcal{A}}, *)$. This concludes the proof. □

Now we apply Theorem 4 to deduce that Theorem 2 also allows us to characterize fuzzy completeness for two large classes of fuzzy metric spaces.

Theorem 5. *Let $(\mathcal{X}, \mathcal{M}, *)$ be a fuzzy metric space such that $*_L \leq *$ or $*$ is of H-type. Then $(\mathcal{X}, \mathcal{M}, *)$ is complete if and only if every Miheţ contraction on any closed subset of $(\mathcal{X}, \mathcal{M}, *)$ has a fixed point.*

Proof of Theorem 5. Suppose that $(\mathcal{X}, \mathcal{M}, *)$ is complete and let $\mathcal{A}$ be a closed subset of $(\mathcal{X}, \mathcal{M}, *)$. Then $(\mathcal{A}, \mathcal{M}_{|\mathcal{A}}, *)$ is a complete fuzzy metric space. It follows from Theorem 2 that every Miheţ contraction on $\mathcal{A}$ has a fixed point.

Conversely, let $\mathcal{A}$ be a closed subset of $(\mathcal{X}, \mathcal{M}, *)$ and suppose that every Miheţ contraction on $\mathcal{A}$ has a fixed point. Since, by Proposition 1, every Hicks contraction is a Miheţ contraction, we deduce that every Hicks contraction on $\mathcal{A}$ has a fixed point. Hence $(\mathcal{X}, \mathcal{M}, *)$ is complete by Theorem 4. □

Next, we shall deduce the classical Hu theorem from our main result. Two simple auxiliary results will be useful to this end.

If $(\mathcal{X}, \sigma)$ is a metric space, we denote by σ_1 the metric defined on $\mathcal{X}$ by $\sigma_1(\xi, \eta) = \min\{1, \sigma(\xi, \eta)\}$ for all $\xi, \eta \in \mathcal{X}$. Obviously, the topologies induced by σ and σ_1 coincide, and $(\mathcal{X}, \sigma)$ is complete if and only if (X, σ_1) is complete. Therefore $(\mathcal{X}, \mathcal{M}_{01}^{\sigma}, \wedge)$ is complete if and only if $(\mathcal{X}, \mathcal{M}_{01}^{\sigma_1}, \wedge)$ is complete.

Proposition 2. *Let $(\mathcal{X}, \sigma)$ be a metric space. Then, every Banach contraction on $(\mathcal{X}, \sigma_1)$ is a Banach contraction on $(\mathcal{X}, \sigma)$.*

Proof of Proposition 2. Let $\mathcal{T}$ be a Banach contraction on $(\mathcal{X}, \sigma_1)$. Then, there exists $c \in (0, 1)$ such that $\sigma_1(\mathcal{T}\xi, \mathcal{T}\eta) \leq c\sigma_1(\xi, \eta)$, for all $\xi, \eta \in \mathcal{X}$.

If $\sigma_1(\mathcal{T}\xi, \mathcal{T}\eta) = 1$ for some $\xi, \eta \in \mathcal{X}$, we deduce that $1 \leq c\sigma_1(\xi, \eta) \leq c$, which is not possible. Therefore $\sigma_1(\mathcal{T}\xi, \mathcal{T}\eta) = \sigma(\mathcal{T}\xi, \mathcal{T}\eta) < 1$ for all $\xi, \eta \in \mathcal{X}$, so

$$\sigma(\mathcal{T}\xi, \mathcal{T}\eta) = \sigma_1(\mathcal{T}\xi, \mathcal{T}\eta) \leq c\sigma_1(\xi, \eta) \leq c\sigma(\xi, \eta).$$

We conclude that $\mathcal{T}$ is a Banach contraction on $(\mathcal{X}, \sigma)$. □

Proposition 3. *Let $(\mathcal{X}, \sigma)$ be a metric space with $\sigma \leq 1$. Then, every Hicks contraction on $(\mathcal{X}, \mathcal{M}_{01}^{\sigma}, \wedge)$ is a Banach contraction on $(\mathcal{X}, \sigma)$.*

Proof of Proposition 3. Let $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ be a Hicks contraction on $(\mathcal{X}, \mathcal{M}_{01}^{\sigma}, \wedge)$, with constant c. We want to show that $\sigma(\mathcal{T}\xi, \mathcal{T}\eta) \leq c\sigma(\xi, \eta)$ for all $\xi, \eta \in \mathcal{X}$.

Indeed, assume the contrary. Then, there exist $\xi, \eta \in \mathcal{X}$ such that $\sigma(\mathcal{T}\xi, \mathcal{T}\eta) > c\sigma(\xi, \eta)$. Choose an $\varepsilon > 0$ for which $c(\sigma(\xi, \eta) + \varepsilon) < 1$ and $\sigma(\mathcal{T}\xi, \mathcal{T}\eta) > c(\sigma(\xi, \eta) + \varepsilon)$. Since $\mathcal{M}_{01}^{\sigma}(\xi, \eta, \sigma(\xi, \eta) + \varepsilon) = 1$ and $\mathcal{T}$ is a Hicks contraction we deduce that $\mathcal{M}_{01}^{\sigma}(\mathcal{T}\xi, \mathcal{T}\eta, c(\sigma(\xi, \eta) + \varepsilon)) > 1 - c(\sigma(\xi, \eta) + \varepsilon) > 0$. On the other hand, from $\sigma(\mathcal{T}\xi, \mathcal{T}\eta) > c(\sigma(\xi, \eta) + \varepsilon)$, it follows that $\mathcal{M}_{01}^{\sigma}(\mathcal{T}\xi, \mathcal{T}\eta, c(\sigma(\xi, \eta) + \varepsilon)) = 0$. This contradiction concludes the proof. □

Theorem 6. *(Hu [2]). A metric space $(\mathcal{X}, \sigma)$ is complete if and only if every Banach contraction on any closed subset of $(\mathcal{X}, \sigma)$ has a fixed point.*

Proof of Theorem 6. Since the proof of the "only if" part is obvious, we only show the "if" part. Suppose that every Banach contraction on any closed subset of $(\mathcal{X}, \sigma)$ has a fixed point. Let $\mathcal{A}$ be a closed subset of the fuzzy metric space $(\mathcal{X}, \mathcal{M}_{01}^{\sigma_1}, \wedge)$ and let $\mathcal{T} : \mathcal{A} \to \mathcal{A}$ be a Hicks contraction on $(\mathcal{A}, \mathcal{M}_{01|\mathcal{A}}^{\sigma_1}, \wedge)$. Since $\sigma_1 \leq 1$, it follows from Proposition 3 that $\mathcal{T}$ is a Banach contraction on $(\mathcal{A}, \sigma_{1|\mathcal{A}})$, where $\sigma_{1|\mathcal{A}}$ denotes the restriction of the metric σ_1 to $\mathcal{A} \times \mathcal{A}$. Therefore $\mathcal{T}$ is a Banach contraction on $(\mathcal{A}, \sigma_{|\mathcal{A}})$ by Proposition 2. Hence $\mathcal{T}$ has a fixed point in $\mathcal{A}$. Consequently $(\mathcal{X}, \mathcal{M}_{01}^{\sigma_1}, \wedge)$ is complete by Theorem 4. Thus $(\mathcal{X}, \mathcal{M}_{01}^{\sigma}, \wedge)$ is complete and, hence, $(\mathcal{X}, \sigma)$ is complete. □

Sehgal and Baharucha-Reid proved in [17] the first fixed point theorem for fuzzy metric spaces (actually, they obtained their result in the realm of Menger spaces).

Theorem 7. *(Sehgal and Baharucha-Reid [17]). Let $(\mathcal{X}, \mathcal{M}, \wedge)$ be a complete fuzzy metric space. If $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ is a mapping such that there exists $c \in (0, 1)$ satisfying*

$$\mathcal{M}(\mathcal{T}\xi, \mathcal{T}\eta, ct) \geq \mathcal{M}(\xi, \eta, t),$$

for all $\xi, \eta \in \mathcal{X}$ and $t > 0$, then $\mathcal{T}$ has a unique fixed point.

We finish the paper showing that, contrarily to the case of Hicks contractions, this fundamental theorem is not suitable to obtain a characterization of fuzzy metric completeness. More precisely, we present an example of a non-complete fuzzy metric space $(\mathcal{X}, \mathcal{M}, \wedge)$ such that for any closed subset $\mathcal{A}$, every mapping $\mathcal{T} : \mathcal{A} \to \mathcal{A}$ satisfying the conditions of Theorem 7 is constant and, hence, has a unique fixed point.

Example 3. *Let $\mathcal{X} = \{1/p : p \in \mathbb{N}\}$ and let $\mathcal{M} : \mathcal{X} \times \mathcal{X} \times [0, \infty) \to [0, 1]$ given as $\mathcal{M}(\xi, \eta, 0) = 0$, $\mathcal{M}(\xi, \eta, t) = t/(t + |\xi - \eta|)$ if $t \leq 1$, and $\mathcal{M}(\xi, \eta, t) = 1$ if $t > 1$, for all $\xi, \eta \in \mathcal{X}$. Then $(\mathcal{X}, \mathcal{M}, \wedge)$ is a fuzzy metric space (see [24] (Example on page 2016)). Clearly $(1/p)_{p \in \mathbb{N}}$ is a non convergent Cauchy sequence in $(\mathcal{X}, \mathcal{M}, \wedge)$, so it is not complete. Now let $\mathcal{A}$ be any subset of $\mathcal{X}$ (note that all subsets of $\mathcal{X}$ are closed because $\tau_{\mathcal{M}}$ is the discrete topology on $\mathcal{X}$), and $\mathcal{T} : \mathcal{A} \to \mathcal{A}$ be a mapping such that there exists $c \in (0, 1)$ for which $\mathcal{M}(\mathcal{T}\xi, \mathcal{T}\eta, ct) \geq \mathcal{M}(\xi, \eta, t)$ for all $\xi, \eta \in \mathcal{A}$ and $t > 0$. Take $t_0 \in (1, 1/c)$. Then $ct_0 < 1$, so*

$$\mathcal{M}(\mathcal{T}\xi, \mathcal{T}\eta, ct_0) = \frac{ct_0}{ct_0 + |\mathcal{T}\xi - \mathcal{T}\eta|} \geq \mathcal{M}(\xi, \eta, t_0) = 1.$$

Therefore $|\mathcal{T}\xi - \mathcal{T}\eta| = 0$, and, thus, $\mathcal{T}\xi = \mathcal{T}\eta$ for all $\xi, \eta \in \mathcal{A}$. We conclude that $\mathcal{T}$ is constant and, hence, it has a unique fixed point in $\mathcal{A}$. Note that, by Theorem 4, there exists a (closed) subset $\mathcal{A}$ of $\mathcal{X}$ and a Hicks contraction $\mathcal{T} : \mathcal{A} \to \mathcal{A}$ without fixed points.

Author Contributions: All authors contributed equally in the preparation, development and writing this article. All authors have read and agreed to the published version of the manuscript.

Funding: This research was partially funded by Ministerio de Ciencia, Innovación y Universidades, under grant PGC2018-095709-B-C21 and AEI/FEDER, UE funds.

Acknowledgments: The authors thank the reviewers for their constructive and useful suggestions and comments.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Connell, E.H. Properties of fixed point spaces. *Proc. Am. Math. Soc.* **1959**, *10*, 974–979. [CrossRef]
2. Hu, T.K. On a fixed point theorem for metric spaces. *Am. Math. Mon.* **1967**, *74*, 436–437. [CrossRef]
3. Subrahmanyam, P.V. Completeness and fixed-points. *Mh. Math.* **1975**, *80*, 325–330. [CrossRef]
4. Kirk, A.W. Caristi's fixed point theorem and metric convexity. *Colloq. Math.* **1976**, *36*, 81–86. [CrossRef]
5. Kannan, R. Some results on fixed points. *Bull. Calcutta Math. Soc.* **1968**, *60*, 71–76.
6. Caristi, J. Fixed point theorems for mappings satisfying inwardness conditions. *Trans. Am. Math. Soc.* **1976**, *215*, 241–251. [CrossRef]
7. Suzuki, T.; Takahashi, W. Fixed point theorems and characterizations of metric completeness. *Top. Methods Nonlinear Anal.* **1996**, *8*, 371–382. [CrossRef]
8. Suzuki, T. A generalized Banach contraction principle that characterizes metric completeness. *Proc. Am. Math. Soc.* **2008**, *136*, 1861–1869. [CrossRef]
9. Romaguera, S.; Tirado, P. A characterization of quasi-metric completeness in terms of α-ψ-contractive mappings having fixed points. *Mathematics* **2020**, *8*, 16. [CrossRef]
10. Samet, B.; Vetro, C.; Vetro, P. Fixed point theorems for $\alpha - \psi$-contractive type mappings. *Nonlinear Anal.* **2012**, *75*, 2154–2165. [CrossRef]
11. Abbas, M.; Ali, B.; Romaguera, S. Multivalued Caristi's type mappings in fuzzy metric spaces and a characterization of fuzzy metric completeness. *Filomat* **2015**, *29*, 1217–1222. [CrossRef]
12. Castro-Company, F.; Romaguera, S.; Tirado, P. On the construction of metrics from fuzzy metrics and its application to the fixed point theory of multivalued mappings. *Fixed Point Theory Appl.* **2015**, *2015*, 226. [CrossRef]
13. Kramosil, I.; Michalek, J. Fuzzy metrics and statistical metric spaces. *Kybernetika* **1975**, *11*, 336–344.
14. Miheţ, D. Weak-Hicks contractions. *Fixed Point Theory* **2005**, *6*, 71–78.
15. Hicks, T.L. Fixed point theory in probabilistic metric spaces. *Rev. Res. Fac. Sci. Univ. Novi Sad* **1983**, *13*, 63–72.
16. Radu, V. Some fixed point theorems in probabilistic metric spaces. In *Lectures Notes Math*; Springer: Berlin/Heidelberg, Germany, 1987; Volume 1233, pp. 125-133.
17. Sehgal, V.M.; Bharucha-Reid, A.T. Fixed points of contraction mappings on probabilistic metric spaces. *Math. Syst. Theory* **1972**, *6*, 97–102. [CrossRef]
18. Engelking, R.E. *General Topology*; Sigma Series in Pure Mathematics; Heldermann: Berlin, Germany, 1989; Volume 6.
19. Klement, E.; Mesiar, R.; Pap, E. *Triangular Norms*; Kluwer Academic: Dordrecht, The Netherlands, 2000.
20. Hadžić, O.; Pap, E. *Fixed Point Theory in Probabilistic Metric Spaces*; Kluwer Academic: Dordrecht, The Netherlands, 2001.
21. Ricarte, L.A.; Romaguera, S. On φ-contractions on fuzzy metric spaces with application to the intuitionistic setting. *Iran. J. Fuzzy Syst.* **2013**, *10*, 63–72.
22. Creede, G.D. Concerning semi-stratifiable spaces. *Pac. J. Math.* **1970**, *32*, 47–54. [CrossRef]
23. Radu, V. Some suitable metrics on fuzzy metric spaces. *Fixed Point Theory* **2004**, *5*, 323–347.
24. Ćirić, L. Solving the Banach fixed point principle for nonlinear contractions in probabilistic metric spaces. *Nonlinear Anal.* **2010**, *72*, 2009–2018. [CrossRef]