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Large Constant-Sign Solutions of Discrete Dirichlet Boundary Value Problems with p-Mean Curvature Operator

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Abstract: In this paper, we consider the existence of infinitely many large constant-sign solutions for a discrete Dirichlet boundary value problem involving p-mean curvature operator. The methods are based on the critical point theory and truncation techniques. Our results are obtained by requiring appropriate oscillating behaviors of the non-linear term at infinity, without any symmetry assumptions.

Keywords: discrete Dirichlet boundary value problem; p-mean curvature operator; constant-sign solutions; discrete maximum principle; critical point theory

1. Introduction

Let $\mathbb{Z}, \mathbb{N}$ and $\mathbb{R}$ denote the sets of integer numbers, natural numbers and real numbers, respectively. For $a, b \in \mathbb{Z}$, define $\mathbb{Z}(a) = \{a, a+1, \cdots\}$, and $\mathbb{Z}(a, b) = \{a, a+1, \cdots, b\}$ when $a \leq b$.

Consider the following Dirichlet boundary value problem of the nonlinear difference equation

$$(D_p^{\lambda,f}) \begin{cases} -\triangle\left(\phi_{p,c}\left(\triangle u(k-1)\right)\right) = \lambda f(k, u(k)), & k \in \mathbb{Z}(1, T), \\ u(0) = u(T+1) = 0, \end{cases}$$

where T is a given positive integer, λ is a positive real parameter, $\triangle$ is the forward difference operator defined by $\triangle u(k) = u(k+1) - u(k)$, $f(k, \cdot) : \mathbb{R} \to \mathbb{R}$ is a continuous function for each $k \in \mathbb{Z}(1, T)$ and $\phi_{p,c}(s) := (1 + |s|^2)^{\frac{p-2}{2}} s$, $p \in [1, +\infty)$. Here, $\triangle\left(\phi_{p,c}\left(\triangle u(k-1)\right)\right)$ may be seen as a discretization of the p-mean curvature operator.

We may think problem $(D_p^{\lambda,f})$ as being a discrete analog of one-dimensional case of the following problem

$$\begin{cases} -\text{div}\left(\phi_{p,c}\left(\nabla u\right)\right) = \lambda f(x, u), & x \in \Omega \subset \mathbb{R}^n, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1}$$

where div $\left(\phi_{p,c}\left(\nabla u\right)\right)$ is named p-mean curvature operator, which is a generalization of mean curvature operator; see [1,2]. If $p = 1$, it reduces to the mean curvature operator. If $p = 2$, it reduces to the Laplacian operator. The above problem arises from differential geometry and physics such as capillarity; see [3–5] and references therein. When $p = 1$ and $f(x, u) = u$, the above problem describes the free surface of a pendent drop filled with liquid under gravitational field [4]. In the past decades, several authors have discussed the existence and multiplicity of solutions of Problem (1); see [1,6–12]. For example, Chen and Shen in [1] have obtained the existence of infinitely many solutions of Problem (1) with $\lambda = 1$ via a symmetric version of Mountain Pass Theorem. When $p = 1$ and $\Omega = (0, 1)$, Obersnel and Omari in [11] have established the existence and multiplicity of positive solutions of Problem (1), which depend on the behavior of f at zero or at infinity. G. A. Afrouzi et al. in [6] have acquired a sequence of nonnegative and nontrivial solutions strongly converging to zero in $C^1([0, 1])$, under suitable oscillating behavior of the nonlinear term f at zero. However, the results on the existence of solutions for problem $(D_p^{\lambda,f})$ are scarce in the literature besides the case of $p = 1$.

Nonlinear discrete problems appear in many mathematical models, such as computer science, mechanical engineering, control systems, artificial or biological neural networks, economics, fluid mechanics and many others; see [13–17]. Many authors have discussed the existence and multiplicity of solutions for difference equations through classical tools of nonlinear analysis: Fixed point theorems, upper and lower solutions techniques; see [7,9] and the references given therein. Since 2003, by starting from the seminal paper [18], variational methods have been used to investigate nonlinear difference equations, which have obtained various results; see [19–34].

In paper [35], the authors have considered problem $(D_1^{\lambda,f})$, obtaining infinitely many positive solutions when λ belongs to a precise real interval. It is worth noticing that the suitable oscillating behaviors of the nonlinear term f at infinity play a key role. Inspired by [19,32,35–40], the main purpose of this paper is to investigate the existence conditions of infinitely many constant-sign solutions for problem $(D_p^{\lambda,f})$, without any symmetry hypothesis. Here, a solution $\{u(k)\}$ of $(D_p^{\lambda,f})$ is called a constant-sign solution, if $u(k) > 0$ for all $k \in \mathbb{Z}(1, T)$ or $u(k) < 0$ for all $k \in \mathbb{Z}(1, T)$. Compared to problem $(D_1^{\lambda,f})$, problem $(D_p^{\lambda,f})$ is more difficult to handle. To facilitate the analysis, we have to divide the problem into two categories: $1 \le p < 2$ and $2 \le p < +\infty$. We believe that this is the first time to discuss the existence of infinitely many solutions for a non-linear second order difference equation with p-mean curvature operator.

A special case of our results is the following.

Theorem 1. *Let $g : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $g(t)t \ge 0$ for $t \ne 0$. Assume that*

$$\liminf_{t\to\infty} \frac{\int_0^t g(\tau)d\tau}{|t|^p} = 0, \text{ and } \limsup_{t\to\infty} \frac{\int_0^t g(\tau)d\tau}{|t|^p} = +\infty.$$

Then, for every $\lambda > 0$, the problem

$$\begin{cases} -\triangle\left(\phi_{p,c}\left(\triangle u(k-1)\right)\right) = \lambda g(u(k)), & k \in \mathbb{Z}(1, T), \\ u(0) = u(T+1) = 0, \end{cases}$$

admits two unbounded sequences of constant-sign solutions (one positive and one negative).

This paper is organized as follows. In Section 2, we introduce the the suitable Banach space and appropriate functional corresponding to problem $(D_p^{\lambda,f})$. To obtain sequences of constant-sign solutions of problem $(D_p^{\lambda,f})$, three basic lemmas are introduced. In Section 3, under suitable hypotheses on f, we obtain the existence of infinitely many constant-sign solutions for problem $(D_p^{\lambda,f})$. In Section 4, we give two examples to demonstrate our results. Finally, conclusions are given for this paper.

2. Mathematical Background

To solve problem $(D_p^{\lambda,f})$, we naturally select the T-dimensional Banach space

$$X = \{u : \mathbb{Z}(0, T+1) \to \mathbb{R} : u(0) = u(T+1) = 0\},$$

endowed with the norm

$$\|u\| := \left(\sum_{k=1}^{T} (\triangle u(k))^2\right)^{\frac{1}{2}} \text{ for all } u \in X.$$

Another useful norm on X is

$$||u||_\infty := \max_{k\in\mathbb{Z}(1,T)} |u(k)| \text{ for all } u \in X.$$

In the sequel, we will use the following inequalities.

For $0 < r < s,\ x_k \geq 0, k \in \mathbb{Z}(1,\ n)$, one has

$$\left(\sum_{k=1}^{n} x_k^s\right)^{1/s} \leq \left(\sum_{k=1}^{n} x_k^r\right)^{1/r}, \tag{2}$$

see [41].

$$||u||_\infty \leq \frac{\sqrt{T+1}}{2}\|u\|, \tag{3}$$

for every $u \in X$, it can follow from Lemma 2.2 of [42].

For all $u \in X$, let

$$\Phi(u) := \frac{1}{p}\sum_{k=0}^{T}\left(\left(1+(\triangle u(k))^2\right)^{\frac{p}{2}}-1\right), \text{ and } \Psi(u) := \sum_{k=1}^{T} F(k, u(k)), \tag{4}$$

where $F(k,t) := \int_0^t f(k,\tau)\,d\tau$ for every $t \in \mathbb{R}$ and $k \in \mathbb{Z}(1,\ T)$. Further, let us denote $I_\lambda(u) := \Phi(u) - \lambda\Psi(u)$ for $u \in X$. Through standard arguments, we follow that $I_\lambda \in C^1(S, \mathbb{R})$, and the critical points of I_λ are exactly the solutions of problem $(D_p^{\lambda,f})$. In fact, one has

$$\begin{aligned} I'_\lambda(u)(v) &= \sum_{k=0}^{T}\left(\phi_{p,c}(\triangle u(k)\right)\triangle v(k) - \lambda\sum_{k=1}^{T} f(k,u(k))v(k) \\ &= \sum_{k=0}^{T}\left(\phi_{p,c}(\triangle u(k)\right) v(k+1) - \sum_{k=0}^{T}\left(\phi_{p,c}(\triangle u(k)\right) v(k) - \lambda\sum_{k=1}^{T} f(k,u(k))v(k) \\ &= \sum_{k=1}^{T}\left(\phi_{p,c}(\triangle u(k-1)\right) v(k) - \sum_{k=1}^{T}\left(\phi_{p,c}(\triangle u(k)\right) v(k) - \lambda\sum_{k=1}^{T} f(k,u(k))v(k) \\ &= -\sum_{k=1}^{T}[\triangle((\phi_{p,c}(\triangle u(k)) - \lambda f(k,u(k))]v(k), \end{aligned}$$

for all $u, v \in X$.

Next, we need to establish the following strong maximum principle to obtain the positive solutions of problem $(D_p^{\lambda,f})$, i.e., $u(k) > 0$ for each $k \in \mathbb{Z}(1,\ T)$.

Lemma 1. *Assume $u \in X$ such that either*

$$u(k) > 0 \quad or \quad -\triangle\left(\varphi_{p,c}(\triangle u(k-1)\right) \geq 0, \tag{5}$$

for any $k \in \mathbb{Z}(1,\ T)$. Then, either $u > 0$ in $\mathbb{Z}(1,\ T)$ or $u \equiv 0$.

Proof. For $u \in X$, put $m = \min\{u(k), k \in \mathbb{Z}(0,\ T+1)\}$, then $m \leq 0$.

If there exists $j \in \mathbb{Z}(1,\ T)$ such that $u(j) = m$, we claim that $u \equiv 0$. Indeed, since $\Delta u(j-1) = u(j) - u(j-1) \leq 0$ and $\Delta u(j) = u(j+1) - u(j) \geq 0$, $\varphi_{p,c}(s)$ is strictly monotone increasing in s, and $\varphi_{p,c}(0) = 0$, we have

$$\varphi_{p,c}(\triangle u(j)) \geq 0 \geq \varphi_{p,c}(\triangle u(j-1)). \tag{6}$$

On the other hand, by (5), let $k = j$, we obtain

$$\varphi_{p,c}(\triangle u(j)) \leq \varphi_{p,c}(\triangle u(j-1)). \tag{7}$$

Combining inequalities (6) and (7), we get that $\varphi_{p,c}(\triangle u(j)) = 0 = \varphi_{p,c}(\triangle u(j-1))$. That is $u(j+1) = u(j-1) = u(j) = m$. By iterating this argument, we obtain easily $u(0) = u(1) = u(2) = \ldots = u(T) = u(T+1)$. Thus $u \equiv 0$.

If $u(j) > m$ for every $j \in \mathbb{Z}(1, T)$, then $u(0) = u(T+1) = m = 0$. It follows that $u(j) > 0$, for all $j \in \mathbb{Z}(1, T)$. The proof is complete.

In the same way, we have the following result to get negative solutions problem $(D_p^{\lambda,f})$, i.e., $u(k) < 0$ for each $k \in \mathbb{Z}(1, T)$.

Lemma 2. *Assume $u \in X$ such that either*

$$u(k) < 0 \quad \text{or} \quad -\triangle\left(\varphi_{p,c}(\triangle u(k-1)\right) \leq 0, \tag{8}$$

for any $k \in \mathbb{Z}(1, T)$. Then, either $u < 0$ in $\mathbb{Z}(1, T)$ or $u \equiv 0$.

Truncation techniques are usually used to discuss the existence of constant-sign solutions. To the end, we introduce the following truncations of the functions $f(k,t)$ for every $k \in \mathbb{Z}(1, T)$.
If $f(k,0) \geq 0$ for each $k \in \mathbb{Z}(1, T)$. Set

$$f^+(k,t) := \begin{cases} f(k,t), & \text{if } t \geq 0, \\ f(k,0), & \text{if } t < 0. \end{cases}$$

Clearly, $f^+(k,\cdot)$ is also continuous, for every $k \in \mathbb{Z}(1, T)$. By Lemma 1, all solutions of problem (D_p^{λ,f^+}) are also solutions of problem $(D_p^{\lambda,f})$. Therefore, when problem (D_p^{λ,f^+}) has non-zero solutions, then problem $(D_p^{\lambda,f})$ possesses positive solutions.

If $f(k,0) \leq 0$ for each $k \in \mathbb{Z}(1, T)$. Set

$$f^-(k,t) := \begin{cases} f(k,0), & \text{if } t > 0, \\ f(k,t), & \text{if } t \leq 0. \end{cases}$$

When problem (D_p^{λ,f^-}) has non-zero solutions, then problem $(D_p^{\lambda,f})$ possesses negative solutions.

Here, we introduce a lemma (Theorem 4.3 of [38]) which is the main tool used to research problem $(D_p^{\lambda,f})$.

Lemma 3. *Let X be a finite dimensional Banach space and let $I_\lambda : X \to \mathbb{R}$ be a function satisfying the following structure hypothesis:*

(H) *$I_\lambda(u) := \Phi(u) - \lambda\Psi(u)$ for all $u \in X$, where $\Phi, \Psi : X \to \mathbb{R}$ be two continuously Gâteux differentiable functions with Φ coercive, i.e., $\lim_{\|u\| \to +\infty} \Phi(u) = +\infty$, and such that $\inf_X \Phi = \Phi(0) = \Psi(0) = 0$.*

For all $r > 0$, put

$$\varphi(r) := \frac{\sup\limits_{\Phi^{-1}[0,\, r]} \Psi}{r}, \text{ and } \varphi_\infty := \liminf_{r \to +\infty} \varphi(r).$$

Assume that $\varphi_\infty < +\infty$ and for each $\lambda \in (0, \frac{1}{\varphi_\infty})$ I_λ is unbounded from below. Then, there is a sequence $\{u_n\}$ of critical points (local minima) of I_λ such that $\lim\limits_{n \to +\infty} \Phi(u_n) = +\infty$.

3. Main Results

In the following, we will discuss the existence of constant-sign solutions of problem $(D_p^{\lambda,f})$. Our purpose is to apply Lemma 3 to the function $I_\lambda^{\pm} : X \to \mathbb{R}$, $I_\lambda^{\pm}(u) := \Phi(u) - \lambda\Psi^{\pm}(u)$, where $\Psi^{\pm}(u) = \sum_{k=1}^{T} F^{\pm}(k, u(k))$ and $F^{\pm}(k,t) := \int_0^t f^{\pm}(k,\tau)\,d\tau$ for every $k \in \mathbb{Z}(1, T)$ and then exploit Lemma 1 or Lemma 2 to get our results.

Let

$$A_{\pm\infty} := \liminf_{t\to+\infty} \frac{\sum_{k=1}^{T} \max_{0\le s\le t} F(k, \pm s)}{t^p}, \text{ and } B^{\pm\infty} := \limsup_{t\to\pm\infty} \frac{\sum_{k=1}^{T} F(k,t)}{|t|^p}.$$

Considering the functional I_λ^{+}, we have the following conclusions.

Theorem 2. *Let $1 \le p < 2$ and $f(k,\cdot) : \mathbb{R} \to \mathbb{R}$ to be a continuous function with $f(k,0) \ge 0$ for each $k \in \mathbb{Z}(1, T)$. Assume that*

$$(i_1) A_{+\infty} < \frac{2^{p-1}}{(T+1)^{\frac{p}{2}}} B^{+\infty}.$$

Then, for each $\lambda \in \left(\frac{2}{pB^{+\infty}}, \frac{2^p}{p(T+1)^{\frac{p}{2}} A_{+\infty}}\right)$, problem $(D_p^{\lambda,f})$ has an unbounded sequence of positive solutions.

Proof. Consider the auxiliary problem

$$(D_p^{\lambda,f^+}) \begin{cases} -\triangle\left(\phi_{p,c}\left(\triangle u(k-1)\right)\right) = \lambda f^{+}(k, u(k)), & k \in \mathbb{Z}(1, T), \\ u(0) = u(T+1) = 0. \end{cases}$$

Obviously Φ and Ψ^{+} satisfy hypothesis required in Lemma 3. For $t > 0$, set

$$r = \frac{1}{p}\left(\sqrt{\frac{4t^2}{T+1} + (T+1)^{\frac{2p-2}{p}}} - (T+1)^{\frac{p-1}{p}}\right)^{p}.$$

Assume $u \in X$ and

$$\Phi(u) = \frac{1}{p}\sum_{k=0}^{T}\left(\left(1 + (\triangle u(k))^2\right)^{\frac{p}{2}} - 1\right) \le r.$$

Put $v(k) = \left(1 + (\triangle u(k))^2\right)^{\frac{p}{2}} - 1$, for every $k \in \mathbb{Z}(0, T)$, then $\sum_{k=0}^{T} v(k) \le p\,r$.

By (2) and Hölder inequality as well, we have

$$\begin{aligned} \sum_{k=0}^{T}(\triangle u(k))^2 &= \sum_{k=0}^{T}\left(\left((1+v(k))^{\frac{1}{p}}\right)^2 - 1\right) \\ &\le \left(\sum_{k=0}^{T} v(k)\right)^{\frac{2}{p}} + 2(T+1)^{\frac{p-1}{p}}\left(\sum_{k=0}^{T} v(k)\right)^{\frac{1}{p}} \\ &\le (pr)^{\frac{2}{p}} + 2(T+1)^{\frac{p-1}{p}}(pr)^{\frac{1}{p}} \\ &= \frac{4t^2}{T+1}. \end{aligned}$$

Owing to (3), it follows

$$||u||_\infty \le \frac{\sqrt{T+1}}{2}\left(\sum_{k=0}^{T}(\triangle u(k))^2\right)^{\frac{1}{2}} \le t.$$

Thus, one has $\Phi^{-1}[0,\ r] \subseteq \{u \in X : ||u||_\infty \leq t\}$
By the definition of φ, we obtain

$$\varphi(r) = \frac{\sup\limits_{\Phi^{-1}[0,\ r]} \Psi^+}{r} \leq \frac{\sup\limits_{||u||_\infty \leq t} \sum\limits_{k=0}^{T} F^+(k, u(k))}{r} \leq \frac{p \sum\limits_{k=1}^{T} \max\limits_{0 \leq s \leq t} F(k,s)}{\left(\sqrt{\frac{4t^2}{T+1} + (T+1)^{\frac{2p-2}{p}}} - (T+1)^{\frac{p-1}{p}}\right)^p}.$$

Bearing in mind condition(i_1), we follow that $\varphi_\infty \leq \frac{p\,(T+1)^{\frac{p}{2}}}{2^p} A_{+\infty} < +\infty$.

In the next step, we need to prove that I_λ^+ is unbounded from below. To this end, we consider two cases: $B^{+\infty} = +\infty$ and $B^{+\infty} < +\infty$. If $B^{+\infty} = +\infty$, let $\{c_n\}$ be a sequence of positive numbers, with $\lim_{n \to +\infty} c_n = +\infty$, such that

$$\sum_{k=1}^{T} F^+(k, c_n) = \sum_{k=1}^{T} F(k, c_n) \geq \frac{(2+p)}{\lambda p}\, c_n^p, \text{ for every } n \in \mathbb{N}.$$

In the following, we take in X the sequence $\{\omega_n\}$ defined by putting $\omega_n(k) = c_n$, for $k \in \mathbb{Z}(1,\ T)$. Using again (2), one has

$$I_\lambda^+(\omega_n) = \frac{2}{p}\left(\left(1 + c_n^2\right)^{\frac{p}{2}} - 1\right) - \lambda \sum_{k=1}^{T} F^+(k, c_n) \leq \frac{2}{p}\, c_n^p - \frac{2+p}{p}\, c_n^p = -c_n^p,$$

which implies that $\lim_{n \to +\infty} I_\lambda^+(\omega_n) = -\infty$. If $B^{+\infty} < +\infty$, since $\lambda > \frac{2}{pB^{+\infty}}$, we may take $\epsilon_0 > 0$ such that $\frac{2}{p} - \lambda B^{+\infty} + \lambda\epsilon_0 < 0$. Then there exists a sequence of positive numbers $\{c_n\}$ such that $\lim_{n \to +\infty} c_n = +\infty$ and

$$(B^{+\infty} - \epsilon_0)c_n^p \leq \sum_{k=1}^{T} F^+(k, c_n) = \sum_{k=1}^{T} F(k, c_n) \leq (B^{+\infty} + \epsilon_0)c_n^p.$$

Arguing as before and by choosing $\{\omega_n\}$ in X as above, we have

$$I_\lambda^+(\omega_n) = \frac{2}{p}\left(\left(1 + c_n^2\right)^{\frac{p}{2}} - 1\right) - \lambda \sum_{k=1}^{T} F^+(k, c_n) \leq \frac{2}{p}\, c_n^p - \lambda(B^{+\infty} - \epsilon_0)c_n^p = (\frac{2}{p} - \lambda B^{+\infty} + \lambda\epsilon_0)c_n^p.$$

Since $\frac{2}{p} - \lambda B^{+\infty} + \lambda\epsilon_0 < 0$, it is clear that $\lim_{n \to +\infty} I_\lambda^+(\omega_n) = -\infty$. Considering the above two cases, we follow that I_λ^+ is unbounded from below.

According to Lemma 3, there exist a sequence $\{u_n\}$ of critical points (local minima) of I_λ^+ such that $\lim\limits_{n \to +\infty} \Phi(u_n) = +\infty$. Hence, for every $n \in \mathbb{N}$, u_n is a non-zero solution of problem (D_p^{λ, f^+}), by Lemma 1, u_n is a positive solution of problem $(D_p^{\lambda, f})$. Since Φ is bounded on bounded sets and $\lim\limits_{n \to +\infty} \Phi(u_n) = +\infty$, $\{u_n\}$ must be unbounded. So Theorem 2 holds and the proof is complete.

Theorem 3. *Let* $2 \leq p < +\infty$ *and* $f(k, \cdot) : \mathbb{R} \to \mathbb{R}$ *to be a continuous function with* $f(k, 0) \geq 0$ *for each* $k \in \mathbb{Z}(1,\ T)$. *Assume that*

$$(i_2)\ A_{+\infty} < \frac{(\sqrt{2})^p}{(T+1)^{p-1}} B^{+\infty}.$$

Then, for each $\lambda \in (\frac{(\sqrt{2})^p}{pB^{+\infty}}, \frac{2^p}{p(T+1)^{p-1} A_{+\infty}})$, *problem* $(D_p^{\lambda, f})$ *has an unbounded sequence of positive solutions.*

Proof. We sketch only the differences with the proof of Theorem 2. For $t > 0$, make

$$r = \frac{(2t)^p}{p\,(T+1)^{p-1}}.$$

Assume $u \in X$ and

$$\Phi(u) = \frac{1}{p}\sum_{k=0}^{T}\left(\left(1+(\triangle u(k))^2\right)^{\frac{p}{2}} - 1\right) \le r.$$

Denote $v(k) = \left(1+(\triangle u(k))^2\right)^{\frac{p}{2}} - 1,$ for every $k \in \mathbb{Z}(0,\, T)$, then $\sum\limits_{k=0}^{T} v(k) \le p\, r$.

Noting the inequality $(x+y)^\theta \le x^\theta + y^\theta,$ for $0 < \theta \le 1, x \ge 0, y \ge 0$ and Hölder inequality, one has

$$\begin{aligned}
\sum_{k=0}^{T}(\triangle u(k))^2 &= \sum_{k=0}^{T}(1+v(k))^{\frac{2}{p}} - 1) \\
&\le \sum_{k=0}^{T}(v(k))^{\frac{2}{p}} \\
&\le (T+1)^{\frac{p-2}{p}}\left(\sum_{k=0}^{T} v(k)\right)^{\frac{2}{p}} \\
&\le (T+1)^{\frac{p-2}{p}}(pr)^{\frac{2}{p}} = \frac{4t^2}{T+1}.
\end{aligned}$$

Applying (3), we have

$$||u||_\infty \le \frac{\sqrt{T+1}}{2}\left(\sum_{k=0}^{T}(\triangle u_k)^2\right)^{\frac{1}{2}} \le t.$$

By the definition of φ, we have

$$\varphi(r) = \frac{\sup\limits_{\Phi^{-1}[0,\, r]} \Psi^+}{r} \le \frac{\sup\limits_{||u||_\infty \le t} \sum\limits_{k=0}^{T} F^+(k, u(k))}{r} \le \frac{p(T+1)^{p-1}\sum\limits_{k=1}^{T}\max\limits_{0\le s\le t} F(k,s)}{2^p t^p}.$$

Using condition(i_2), $\varphi_\infty \le \frac{p(T+1)^{p-1}}{2^p} A_{+\infty} < +\infty$ holds.

Now, we verify that I_λ^+ is unbounded form blow. Fist, assume that $B^{+\infty} = +\infty$. Let $\{c_n\}$ be a sequence of positive numbers, with $\lim_{n\to+\infty} c_n = +\infty$, such that

$$\sum_{k=1}^{T} F^+(k, c_n) = \sum_{k=1}^{T} F(k, c_n) \ge \frac{(\sqrt{2}\,)^p + p}{\lambda p}\, c_n^p, \text{ for } n \in \mathbb{N}.$$

Picking the sequence $\{\omega_n\}$ in X by $\omega_n(k) = c_n$, for $k \in \mathbb{Z}(1,\, T)$. Exploiting the inequality $(x+y)^\theta \le 2^{\theta-1}(x^\theta + y^\theta)$ for $\theta \ge 1, x \ge 0, y \ge 0$, we get

$$\begin{aligned}
I_\lambda^+(\omega_n) &= \tfrac{2}{p}\left(\left(1+c_n^2\right)^{\frac{p}{2}} - 1\right) - \lambda \textstyle\sum_{k=1}^{T} F^+(k, c_n) \le \frac{(\sqrt{2}\,)^p}{p}\, c_n^p + \frac{(\sqrt{2}\,)^p - 2}{p} - \frac{(\sqrt{2}\,)^p + p}{p}\, c_n^p \\
&= -c_n^p + \tfrac{(\sqrt{2}\,)^p - 2}{p},
\end{aligned}$$

which implies that $\lim_{n\to+\infty} I_\lambda(\omega_n) = -\infty$.

Next, assume that $B^{+\infty} < +\infty$. Since $\lambda > \frac{(\sqrt{2}\,)^p}{pB^{+\infty}}$, we may take $\epsilon_0 > 0$ such that $\frac{(\sqrt{2}\,)^p}{p} - \lambda B^{\infty} + \lambda\epsilon_0 < 0$. Then there exists a sequence of positive numbers $\{c_n\}$ such that $\lim_{n\to+\infty} c_n = +\infty$ and

$$(B^{+\infty} - \epsilon_0)c_n^p \le \sum_{k=1}^{T} F^+(k, c_n) = \sum_{k=1}^{T} F(k, c_n) \le (B^{+\infty} + \epsilon_0)c_n^p.$$

Define the sequence $\{\omega_n\}$ in S as above, we obtain

$$\begin{aligned} I_\lambda^+(\omega_n) &= \tfrac{2}{p}\left(\left(1 + c_n^2\right)^{\frac{p}{2}} - 1\right) - \lambda \textstyle\sum_{k=1}^{T} F^+(k, c_n) \le \frac{(\sqrt{2}\,)^p}{p} c_n^p + \frac{(\sqrt{2}\,)^{p-2}}{p} - \lambda(B^{+\infty} - \epsilon_0)c_n^p \\ &= (\tfrac{(\sqrt{2}\,)^p}{p} - \lambda B^{+\infty} + \lambda\epsilon_0)c_n^p + \tfrac{(\sqrt{2}\,)^{p-2}}{p}. \end{aligned}$$

Since $\frac{(\sqrt{2}\,)^p}{p} - \lambda B^{+\infty} + \lambda\epsilon_0 < 0$, it is obvious that $\lim_{n\to+\infty} I_\lambda^+(\omega_n) = -\infty$.

Thus, we follow that I_λ^+ is unbounded from below. According to Lemmas 1 and 3, we have finished the proof of the theorem.

Similarly, considering the functional I_λ^-, we can achieve the following results.

Theorem 4. *Let $1 \le p < 2$ and $f(k,\cdot) : \mathbb{R} \to \mathbb{R}$ to be a continuous function with $f(k, 0) \le 0$ for each $k \in \mathbb{Z}(1, T)$. Assume that*

$$(i_3)\ A_{-\infty} < \frac{2^{p-1}}{(T+1)^{\frac{p}{2}}} B^{-\infty}.$$

Then, for each $\lambda \in (\frac{2}{pB^{-\infty}}, \frac{2^p}{p(T+1)^{\frac{p}{2}} A_{-\infty}})$, problem $(D_p^{\lambda,f})$ has an unbounded sequence of negative solutions.

Theorem 5. *Let $2 \le p < +\infty$ and $f(k,\cdot) : \mathbb{R} \to \mathbb{R}$ to be a continuous function with $f(k, 0) \le 0$ for each $k \in \mathbb{Z}(1, T)$. Assume that*

$$(i_4)\ A_{-\infty} < \frac{(\sqrt{2})^p}{(T+1)^{p-1}} B^{-\infty}.$$

Then, for each $\lambda \in (\frac{(\sqrt{2})^p}{pB^{-\infty}}, \frac{2^p}{p(T+1)^{p-1} A_{-\infty}})$, problem $(D_p^{\lambda,f})$ has an unbounded sequence of negative solutions.

Combining Theorems 2 and 4, we have the following corollary.

Corollary 1. *Let $1 \le p < 2$ and $f(k,\cdot) : \mathbb{R} \to \mathbb{R}$ to be a continuous function with $f(k, 0) = 0$ for each $k \in \mathbb{Z}(1, T)$. Assume that*

$$(i_5)\ \max\{A_{+\infty}, A_{-\infty}\} < \frac{2^{p-1}}{(T+1)^{\frac{p}{2}}} \min\{B^{+\infty}, B^{-\infty}\}.$$

Then, for each $\lambda \in (\frac{2}{p\min\{B^{+\infty}, B^{-\infty}\}}, \frac{2^p}{p(T+1)^{\frac{p}{2}} \max\{A_{+\infty}, A_{-\infty}\}})$, problem $(D_p^{\lambda,f})$ admits two unbounded sequences of constant-sign solutions (one positive and one negative).

Similarly, combining Theorems 3 and 5, we have the following corollary.

Corollary 2. *Let $2 \le p < +\infty$ and $f(k,\cdot) : \mathbb{R} \to \mathbb{R}$ to be a continuous function with $f(k, 0) = 0$ for each $k \in \mathbb{Z}(1, T)$. Assume that*

$$(i_6)\ \max\{A_{+\infty}, A_{-\infty}\} < \frac{(\sqrt{2})^p}{(T+1)^{p-1}} \min\{B^{+\infty}, B^{-\infty}\}.$$

Then, for each $\lambda \in \left(\frac{(\sqrt{2})^p}{p\min\{B^{+\infty},B^{-\infty}\}}, \frac{2^p}{p(T+1)^{p-1}\max\{A_{+\infty},A_{-\infty}\}}\right)$*, problem* $(D_p^{\lambda,f})$ *admits admits two unbounded sequences of constant-sign solutions (one positive and one negative).*

Remark 1. *If we let* $p \to 2^-$ *in Theorem 2, we find that the conditions and consequence of Theorem 2 is the same as those of Theorem 3 for* $p = 2$*. Moreover the results are consistent with results in [37]. For the special case,* $p = 1$*, Theorem 2 reduces to Corollary 2.1 of [35].*

Remark 2. *We note that, if for each* $k \in \mathbb{Z}(1,T), f(k,\cdot) : \mathbb{R} \to \mathbb{R}$ *is a continuous function satisfying* $f(k,t)t \geq 0$ *for all* $t \in \mathbb{R} \setminus \{0\}$*, then*

$$A_{+\infty} = \liminf_{t\to+\infty} \frac{\sum_{k=1}^{T} F(k,t)}{t^p}, \text{ and } A_{-\infty} = \liminf_{t\to-\infty} \frac{\sum_{k=1}^{T} F(k,t)}{|t|^p}.$$

Consequently, Theorem 1 immediately follows by Corollaries 1 and 2.

4. Two Examples

Example 1. *For* $1 \leq p < 2$*, we consider the boundary value problem* $(D_p^{\lambda,f})$ *with*

$$f(k,t) = p|t|^{p-1}\text{sign}(t)\left(\frac{T+1}{T} + \sin\left(\frac{1}{2T}\ln(|t|^p+1)\right) + \frac{1}{2T}\cos\left(\frac{1}{2T}\ln(|t|^p+1)\right)\right), \quad (9)$$

for $k \in \mathbb{Z}(1,\, T)$*, then*

$$F(k,t) = \int_0^t f(k,\tau)d\tau = \frac{T+1}{T}|t|^p + (|t|^p+1)\sin\left(\frac{1}{2T}\ln(|t|^p+1)\right), \text{ for } t \in \mathbb{R}.$$

Since $f(k,t) \geq p\, t^{p-1}\left(\frac{T+1}{T} - 1 - \frac{1}{2T}\right) = \frac{p}{2T}\, t^{p-1} > 0$*, for* $t > 0$ *and* $f(k,0) = 0$*, we follow that for each fixed* $k \in \mathbb{Z}(1,\, T)$*,* $F(k,t)$ *is strictly monotone increasing on* $[0,+\infty)$*. One has* $\max_{0\leq s\leq t} F(k,s) = F(k,t)$*, for each* $t \geq 0$*. Clearly,*

$$A_{+\infty} = \liminf_{t\to+\infty} \frac{TF(k,t)}{t^p} = \liminf_{t\to+\infty} \frac{(T+1)t^p + T(t^p+1)\sin(\frac{1}{2T}\ln(t^p+1))}{t^p} = 1,$$

and

$$B^{+\infty} = \limsup_{t\to+\infty} \frac{TF(k,t)}{t^p} = \limsup_{t\to+\infty} \frac{(T+1)t^p + T(t^p+1)\sin(\frac{1}{2T}\ln(t^p+1))}{t^p} = 2T+1.$$

In view of $1 \leq p < 2$ *, we follow that* $A_{+}\infty < \frac{2^{p-1}}{(T+1)^{\frac{p}{2}}}B^{+\infty}$*. Applying to Theorem 2, problem* $(D_p^{\lambda,f})$ *admits an unbounded sequence of positive solutions.*

Let us consider another example.

Example 2. *Let* $T = 4$*,* $p = 3$ *and* f *be a function defined as follows*

$$f(k,t) = 3|t|t\left(\frac{5}{4} + \sin(\frac{1}{8}\ln(|t|^3+1)) + \frac{1}{8}\cos(\frac{1}{8}\ln(|t|^3+1))\right), k \in \mathbb{Z}(1,\, 4)$$

Then, for every $\lambda \in (\frac{2\sqrt{2}}{27}, \frac{8}{75})$*, the problem*

$$\begin{cases} -\triangle\left(\phi_{3,c}\left(\triangle u(k-1)\right)\right) = \lambda f(k, u(k)), \quad k \in \mathbb{Z}(1, 4), \\ u(0) = u(5) = 0, \end{cases} \tag{10}$$

Admits an unbounded sequence of positive solutions and an unbounded sequence of negative solutions. Indeed, $f(k,t) \geq 3t^2\left(\frac{5}{4} - 1 - \frac{1}{8} = \frac{3}{8}t^2\right) > 0$*, for* $t > 0$ *and* $f(k,0) = 0$*.*

$$F(k, t) = \int_0^t f(k, \tau)d\tau = \frac{5}{4}|t|^3 + (|t|^3 + 1)\sin(\frac{1}{8}\ln(|t|^3 + 1)), \textit{ for } t \in \mathbb{R}.$$

Since $f(k,t) \geq 3\,t^2\left(\frac{5}{4} - 1 - \frac{1}{8}\right) = \frac{3}{8}t^2 > 0$*, for* $t > 0$*, we follow that for each fixed* $k \in \mathbb{Z}(1, 4)$*,* $F(k,t)$ *is strictly monotone increasing on* $[0, +\infty)$*. Thus,* $\max_{|s| \leq t} F(k,s) = F(k,t)$*, for each* $t \geq 0$*. Obviously,*

$$A_{\pm\infty} = \liminf_{t\to+\infty} \frac{4F(k,t)}{t^3} = \liminf_{t\to+\infty} \frac{5t^3 + 4(t^3+1)\sin(\frac{1}{8}\ln(t^3+1))}{t^3} = 1,$$

and

$$B^{\pm\infty} = \limsup_{t\to+\infty} \frac{4F(k,t)}{t^3} = \limsup_{t\to+\infty} \frac{5t^3 + 4(t^3+1)\sin(\frac{1}{8}\ln(t^3+1))}{t^3} = 9.$$

Through simple computation, $\max\{A_{+\infty}, A_{-\infty}\} < \frac{(\sqrt{2})^p}{(T+1)^{p-1}} \min\{B^{+\infty}, B^{-\infty}\}$ *holds. Corollary 2 ensures our claim.*

5. Conclusions

In this paper, we have discussed the Dirichlet boundary value problem of the difference equation with p-mean curvature operator. Some sufficient conditions are derived for the existence of sequences of constant-sign solutions to the problem. Two examples are given to show the effectiveness of our results.

To solve problem $(D_p^{\lambda,f})$, we further develop the methods adopted in [23]. The approaches can be used for the boundary value problems of differential equations involving p-mean curvature operator. Therefore, our work has both theoretical and practical significance.

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