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The Homomorphism Theorems of M-Hazy Rings and Their Induced Fuzzifying Convexities

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Abstract: In traditional ring theory, homomorphisms play a vital role in studying the relation between two algebraic structures. Homomorphism is essential for group theory and ring theory, just as continuous functions are important for topology and rigid movements in geometry. In this article, we propose fundamental theorems of homomorphisms of M-hazy rings. We also discuss the relation between M-hazy rings and M-hazy ideals. Some important results of M-hazy ring homomorphisms are studied. In recent years, convexity theory has become a helpful mathematical tool for studying extremum problems. Finally, M-fuzzifying convex spaces are induced by M-hazy rings.

Keywords: M-hazy group; M-hazy ring; M-hazy ideal; M-hazy ring homomorphism; M-fuzzifying convex space

1. Introduction

Fuzzy set theory was proposed by Zadeh in [1], which is a pioneering study that opens up new avenues for applying it to many other useful directions. Many researchers applied fuzzy sets to algebraic structures for their characterization, such as for groups, rings, modules, vector spaces, topologies, and many more. Rosenfeld introduced fuzzy groups in [2], in which he defined the fuzzy subgroup of a group and discussed some of its properties. In Rosenfeld's approach, only the subset of a group $(G, \circ)$ is fuzzy and the binary operation is crisp. Demirci [3] introduced "vague groups" by using vague binary operation. Years later, Demirci [4] defined "smooth groups" by using fuzzy functions and fuzzy equalities [5–7]. Since then, many researchers have worked in various directions, such as with smooth subgroups and smooth homomorphisms, generalized vague groups, vague rings, vague ideals, fuzzy multi-polygroups, fuzzy multi-H_v-ideals, and other fuzzy algebra [8–19].

Liu [20,21] proposed and obtained fundamental results of fuzzy subrings and fuzzy ideals of a ring. Dixit et al. [22] discussed some basic properties of fuzzy rings. Crist and Mordeson [23] gave the notion of "vague rings" by using a vague binary operation. Aktas and Cagman [24] defined a type of fuzzy ring, in which they introduced fuzzy rings based on fuzzy binary operations. Xu [25] defined "smooth rings" by using a smooth binary operation. Sezer [26] discussed some properties of "vague rings" by using the concept of fuzzy equality. Shen [27] defined a fuzzifying group based on complete residuated lattice-valued logic. Liu and Shi [28] introduced an M-hazy group by using an M-fuzzy function and an M-hazy operation. In this approach, the concept of the M-hazy associative law, completely residuated lattices, and the fuzzy implications are used. Furthermore, by using two binary operations, namely the M-hazy addition operation and the M-hazy multiplication operation, Mehmood et al. [29] introduced a new type of ring called M-hazy rings and discussed their various properties. It is important to mention that a new fuzzy distributive law called the M-hazy distributive law has been introduced in order to obtain M-hazy rings.

In classical ring theory, homomorphisms of two algebraic structures are meant to be the relation between the algebraic structure of one type to another algebraic structure of the same type. Homomorphism is essential for group theory and ring theory, just as continuous functions are important for topology and rigid movements in geometry. To say that a homomorphism preserves structure is the same thing as saying that, if I took all of the copper beams from a skyscraper and made three schools, I would have "preserved the structure" of the skyscraper. Saying that "some of the structure" is preserved is okay, though, and is completely different from "preserving the structure". Nonetheless, the measure of "exactly how much" of the structure is preserved by a homomorphism is given by the isomorphism theorem. It has a lot of uses in image understanding, image processing, and general applications in pattern analysis—for example, in computer vision. By using the concept of classical homomorphism and fuzzy sets, Xuan [30] introduced the concept of fuzzy homomorphism and fuzzy isomorphism between two fuzzy groups. Ajmal [31] introduced the fundamental theorem of homomorphism for fuzzy subgroups and also discussed its related properties. Dobritsa and Yakh yaeva [32] introduced homomorphisms of fuzzy groups by using the concept of fuzzy binary operation, that is, the result of the operation is not an element but part of the original collection. Rasouli and Davvaz [33] considered the homomorphisms, dual homomorphisms, and strong homomorphisms between hyper-MV algebras, and they defined hyper-MV ideals with their properties.

As a useful mathematical instrument, convexity theory has been quickly applied in recent years to the study of extreme problems. A convexity on a set is a family of subsets stable for intersection and for nested unions. Each subset is included in the smallest convex set, the convex hull; the hull of a finite set is called a polytope. Sets of this type determine the convexity. Various classes of examples have been selected for intensive use. Other convexities are obtained from certain natural constructions, such as generation by subbases or formation of subspaces, products, convex hyperspaces, quotients, etc. The notion of convex structures has also been promoted to fuzzy settings. With the constant expansion of convexity theory, fuzzy convex structures have become an important topic of research, such as in [34–43]. Shi and Xiu [44] proposed a new method for the fuzzification of a convex structure, which is called an M-fuzzifying convex structure. Liu and Shi [45] proposed the notion of M-fuzzifying median algebra, in which they defined the fuzzy associative law, which gives the inspiration to work on other algebraic structures, such as groups, lattices, and rings. Furthermore, Liu and Shi [46] introduced a new approach to the fuzzification of lattices, which is called an M-hazy lattice, and also discussed its relation with M-fuzzifying convex spaces. The convexity of an M-fuzzy set on a set X is an M-fuzzy set on a power set with certain properties; therefore, each subset of X can be regarded as a convex set to some extent. Keeping in view the importance of this fact, we gave an approach to inducing M-fuzzifying convex spaces into M-hazy rings.

The article is organized as follows. Section 2 consists of the fundamental notions of completely residuated lattices, M-hazy groups, M-hazy rings, and M-hazy ideals with their basic properties. The aim of Section 3 is to present homomorphism theorems of M-hazy rings. We discuss the relation between M-hazy rings and M-hazy ideals. Then, various properties of homomorphisms of M-hazy rings are studied. Section 4 gives an approach to inducing M-fuzzifying convex spaces using M-hazy rings. Section 5 concludes the paper.

2. Preliminaries

In this section, we will review some basic definitions and results about complete residuated lattices, M-hazy groups, M-hazy rings, M-hazy ideals, and M-fuzzifying convexity.

Throughout this article, $(M, \vee, \wedge, \star, \rightarrow, \perp, \top)$ expresses a completely residuated lattice, where the partial order of M is $\leq$. For $D \subseteq M$, we denote $\bigvee D$ for the least upper bound of D and $\bigwedge D$ for the greatest lower bound of D. In general, we use the notion that $\bigvee \emptyset = \perp$ and $\bigwedge \emptyset = \top$. Note that

2^P (respectively, M^P) represents the set of all subsets (respectively, M-subsets) on a nonempty set P. A family $\{A_i \mid i \in \omega\}$ is up-directed given that for every $A_1, A_2 \in \{A_i \mid i \in \omega\}$, there is an element $A_3 \in \{A_i \mid i \in \omega\}$ such that $A_1 \subseteq A_3$ and $A_2 \subseteq A_3$, denoted by: $\{A_i \mid i \in \omega\} \overset{dir}{\subseteq} 2^P$.

A mapping $F : P \longrightarrow M$ is defined as an M-fuzzy set on P. The collection of all M-fuzzy sets on P is defined by M^P. For $\lambda \in M$ and $A \in M^P$, define an M-fuzzy set as follows:

$$(A \star \lambda)(p) = A(p) \star \lambda.$$

When $A = \chi_{\{p\}}, \chi_{\{p\}} \star \lambda$ is denoted by p_λ, which is called an M-fuzzy point. It is easy to check that

$$p_\lambda(q) = \begin{cases} \lambda, & p = q, \\ \bot, & p \neq q. \end{cases}$$

Definition 1 ([47]). *Let $\star : M \times M \longrightarrow M$ be a mapping. If $\star$ satisfies the following conditions;*

(1) $k \star l = l \star k$,
(2) $(k \star l) \star m = k \star (l \star m)$,
(3) $k \leq m, l \leq d$ *implies* $k \star l \leq m \star d$,
(4) $k \star 1 = k$ *for all* $k \in M$, *then* $\star$ *is called a triangular norm (for short, t-norm) on* M.

Definition 2 ([47]). *Let $\rightarrow: M \times M \longrightarrow M$ be a mapping and let $\star$ be a t-norm in M. If for all $k, l, m \in M$,*

$$k \leq l \rightarrow m \Leftrightarrow k \star l \leq m,$$

then $\rightarrow$ is called the residuum of $\star$.

Definition 3 ([48]). *Let $(M, \vee, \wedge, \bot, \top)$ be a bounded lattice with the smallest element $\bot$ and the greatest element $\top$, $\star$ a t-norm on M, and $\rightarrow$ the residuum of $\star$. Then, $(M, \vee, \wedge, \star, \rightarrow, \bot, \top)$ is called a residuated lattice.*

A residuated lattice is known to be complete if the paramount lattice is complete. In addition, we denote $k \leftrightarrow l = (k \rightarrow l) \wedge (l \rightarrow k)$. In the Proposition given below, some properties of the implication operation are shown.

Proposition 1 ([49,50]). *Let $(M, \vee, \wedge, \star, \rightarrow, \bot, \top)$ be a complete residuated lattice. Then, for all $k, l, m \in M$, $\{k_i\}_{i \in I}, \{l_i\}_{i \in I} \subseteq M$, the following statements holds:*

(1) $k \rightarrow l = \bigvee\{m \in M \mid k \star m \leq l\}$,
(2) $l \leq k \rightarrow l, \top \rightarrow k = k$,
(3) $k \leq l$ *implies* $k \rightarrow l = \top$,
(4) $k \rightarrow (l \rightarrow m) = l \rightarrow (k \rightarrow m) = (k \star l) \rightarrow m$,
(5) $k \rightarrow (\bigwedge_{i \in I} l_i) = \bigwedge_{i \in I} (k \rightarrow l_i)$,
(6) $(\bigvee_{i \in I} k_i) \rightarrow l = \bigwedge_{i \in I} (k_i \rightarrow l)$.

Definition 4 ([28]). *A mapping $+ : P \times P \longrightarrow M^P$ is called an M-hazy operation on P, provided that it satisfies the following two conditions:*

(MH1) $\forall k, l \in P$, *we have* $\bigvee_{p \in P} (k + l)(p) \neq \bot$,
(MH2) $\forall k, l, p, q \in P$, $(k + l)(p) \star (k + l)(q) \neq \bot \Rightarrow p = q$.

Definition 5 ([28]). *Let* $+ : P \times P \longrightarrow M^P$ *be an M-hazy operation on P. The pair* $(P,+)$ *is called an M-hazy (semigroup) group if + satisfies the following conditions ((MG1)) (MG1)-(MG3):*

(MG1) $\forall k,\ l,\ m,\ p,\ q \in P,\ (k+l)(p) \star (l+m)(q) \le \bigwedge_{r\in P}((p+m)(r) \leftrightarrow (k+q)(r))$, *i.e., the M-hazy associative law holds.*
(MG2) *There exists an element* $o \in P$ *such that* $o + k = k_\top$ *for all* $k \in P$, *where o is called the left identity element of P.*
(MG3) *For every* $k \in P$, *there exists an element* $l \in P$ *such that* $l + k = o_\top$, *where l is called the left inverse of k and is denoted by* $-k$.
An M-hazy group $(P,+)$ *is said to be abelian if + satisfies the following condition:*
(MG4) $k + l = l + k$ *for all* $k, l \in P$.

Proposition 2 ([28]). *In an M-hazy group* $(P,+)$, *the following equations hold for all* $k, l, m \in P$:

$$(k+l)(m) = ((-k)+m)(l) = (m+(-l))(k) = (l+(-m))(-k)$$
$$= ((-m)+k)(-l) = ((-l)+(-k))(-m).$$

Theorem 1 ([28]). *Let H be a nonempty subset of an M-hazy group* $(P,+)$, *then* $(H,+)$ *is an M-hazy subgroup of* $(P,+)$ *if and only if the following two conditions hold:*

(1) *for all* $k,l \in H$, *we have* $\bigvee_{p\in H}(k+l)(p) \neq \bot$,
(2) *for all* $k \in H$, *we have* $k^{-1} \in H$.

Definition 6 ([28]). *Let* $(N,+)$ *be an M-hazy subgroup of M-hazy group* $(P,+)$. *Then,* $(N,+)$ *is called an M-hazy normal subgroup of* $(P,+)$, *provided that for each* $k \in P$, *we have* $k + N = N + k$.

Definition 7 ([29]). *Let* $+ : R \times R \longrightarrow M^R$ *and* $\bullet : R \times R \longrightarrow M^R$ *be the two M-hazy operations on R. Then, the 3-tuple* $(R,+,\bullet)$ *is called an M-hazy ring if the following three conditions are satisfied:*

(MHR1) $(R,+)$ *is an abelian M-hazy group,*
(MHR2) $(R,\bullet)$ *is an M-hazy semigroup,*
(MHR3) $\forall k, l, m, p, q, r \in R$,
$(k \bullet l)(p) \star (l+m)(q) \star (k \bullet m)(r) \le \bigwedge_{w\in R}((k \bullet q)(w) \leftrightarrow (p+r)(w))$,
i.e., the M-hazy distributive law holds.

In the following, the operations $+$ and $\bullet$ are respectively called the M-hazy addition operation and M-hazy multiplication operation.

Proposition 3 ([29]). *Let* $(R,+,\bullet)$ *be an M-hazy ring and o be the additive identity element of R; then,* $\forall k, l, m \in R$, *we have*

(1) $(k \bullet o)(o) \neq \bot$ *and* $(o \bullet k)(o) \neq \bot$,
(2) *if* $(k \bullet l)(p) \neq \bot$ *then* $(k \bullet (-l))(-p) \neq \bot$ *and* $((-k) \bullet l)(-p) \neq \bot$,
(3) *if* $(k \bullet l)(p) \neq \bot$ *then* $((-k) \bullet (-l))(p) \neq \bot$.

Definition 8 ([29]). *Let A be the nonempty subset of an M-hazy ring* $(R,+,\bullet)$; *then,* $(A,+,\bullet)$ *is an M-hazy subring of* $(R,+,\bullet)$ *if and only if the following conditions hold:*

(1) $\forall\, k, l \in A$, *we have* $\bigvee_{p\in A}(k+l)(p) \neq \bot$,
(2) $\forall\, k, l \in A$, *we have* $\bigvee_{p\in A}(k \bullet l)(p) \neq \bot$,
(3) $\forall\, k \in A$, *we have* $-k \in A$.

Definition 9 ([29]). *A nonempty subset I of an M-hazy ring R is called an M-hazy left (respectively, right) ideal of R if the following conditions are satisfied:*

(1) $\forall\, k, l \in I$, *we have* $\bigvee_{p\in I} (k+l)(p) \neq \perp$,

(2) $\forall\, k \in I$, *we have* $-k \in I$,

(3) $\forall\, k \in I, \forall\, p \in R$, *we have* $\bigvee_{q\in I} (p \bullet k)(q) \neq \perp \left(\bigvee_{q\in I} (k \bullet p)(q) \neq \perp \right)$.

M-hazy ideals are playing exactly the same role as that of M-hazy normal subgroups in an M-hazy group context; in fact, an M-hazy ideal is an M-hazy normal subgroup of an M-hazy additive group of an M-hazy ring. In particular, we can form cosets and consider the quotient R/I_+.

Proposition 4 ([29]). *Let $(R, +, \bullet)$ be an M-hazy ring and I be an M-hazy ideal of R. If $[k +_\perp l]\widetilde{+}m = [k]\widetilde{+}[l +_\perp m]$, $[k \bullet_\perp l]\widetilde{\bullet}m = [k]\widetilde{\bullet}[l \bullet_\perp m]$ and $[k]\widetilde{\bullet}[l +_\perp m] = ([k]\widetilde{\bullet}[l]) +_\perp ([k]\widetilde{\bullet}[m])$ for all $k, l, m \in R$, then R/I_+ is an M-hazy (quotient) ring under the M-hazy operations $\widetilde{+}$ and $\widetilde{\bullet}$.*

We now present the definition of an M-fuzzifying convex space, and we refer to Vel [51] for all context on the theory of convexity that might be necessary.

Definition 10 ([44]). *A mapping $\mathscr{C} : 2^P \to M$ is called an M-fuzzifying convexity on P if it satisfies the following conditions:*

(MC1) $\mathscr{C}(\emptyset) = \mathscr{C}(P) = \top$,
(MC2) *if* $\{A_i | i \in \Omega\} \subseteq 2^P$ *is nonempty, then* $\mathscr{C}(\bigcap_{i\in\Omega} A_i) \geq \bigwedge_{i\in\Omega} \mathscr{C}(A_i)$,
(MC3) *if* $\{A_i | i \in \Omega\} \overset{dir}{\subseteq} 2^P$, *then* $\mathscr{C}(\bigcup_{i\in\Omega} A_i) \geq \bigwedge_{i\in\Omega} \mathscr{C}(A_i)$.

For an M-fuzzifying convex structure $\mathscr{C}$ on P, the pair $(P, \mathscr{C})$ is called an M-fuzzifying convex space.

A mapping $\phi : (P, \mathscr{C}_P) \longrightarrow (Q, \mathscr{C}_Q)$ is said to be M-fuzzifying convexity preserving (M-CP, in short) if $\mathscr{C}_P(\phi^{\leftarrow}(B)) \geq \mathscr{C}_Q(B)$ for each $B \in 2^Q$.

3. Properties of Homomorphisms of M-Hazy Rings

In this section, we discuss the relation between M-hazy rings and M-hazy ideals. Furthermore, the homomorphism theorems of M-hazy rings are proven and their various related properties are studied. We define $\perp$ as $\star$-prime when $k \star b = \perp \Rightarrow k = \perp$ or $b = \perp$ ($k \neq \perp$ and $b \neq \perp \Rightarrow k \star b \neq \perp$).

Let I be an M-hazy ideal of an M-hazy ring $(R, +, \bullet)$, $\perp$ be $\star$-prime, and $\omega = \{k + I \mid k \in R\}$. Define a relation over ω by $k_1 + I \sim k_2 + I \Leftrightarrow \exists\, u \in I$ such that

$$((-k_1) + k_2)(u) \neq \perp.$$

We first prove that the relation "$\sim$" on the set ω is an equivalence relation.
(1) Since $((-k) + k)(o) = \top$ and $o \in I$, we have $k + I \sim k + I\ \forall\, k \in R$.
(2) Let $k_1 + I \sim k_2 + I$; there exists $i \in I$ such that

$$((-k_1) + k_2)(i) \neq \perp,$$

then by Proposition 2, we obtain

$$((-k_2) + k_1)(-i) \neq \perp.$$

Since $i \in I$, so $-i \in I$, and consequently,

$$k_2 + I \sim k_1 + I.$$

(3) Let $k_1 + I \sim k_2 + I$ and $k_2 + I \sim k_3 + I$; then, there exists $i_1, i_2 \in I$ such that

$$((-k_1) + k_2)(i_1) \neq \perp \text{ and } ((-k_2) + k_3)(i_2) \neq \perp.$$

In addition, by Proposition 2, we have

$$(i_1 + (-k_2))(-k_1) \neq \perp.$$

Since $i_1, i_2 \in I$, and I is an M-hazy ideal of R, then there exists $m \in I$ such that $(i_1 + i_2)(m) \neq \perp$. Now, by (MG1), we obtain

$$(i_1 + (-k_2))(-k_1) \star ((-k_2) + k_3)(i_2) \leq \bigwedge_{r \in R} (((-k_1) + k_3)(r) \leftrightarrow (i_1 + i_2)(r)),$$

$$(i_1 + (-k_2))(-k_1) \star ((-k_2) + k_3)(i_2) \star (i_1 + i_2)(m) \leq ((-k_1) + k_3)(m).$$

Since $(i_1 + (-k_2))(-k_1) \neq \perp$, $((-k_2) + k_3)(i_2) \neq \perp$, and $\perp$ is $\star$-prime, whence$((-k_1) + k_3)(m) \neq \perp$, it follows that $k_1 + I \sim k_3 + I$.

Definition 11 ([29])**.** *Let $(R_1, +, \bullet)$ and $(R_2, \oplus, \circ)$ be two M-hazy rings; then, the mapping $\phi : (R_1, +, \bullet) \longrightarrow (R_2, \oplus, \circ)$ is called an M-hazy ring homomorphism (M-RH, in short) if the following conditions hold:*

(1) $\phi_M^{\rightarrow}(k + l) = \phi(k) \oplus \phi(l)$ *for all* $k, l \in R_1$,
(2) $\phi_M^{\rightarrow}(k \bullet l) = \phi(k) \circ \phi(l)$ *for all* $k, l \in R_1$.

Proposition 5 ([29])**.** *Let $\phi : (R_1, +, \bullet) \longrightarrow (R_2, \oplus, \circ)$ be an M-RH of M-hazy rings, and ϕ is surjective. If I and J are M-hazy ideals of R_1 and R_2, respectively, then, the following statements are valid:*

(1) $\phi^{\rightarrow}(I)$ *is an M-hazy ideal of R_2,*
(2) $\phi^{\leftarrow}(J)$ *is an M-hazy ideal of R_1.*

Recall that when we worked with M-hazy groups, the kernel (in short, ker) of a homomorphism was quite important; the "ker" gave rise to M-hazy normal subgroups, which were important in creating M-hazy quotient groups. For M-hazy ring homomorphisms, the situation is very similar. The "ker" of an M-hazy ring homomorphism is still called the "ker" and gives rise to M-hazy quotient rings.

Theorem 2 ([29])**.** *Let ϕ be an M-hazy ring epimorphism of $(R_1, +, \bullet)$ onto $(R_2, \oplus, \circ)$ with $\ker \phi$ and $\widetilde{\phi} : R_1 / \ker \phi \longrightarrow R_2$ defined by $\widetilde{\phi}([k]) = \phi(k)$. Then,*

$$R_1 / \ker \phi \cong R_2.$$

Proposition 6. *Let $\phi : (R_1, +, \bullet) \longrightarrow (R_2, \oplus, \circ)$ be an M-hazy ring homomorphism and o_1, o_2 be the additive identity elements of R_1 and R_2, respectively. Then,*

(1) $\phi(o_1) = o_2$.
(2) $\phi(-k) = -\phi(k)$.

Proof. The proof is similar to the proof in [28] (Proposition 5.3) and is hence omitted. □

Theorem 3. *Let $(R_1, +, \bullet)$ and $(R_2, \oplus, \circ)$ be two M-hazy rings and ϕ be an M-hazy ring homomorphism of R_1 into R_2. Then,*

(1) *If J is an M-hazy ideal of R_2, then $\phi^{\leftarrow}(J)$ is an M-hazy ideal of R_1 containing* $\ker\phi$.
(2) *If I is an M-hazy ideal of R_1 containing* $\ker\phi$*, then $\phi^{\leftarrow}(\phi^{\rightarrow}(I)) = I$.*
(3) *ϕ results in a one-one containment between the M-hazy ideals of R_1 involving* $\ker\phi$ *and the M-hazy ideals of R_2; in this way, if I is an M-hazy ideal of R_1 which contains* $\ker\phi$*, then $\phi^{\rightarrow}(I)$ is the corresponding M-hazy ideal of R_2, and if J is an M-hazy ideal of R_2, then $\phi^{\leftarrow}(J)$ is the corresponding M-hazy ideal of R_1.*

Proof. (1) Since $\phi^{\leftarrow}(J)$ is an M-hazy ideal of R_1 by condition (2) of Proposition 5, we only need to prove that $\ker\phi \subseteq \phi^{\leftarrow}(J)$. Now, let $p \in \ker\phi$. Since J is an M-hazy ideal of R_2, then $\phi(p) = o' \in J$, and so $p \in \phi^{\leftarrow}(J)$. Hence, $\ker\phi \subseteq \phi^{\leftarrow}(J)$.
(2) Since $\phi(p) \in \phi^{\rightarrow}(I)$ for all $p \in I$, we have $p \in \phi^{\leftarrow}(\phi^{\rightarrow}(I))$. Thus, $I \subseteq \phi^{\leftarrow}(\phi^{\rightarrow}(I))$. Let $p \in \phi^{\leftarrow}(\phi^{\rightarrow}(I))$. Therefore, $\phi(p) \in \phi^{\rightarrow}(I)$, and so there exists $k \in I$ such that $\phi(p) = \phi(k)$. Hence, we get

$$(\phi(k) \oplus (-\phi(p)))(o') = \top.$$

Since R_1 is an M-hazy ring and $k, p \in R_1$, then

$$\bigvee_{m \in R_1} (k + (-p))(m) \neq \bot.$$

Since ϕ is an M-hazy ring homomorphism, we obtain

$$\begin{aligned} & \bigvee_{\phi(m)\in\phi^{\rightarrow}(R_1)} (\phi(k) \oplus \phi(-p))(\phi(m)) \\ = & \bigvee_{\phi(m)\in\phi^{\rightarrow}(R_1)} (\phi(k) \oplus (-\phi(p))(\phi(m)) \\ \neq & \ \bot. \end{aligned}$$

Hence, $\phi(m) = o'$, that is, $m \in \ker\phi$. Since $\ker\phi \subseteq I$, we have $m \in I$. Since I is an M-hazy ideal of R_1 and $\bigvee_{m\in I} (k + (-p))(m) \neq \bot$, we get $p \in I$. Hence, $\phi^{\leftarrow}(\phi^{\rightarrow}(I)) \subseteq I$.
(3) Let K_1 be the set of all M-hazy ideals of R_1 containing $\ker\phi$ and K_2 be the set of all M-hazy ideals of R_2.

Let σ be a mapping of K_1 into K_2 defined by $\sigma^{\rightarrow}(I) = \phi^{\rightarrow}(I)$ for all $I \in K_1$. Since ϕ is well defined, σ is well defined.
Let $\sigma^{\rightarrow}(I_1) = \sigma^{\rightarrow}(I_2)$ for $I_1, I_2 \in K_1$. Now, $\phi^{\rightarrow}(I_1) = \phi^{\rightarrow}(I_2)$ for $I_1, I_2 \in K_1$. Since

$$\phi^{\leftarrow}(\phi^{\rightarrow}(I_1)) = \phi^{\leftarrow}(\phi^{\rightarrow}(I_2)),$$

we obtain $I_1 = I_2$ from (2). Therefore, σ is one-one.

Let $J \in K_2$. Then, $\phi^{\leftarrow}(J) \in K_1$ from (1). Let $q \in \phi^{\rightarrow}(\phi^{\leftarrow}(J))$; then, there exists $p \in \phi^{\leftarrow}(J)$ such that $q = \phi(p)$, that is, $\phi(p) \in J$, we have $\phi^{\rightarrow}(\phi^{\leftarrow}(J)) \subseteq J$. On the other hand, let $p \in J$. Since $p \in J \subseteq R_2 = \phi^{\rightarrow}(R_1)$, there exists $k \in R_1$ such that $\phi(k) = p$. Therefore, $\phi(k) \in J$, and so $k \in \phi^{\leftarrow}(J)$. Hence, $p = \phi(k) \in \phi^{\rightarrow}(\phi^{\leftarrow}(J))$, and so $J \subseteq \phi^{\rightarrow}(\phi^{\leftarrow}(J))$. Thus, $\sigma^{\rightarrow}(\phi^{\leftarrow}(J)) = \phi^{\rightarrow}(\phi^{\leftarrow}(J)) = J$. Hence, σ is surjective. □

Theorem 4. *Let $(R, +, \bullet)$ be an M-hazy ring and I be an M-hazy ideal of R. Then, the mapping $\Pi : R \longrightarrow R/I$ by $\Pi(k) = [k + I]$ for all $k \in R$ is an M-hazy ring homomorphism, called an M-hazy canonical homomorphism.*

Proof. Let $k, l, m \in R$ such that

$$\bigvee_{m\in R} (k + l)(m) \neq \bot.$$

Therefore,

$$\begin{aligned} & \left(\Pi(k)\widetilde{+}\Pi(l)\right)\left(\Pi(m)\right) \\ = & \bigvee_{(m+I)\in R/I} \left([k+I]\widetilde{+}[l+I]\right)\left([m+I]\right) \\ = & \bigvee\{(k+l)\,(m) \mid k\in[k+I], l\in[l+I], m\in[m+I]\} \\ \geq & (k+l)(m)\neq\bot, \end{aligned}$$

by Proposition 4.

Now, let $k, l, m \in R$ such that

$$\bigvee_{m\in R}(k\bullet l)(m)\neq\bot.$$

Then,

$$\begin{aligned} & \left(\Pi(k)\widetilde{\bullet}\Pi(l)\right)\left(\Pi(m)\right) \\ = & \bigvee_{(m+I)\in R/I} \left([k+I]\widetilde{\bullet}[l+I]\right)\left([m+I]\right) \\ = & \bigvee\{(k\bullet l)\,(m) \mid k\in[k+I], l\in[l+I], m\in[m+I]\} \\ \geq & (k\bullet l)(m)\neq\bot, \end{aligned}$$

by Proposition 4.

Thus, Π is an M-hazy ring homomorphism from Definition 11. □

Theorem 5. *Let $(R, +, \bullet)$ be an M-hazy ring, and let I_1 and I_2 be the M-hazy ideals of R. Then,*

$$(I_1+I_2)/I_2 \cong I_1/(I_1\cap I_2).$$

Proof. For all $k \in I_2$, $\bigvee_{k\in I_2}(o+k)(k)\neq\bot$. Thus, $I_2 \subseteq I_1 + I_2$, since $o \in I_1$. Therefore, I_2 is an M-hazy ideal of $I_1 + I_2$. Let $\phi : I_1 \longrightarrow (I_1 + I_2)/I_2$ be defined by $\phi(k) = [k + I_2]$. It is clear that ϕ is surjective.
(1) Let $k = l$ for $k, l \in I_1$. Since $(k + (-l))(o) = \top$ and $o \in I_2$, we obtain $k + I_2 \sim l + I_2$, and so $[k + I_2] = [l + I_2]$. Thus, ϕ is well defined.
(2) Let $k, l, m \in I_1$ such that

$$\bigvee_{m\in I_1}(k+l)(m)\neq\bot.$$

Then, we have

$$\begin{aligned} & \bigvee_{[m+I_2]\in(I_1+I_2)/I_2} ([k+I_2]\widetilde{+}[l+I_2])([m+I_2]) \\ = & \bigvee\{(k+l)\,(m) \mid k\in[k+I_2], l\in[l+I_2], m\in[m+I_2]\} \\ \geq & (k+l)(m)\neq\bot. \end{aligned}$$

Thus, $\bigvee_{\phi(m)\in\phi^{\rightarrow}(I_1)}(\phi(k)\widetilde{+}\phi(l))(\phi(m))\neq\bot.$
(3) Let $k, l, m \in I_1$ such that

$$\bigvee_{m\in I_1}(k\bullet l)(m)\neq\bot.$$

Then, we get

$$\begin{aligned}
& \bigvee_{[m+I_2]\in(I_1+I_2)/I_2} ([k+I_2]\widetilde{\bullet}[l+I_2])([m+I_2]) \\
= & \bigvee\{(k \bullet l)\,(m) \mid k \in [k+I_2], l \in [l+I_2], m \in [m+I_2]\} \\
\geq & (k \bullet l)(m) \neq \bot.
\end{aligned}$$

Thus, $\bigvee_{\phi(m)\in\phi^{\rightarrow}(I_1)} (\phi(k)\widetilde{\bullet}\phi(l))(\phi(m)) \neq \bot$.

(4)

$$\begin{aligned}
& \ker\phi = \{k \in I_1 \mid \phi(k) = [o+I_2]\} \\
= & \{k \in I_1 \mid [k+I_2] = [o+I_2]\} \\
= & \{k \in I_1 \mid k+I_2 \sim o+I_2\} \\
= & \left\{ \, k \in I_1 \mid (-k+o)(i) \neq \bot \text{ for some } i \in I_2 \, \right\} \\
= & \{k \in I_1 \mid k \in I_2\} \\
= & I_1 \cap I_2.
\end{aligned}$$

Therefore, we obtain $(I_1+I_2)/I_2 \cong I_1/(I_1 \cap I_2)$ from Theorem 2. □

Theorem 6. *Let $(R, +, \bullet)$ be an M-hazy ring, and let I_1 and I_2 be the M-hazy ideals of R such that $I_1 \subseteq I_2$. Then,*

$$(R/I_1)/(I_2/I_1) \cong R/I_2.$$

Proof. Let $\phi : R/I_1 \longrightarrow R/I_2$ be defined by $\phi([k+I_1]) = [k+I_2]$. It is clear that ϕ is surjective.
(1) Let $[k+I_1] = [l+I_1]$ for $k, l \in R$; then, there exists $i \in I_1$ such that

$$((-k)+l)(i) \neq \bot,$$

since $k+I_1 \sim l+I_1$. Since $I_1 \subseteq I_2$, $i \in I_2$. Hence, $k+I_2 \sim l+I_2$ and so $[k+I_2] = [l+I_2]$.

Thus, ϕ is well defined.

(2) Let

$$\bigvee_{[m+I_1]\in R/I_1} ([k+I_1]\widetilde{+}[l+I_1])([m+I_1]) \neq \bot.$$

Since $I_1 \subseteq I_2$, therefore, $[k+I_1] \subseteq [k+I_2], [l+I_1] \subseteq [l+I_2]$, and $[m+I_1] \subseteq [m+I_2]$. Then, we have

$$\begin{aligned}
& \bigvee_{[m+I_2]\in R/I_2} ([k+I_2]\widetilde{+}[l+I_2])([m+I_2]) \\
\geq & \bigvee\left\{ \, ([k+I_1]\widetilde{+}[l+I_1])([m+I_1]) \, \right\} \\
\geq & ([k+I_1]\widetilde{+}[l+I_1])([m+I_1]) \neq \bot.
\end{aligned}$$

Thus, we get

$$\bigvee_{\phi[m+I_1]\in\phi^{\rightarrow}(R/I_1)} (\phi[k+I_1]\widetilde{+}\phi[l+I_1])(\phi[m+I_1]) \neq \bot.$$

(3) Let

$$\bigvee_{[m+I_1]\in R/I_1} ([k+I_1]\widetilde{\bullet}[l+I_1])([m+I_1]) \neq \bot.$$

Similarly to the proof of (2), we obtain

$$\bigvee_{\phi[m+I_1]\in\phi^{\rightarrow}(R/I_1)} (\phi[k+I_1]\widetilde{\bullet}\phi[l+I_1])(\phi[m+I_1]) \neq \perp.$$

(4)

$$\begin{aligned} & \ker\phi = \{[k+I_1]\in R/I_1 \mid \phi[k+I_1] = [o+I_2]\} \\ = \ & \{[k+I_1]\in R/I_1 \mid [k+I_2] = [o+I_2]\} \\ = \ & \{[k+I_1]\in R/I_1 \mid k+I_2 \sim o+I_2\} \\ = \ & \left\{ \ [k+I_1]\in R/I_1 \mid ((-k)+o)(i) \neq \perp, \text{for some } i\in I_2 \ \right\} \\ = \ & \{[k+I_1]\in R/I_1 \mid k\in I_2\} \\ = \ & I_2/I_1. \end{aligned}$$

Thus, we have $(R/I_1)/(I_2/I_1) \cong R/I_2$ from Theorem 2. □

4. M-Fuzzifying Convex Spaces Induced by M-Hazy Rings

In this section, we give an approach to inducing M-fuzzifying convex spaces using M-hazy rings. Note that in the below Theorem 7 and Theorem 9, the symbols MC1, MC2, and MC3 refer to the three conditions of M-fuzzifying convexity in Definition 10.

Theorem 7. *Let $(R,+,\bullet)$ be an M-hazy ring and define $\mathscr{C}: 2^R \to M$ as follows:*

$$\forall A\in 2^R, \mathscr{C}(A) = \bigwedge_{p\in R}\Big(\big((\bigvee_{k,l\in A}(k+(-l))(p)\big) \to A(p)\big) \wedge \big((\bigvee_{k,l\in A}(k\bullet l)(p)\big) \to A(p)\big)\Big).$$

Then, $(R,\mathscr{C})$ is an M-fuzzifying convex space.

Proof. It suffices to show that $\mathscr{C}$ satisfies **(MC1)**, **(MC2)** and **(MC3)**.
Actually, **(MC1)**: Obviously, $\bigvee_{k,l\in\emptyset}(k+(-l))(p) = \perp$ and $\bigvee_{k,l\in\emptyset}(k\bullet l)(p) = \perp$. Therefore,

$$\mathscr{C}(\emptyset) \ = \ \bigwedge_{p\in R}\Big(\big(\perp\to\emptyset(p)\big)\wedge\big(\perp\to\emptyset(p)\big)\Big) = \bigwedge_{p\in R}\Big(\top\wedge\top\Big) = \bigwedge_{p\in R}\top = \top.$$

If $A=R$, then $A(p)=R(p)=\top$. Therefore,

$$\begin{aligned} & \mathscr{C}(R) = \bigwedge_{p\in R}\Big(\big((\bigvee_{k,l\in R}(k+(-l))(p)\big)\to\top\big)\wedge\big((\bigvee_{k,l\in R}(k\bullet l)(p)\big)\to\top\big)\Big) \\ & = \bigwedge_{p\in R}\Big(\top\wedge\top\Big) = \bigwedge_{p\in R}\top = \top. \end{aligned}$$

Hence, $\mathscr{C}(\emptyset) = \mathscr{C}(R) = \top$.

(MC2): For each $\{A_i \mid i \in \Omega\} \subseteq 2^R$, we have

$$\mathscr{C}(\bigcap_{i\in\Omega} A_i) = \bigwedge_{p\in R} \Big(\big((\bigvee_{k,l\in \bigcap_{i\in\Omega} A_i} (k+(-l))(p) \big) \to \big(\bigcap_{i\in\Omega} A_i\big)(p) \big)$$
$$\wedge \big((\bigvee_{k,l\in \bigcap_{i\in\Omega} A_i} (k\bullet l)(p) \big) \to \big(\bigcap_{i\in\Omega} A_i\big)(p) \big) \Big)$$
$$= \bigwedge_{i\in\Omega}\bigwedge_{p\in R} \Big(\big((\bigvee_{k,l\in \bigcap_{i\in\Omega} A_i} (k+(-l))(p) \big) \to A_i(p) \big)$$
$$\wedge \big((\bigvee_{k,l\in \bigcap_{i\in\Omega} A_i} (k\bullet l)(p) \big) \to A_i(p) \big) \Big)$$
$$\geq \bigwedge_{i\in\Omega}\bigwedge_{p\in R} \Big(\big((\bigvee_{k,l\in A_i} (k+(-l))(p) \big) \to A_i(p) \big)$$
$$\wedge \big((\bigvee_{k,l\in A_i} (k\bullet l)(p) \big) \to A_i(p) \big) \Big)$$
$$= \bigwedge_{i\in\Omega} \mathscr{C}(A_i).$$

(MC3): For each $\{A_i \mid i \in \Omega\} \overset{dir}{\subseteq} 2^R$, we have

$$\mathscr{C}(\bigcup_{i\in\Omega} A_i) = \bigwedge_{p\in R} \Big(\big((\bigvee_{k,l\in \bigcup_{i\in\Omega} A_i} (k+(-l))(p) \big) \to \big(\bigcup_{i\in\Omega} A_i\big)(p) \big)$$
$$\wedge \big((\bigvee_{k,l\in \bigcup_{i\in\Omega} A_i} (k\bullet l)(p) \big) \to \big(\bigcup_{i\in\Omega} A_i\big)(p) \big) \Big)$$
$$= \bigwedge_{p\in R} \Big(\big((\bigvee_{i_1,i_2\in\Omega}\bigvee_{k\in A_{i_1}, l\in A_{i_2}} (k+(-l))(p) \big) \to \big(\bigcup_{i\in\Omega} A_i\big)(p) \big)$$
$$\wedge \big((\bigvee_{i_1,i_2\in\Omega}\bigvee_{k\in A_{i_1}, l\in A_{i_2}} (k\bullet l)(p) \big) \to \big(\bigcup_{i\in\Omega} A_i\big)(p) \big) \Big)$$
$$\geq \bigwedge_{p\in R} \Big(\big((\bigvee_{i_3\in\Omega}\bigvee_{k,l\in A_{i_3}} (k+(-l))(p) \big) \to \big(\bigcup_{i\in\Omega} A_i\big)(p) \big)$$
$$\wedge \big((\bigvee_{i_3\in\Omega}\bigvee_{k,l\in A_{i_3}} (k\bullet l)(p) \big) \to \big(\bigcup_{i\in\Omega} A_i\big)(p) \big) \Big)$$
$$\geq \bigwedge_{i\in\Omega}\bigwedge_{p\in R} \Big(\big((\bigvee_{k,l\in A_i} (k+(-l))(p) \big) \to A_i(p) \big)$$
$$\wedge \big((\bigvee_{k,l\in A_i} (k\bullet l)(p) \big) \to A_i(p) \big) \Big)$$
$$= \bigwedge_{i\in\Omega} \mathscr{C}(A_i).$$

Hence, $(R, \mathscr{C})$ is an M-fuzzifying convex space. □

Theorem 8. *If $\phi : (R_1, +, \bullet) \longrightarrow (R_2, \oplus, \circ)$ is an M-RH mapping, then $\phi : (R_1, \mathscr{C}_{R_1}) \longrightarrow (R_2, \mathscr{C}_{R_2})$ is an M-CP mapping.*

Proof. Since $\phi : (R_1, +, \bullet) \longrightarrow (R_2, \oplus, \circ)$ is an M-RH mapping, this means that:

(1) $\phi^{\rightarrow}_M(k+l) = \phi(k) \oplus \phi(l)$ for all $k, l \in R_1$,
(2) $\phi^{\rightarrow}_M(k\bullet l) = \phi(k) \circ \phi(l)$ for all $k, l \in R_1$.

Then, for all $B \in 2_{R_2}$, we have

$$
\begin{aligned}
&\mathscr{C}_{R_1}(\phi^{\leftarrow}(B)) = \bigwedge_{p\in R_1}\Big(\big((\bigvee_{k,l\in\phi^{\leftarrow}(B)}(k+(-l))(p)\big) \to (\phi^{\leftarrow}(B))(p)\big) \\
&\wedge\big((\bigvee_{k,l\in\phi^{\leftarrow}(B)}(k\bullet l)(p)\big) \to (\phi^{\leftarrow}(B))(p)\big)\Big) \\
&\geq \bigwedge_{p\in R_1}\Big(\big((\bigvee_{k,l\in\phi^{\leftarrow}(B)}(\phi_M^{\leftarrow}\cdot\phi_M^{\rightarrow}(k+(-l)))(p)\big) \to (\phi^{\leftarrow}(B))(p)\big) \\
&\wedge\big((\bigvee_{k,l\in\phi^{\leftarrow}(B)}(\phi_M^{\leftarrow}\cdot\phi_M^{\rightarrow}(k\bullet l))(p)\big) \to (\phi^{\leftarrow}(B))(p)\big)\Big) \\
&= \bigwedge_{p\in R_1}\Big(\big((\bigvee_{k,l\in\phi^{\leftarrow}(B)}(\phi_M^{\leftarrow}(\phi(k)\oplus\phi(-l)))(p)\big) \to (\phi^{\leftarrow}(B))(p)\big) \\
&\wedge\big((\bigvee_{k,l\in\phi^{\leftarrow}(B)}(\phi_M^{\leftarrow}(\phi(k)\circ\phi(l)))(p)\big) \to (\phi^{\leftarrow}(B))(p)\big)\Big) \\
&= \bigwedge_{p\in R_1}\Big(\big((\bigvee_{\phi(k),-\phi(l)\in B}((\phi(k)\oplus(-\phi(l)))(\phi(p))\big) \to (\phi^{\leftarrow}(B))(p)\big) \\
&\wedge\big((\bigvee_{\phi(k),\phi(l)\in B}((\phi(k)\circ\phi(l)))(\phi(p))\big) \to (\phi^{\leftarrow}(B))(p)\big)\Big) \\
&\geq \bigwedge_{q\in R_2}\Big(\big((\bigvee_{q_1,-q_2\in B}(q_1\oplus(-q_2))(q)\big) \to (B)(q)\big) \\
&\wedge\big((\bigvee_{q_1,q_2\in B}(q_1\circ q_2)(q)\big) \to (B)(q)\big)\Big) \\
&= \mathscr{C}_{R_2}(B).
\end{aligned}
$$

This implies that $\phi : (R_1, \mathscr{C}_{R_1}) \longrightarrow (R_2, \mathscr{C}_{R_2})$ is an M-CP mapping. □

Theorem 9. *Let $(R, +, \bullet)$ be an M-hazy ring, and define $\mathscr{C} : 2^R \to M$ as follows:*

$$
\begin{aligned}
\forall A \in 2^R, \mathscr{C}(A) \quad &= \bigwedge_{p\in R}\Big(\big((\bigvee_{k,l\in A}(k+(-l))(p)\big) \to A(p)\big) \\
&\wedge\big((\bigvee_{k\in R,l\in A}(k\bullet l)(p)\big) \to A(p)\big) \\
&\wedge\big((\bigvee_{k\in R,l\in A}(l\bullet k)(p)\big) \to A(p)\big)\Big).
\end{aligned}
$$

Then, $(R, \mathscr{C})$ is an M-fuzzifying convex space.

Proof. The proof is similar to the proof of the Theorem 7 and is hence omitted. □

5. Conclusions

Recently, Liu and Shi [28] introduced M-hazy groups by a new type of fuzzy associative law called the M-hazy associative law. The basic idea is to induce the binary operation of traditional algebra to an M-hazy binary operation. Furthermore, Mehmood et al. [29] investigated M-hazy rings by a new M-hazy distributive law on M-hazy binary operations and also discussed their various properties. Homomorphism is a wide concept, even outside of the mathematical subject we usually call abstract algebra. However, it is always a mapping between two structured objects of the same kind, and that map is structure-preserving. By structure, we mean an operation. In our work, we discuss the relation between M-hazy rings and M-hazy ideals. We discuss the various related properties of homomorphisms of M-hazy rings. Then, we provide the fundamental homomorphism theorems of M-hazy rings. The convexity of an M-fuzzy set on a set X is an M-fuzzy set on a power set with certain properties; therefore, each subset of X can be regarded as a convex set to some extent. Keeping in view the importance of this fact, finally, an approach to inducing M-fuzzifying convex spaces using M-hazy rings is given.

For the future, one possible idea is to study behavior from an M-hazy perspective, the homomorphism of prime ideals and maximal ideals, and fuzzy homogeneity that involves various M-hazy ideals on M-hazy lattices.

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