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Gradient Structures Associated with a Polynomial Differential Equation

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Abstract: In this paper, by using the characteristic system method, the kernel of a polynomial differential equation involving a derivation in R^n is described by solving the Cauchy Problem for the corresponding first order system of PDEs. Moreover, the kernel representation has a special significance on the space of solutions to the corresponding system of PDEs. As very important applications, it has been established that the mathematical framework developed in this work can be used for the study of some second-order PDEs involving a finite set of derivations.

Keywords: scalar derivation; Lie algebra; gradient system; polynomial differential equation; flow; kernel

MSC: 34A26; 15A03; 35C99

1. Introduction

Throughout time, gradient-type representations for some solutions, gradient systems in a Lie algebra and the algebraic representation of gradient systems, have been investigated, with remarkable results, by Vârsan [1]. Moreover, stochastic partial differential equations (SPDEs) of Hamilton–Jacobi type including non F_T -adapted solutions have been studied in Ijacu and Vârsan [2]. By using the commuting property of the drift and diffusion vector fields with respect to the usual Lie bracket, a representation for a classical solution of some nonlinear SPDEs driven by Fisk–Stratonovich stochastic integral was constructed by Iftimie et al. [3]. Furthermore, sufficient conditions for linear subspaces of smooth vector fields in order to be written as a kernel of some linear first order partial differential equations have been formulated and proved in Parveen and Akram [4]. Further, Treanță and Vârsan [5] proved that solutions associated with an extended affine control system can be obtained as a limit process using solutions for a parameterized affine control system and weak small controls. Recently, Treanță [6] studied affine control systems with jumps for which the ideal generated by the drift vector field can be imbedded as a kernel of a linear first-order partial differential equation. Mainly, these references motivate the present study. For other different but connected viewpoints regarding this subject, the reader is directed to Friedman [7], Sussmann [8], Crandall and Souganidis [9], Sontag [10], Bressan and Shen [11], Nonlaopon [12], Saira et al. [13] and Treanță [14–16].

In this paper, taking into account the results included in the quite recently work Treanță [17] (the kernel of a polynomial of scalar derivations is described by solving *Cauchy Problems* for the corresponding system of ODEs; also, a *gradient representation* for the associated Cauchy Problem solution is derived), we investigate the kernel of a polynomial differential equation involving a derivation in R^n by solving the Cauchy Problem for the corresponding first order system of PDEs. Furthermore, we extend a solution

by considering Radon measures and their bounded variation functions, or Wiener and Levy processes. Moreover, it is established that the kernel representation has a special significance on the space of solutions to the corresponding system of PDEs.

This paper is organized as follows. In Section 2, in order to delineate certain steps in the solving algorithm proposed for the main result (Theorem 1), some preliminary results are formulated. More precisely, two crucial lemmas for the present paper are mentioned. Further, we establish two important remarks. A gradient structure for the associated Cauchy Problem solution is provided by Remark 1. The final part of this section, including Remark 2, extends a solution considering Radon measures and their bounded variation functions, or using Wiener (or Levy) processes. The aim of Section 3 is to provide a characterization for the kernel of a polynomial differential equation involving a derivation in R^n . Specifically, through the use of the characteristic system method and some results formulated in Section 2, the associated Cauchy Problem solution is derived (see Theorem 1). Moreover, this solution has a special significance on the space of solutions to the corresponding first order system of PDEs. Finally, Section 4 concludes the paper.

2. Preliminary Results

In this section, taking into account a very recent work (see Treanță [17]), some auxiliary results are formulated.

Let $0 \in I \subseteq \mathbb{R}$ be an open interval. Consider a polynomial of the scalar derivation $\frac{d}{dt}$,

$$P_m \left(t; \frac{d}{dt} \right) = a_1(t) + a_2(t) \left(\frac{d}{dt} \right) + \cdots + a_m(t) \left(\frac{d}{dt} \right)^{m-1} - \left(\frac{d}{dt} \right)^m, \quad (1)$$

where $m \geq 1$, $a_j \in L_\infty(I)$, $j \in \{1, 2, \dots, m\}$. Define

$$H_\infty^m(I) = \left\{ h \in C^{m-1}(I) : \left(\frac{d}{dt} \right)^m (h) \in L_\infty(I) \right\} \quad (2)$$

and consider $\text{Ker}(P_m) \subseteq H_\infty^m(I)$, where

$$\text{Ker}(P_m) = \left\{ h \in H_\infty^m(I) : P_m \left(t; \frac{d}{dt} \right) (h)(t) = 0, \text{ a.e. } t \in I \right\}. \quad (3)$$

The procedure of characteristic systems (see Friedman [7], Vărsan [1]) allows us to describe $\text{Ker}(P_m)$ by solving *Cauchy Problems* for the corresponding system of ODEs using a vector variable

$$y = \text{col}(y_1, y_2, \dots, y_m), \quad \frac{dy}{dt} = Ay + \sum_{i=1}^m a_i(t) B_i y, \quad y(0) = y_0 \in \mathbb{R}^m. \quad (4)$$

Here, the $(m \times m)$ constant matrices A and B_i , $i = \overline{1, m}$, are defined by

$$A = [0 \ e_1 \ \cdots \ e_{m-1}], \quad B_1 = [e_1 \ 0 \ \cdots \ 0], \ \dots, \ B_m = [0 \ 0 \ \cdots \ e_m], \quad (5)$$

where $\{e_1, \dots, e_m\}$ is the canonical basis and $0 \in \mathbb{R}^m$ is the origin. By definition

$$[B_i, B_j] := B_j B_i - B_i B_j, \quad i, j \in \{1, 2, \dots, m\}, \quad (\text{Lie bracket}), \quad (6)$$

and making a direct computation, we get

$$O = [B_i, B_j], \quad i, j \in \{1, 2, \dots, m\}, \quad (7)$$

with O —null matrix, and

$$A^m = O, \quad (A \text{ is a nilpotent matrix}). \quad (8)$$

The Cauchy Problem solution for (4) is represented by

$$y(t; y_0) = [\exp At] \hat{y}(t; y_0), \quad t \in I, \quad (9)$$

where $\{\hat{y}(t; y_0) : t \in I\}$ fulfils the following linear system (initial value problem)

$$\frac{dy}{dt} = \sum_{i=1}^m a_i(t) A_i(t) y, \quad t \in I, \quad y(0) = y_0 \in R^m. \quad (10)$$

Write the $(m \times m)$ matrices

$$A_i(t) := [\exp(-tA)] B_i [\exp tA], \quad i \in \{1, 2, \dots, m\}, \quad (11)$$

as follows

$$A_i(t) = [\exp t \operatorname{ad}_A] (B_i), \quad i \in \{1, 2, \dots, m\}, \quad t \in I, \quad (12)$$

where the linear mapping $\operatorname{ad}_A : M_{m \times m} \rightarrow M_{m \times m}$ is given by $\operatorname{ad}_A(B) := BA - AB$ (see $[A, B]$). In addition, using (7), (8) and (12), we get

$$A_i(t) = B_i + \frac{t}{1!} \operatorname{ad}_A(B_i) + \dots + \frac{t^{m-1}}{(m-1)!} (\operatorname{ad}_A)^{m-1}(B_i), \quad i \in \{1, \dots, m\}. \quad (13)$$

Denote $N = m^2$ and define N matrices $\{C_1, C_2, \dots, C_N\} \subseteq M_{m \times m}$, as follows

$$\begin{aligned} \{C_1, C_2, \dots, C_N\} &= \left\{ \operatorname{ad}_A^k(B_1) : k \in \{0, 1, 2, \dots, m-1\} \right\} \cup \dots \cup \\ &\quad \left\{ \operatorname{ad}_A^k(B_m) : k \in \{0, 1, 2, \dots, m-1\} \right\}. \end{aligned} \quad (14)$$

Furthermore, let $\{\alpha_1(t), \alpha_2(t), \dots, \alpha_N(t) : t \in I\}$ be given by

$$\begin{aligned} \{\alpha_1(t), \dots, \alpha_m(t)\} &= a_1(t) \left[1, \frac{t}{1!}, \dots, \frac{t^{m-1}}{(m-1)!} \right], \\ &\vdots \\ \{\alpha_{N-m+1}(t), \dots, \alpha_N(t)\} &= a_m(t) \left[1, \frac{t}{1!}, \dots, \frac{t^{m-1}}{(m-1)!} \right]. \end{aligned} \quad (15)$$

With these notations, we write ODE (10) as follows

$$\frac{dy}{dt} = \sum_{j=1}^N \alpha_j(t) Y_j(y), \quad t \in I, \quad y(0) = y_0, \quad (16)$$

where $Y_j(y) := C_j y$, $j \in \{1, 2, \dots, N\}$.

Lemma 1 ([17]). Consider $\{C_1, C_2, \dots, C_N\}$ defined in (14), with $N = m^2$. Then $\{C_1, C_2, \dots, C_N\}$ is a basis for $M_{m \times m}$ and

$$\{Y_1(y) = C_1 y, Y_2(y) = C_2 y, \dots, Y_N(y) = C_N y\}$$

is a system of generators for the Lie algebra $L(Y_1, \dots, Y_N) \subseteq C^\infty(R^m; R^m)$ generated by $\{Y_1, \dots, Y_N\}$.

Lemma 2 ([17]). Assume $a_j(t) \equiv 0$, $j \in \{1, 2, \dots, i\}$, for some $1 \leq i \leq m$. Define a subspace $S_i = \text{span}\{e_1, \dots, e_i\} \subseteq R^m$ and its orthogonal complement $S_i^\perp \subseteq R^m$, where $\{e_1, \dots, e_m\} \subseteq R^m$ is the canonical basis. Then

$$\forall y_0 \in S_i \text{ is a stationary point for ODE (10).} \quad (17)$$

In addition, for each $y_0 \in S_i^\perp$, $y_0 \neq 0$, the following statements are valid:

$$\dim L(Y_{j(i)}, \dots, Y_N)(y_0) = \dim [\text{span}\{Y_{j(i)}(y_0), \dots, Y_N(y_0)\}] = m - i, \quad (18)$$

$$\begin{aligned} \{C_{j(i)}, \dots, C_N\} \text{ is a basis for } M_{m \times m}^i := L(C_{j(i)}, \dots, C_N), \quad j(i) := mi + 1 \\ \hat{y}(t, y_0) \in S_i^\perp, \quad t \in I, \text{ for any solution of ODE (10).} \end{aligned} \quad (19)$$

In the following, we establish two important remarks for the main result associated with this paper.

Remark 1. Any solution of ODE (4) can be represented using

$$p(t) := (t_1(t), \dots, t_m(t)), \quad t_j(t) = \int_0^t a_j(s) ds, \quad j \in \{1, 2, \dots, m\}, \quad t \in I. \quad (20)$$

In this regard, notice that the matrices $\{B_1, \dots, B_m\} \subseteq M_{m \times m}$ commute (see (6)) and the linear mapping

$$G(p)\lambda := [\exp t_1 B_1] \cdots [\exp t_m B_m] \lambda, \quad p = (t_1, \dots, t_m) \in R^m, \quad \lambda \in R^m \quad (21)$$

satisfies a gradient system

$$\partial_{t_1} G(p)\lambda = B_1 G(p)\lambda, \dots, \partial_{t_m} G(p)\lambda = B_m G(p)\lambda. \quad (22)$$

In particular, for $p = p(t)$, $t \in I$ (see (20)), we get that any $\{y(t, y_0) : t \in I\}$ satisfying ODE (4) is represented by

$$y(t; y_0) = G(p(t)) \hat{y}(t; y_0), \quad t \in I, \quad y_0 \in R^m, \quad (23)$$

where $\{\hat{y}(t; y_0) : t \in I\} \subseteq C(I, R^m)$ fulfils

$$\frac{dy}{dt} = G(-p(t)) A G(p(t)) y, \quad y(0) = y_0, \quad t \in I. \quad (24)$$

In addition, $A(t) := G(-p(t)) A G(p(t))$, $t \in I$, defined in (24), can be written as (see “ $\circ$ ” as the symbol for composing functions)

$$A(t) = [\exp t_1(t) \text{ad}_{B_1}] \circ \dots \circ [\exp t_m(t) \text{ad}_{B_m}](A), \quad t \in I, \quad (25)$$

where the adjoint mapping $\text{ad}_B: M_{m \times m} \rightarrow M_{m \times m}$ is defined by

$$\text{ad}_B(C) = [B, C] = CB - BC, \quad (\text{Lie bracket}). \quad (26)$$

A direct computation leads us to

$$[\exp t \text{ ad}_{B_m}](A) = A + \frac{t}{1!} \text{ad}_{B_m}(A) + \cdots + \frac{t^k}{k!} \text{ad}_{B_m}^k(A) + \cdots, \quad (27)$$

where $\text{ad}_{B_m}^k(A) = C_m := [0 \dots 0 e_{m-1}]$, for any $k \geq 1$. Rewrite (27) as

$$[\exp t \text{ ad}_{B_m}](A) = A + [\exp t - 1] C_m, \quad C_m = [0 \dots 0 e_{m-1}]. \quad (28)$$

Notice that

$$\text{ad}_{B_{m-1}}(C_m) = C_{m-1} - B_m, \quad \text{ad}_{B_{m-1}}(C_{m-1}) = O, \quad (29)$$

where $C_{m-1} := [0 \dots 0 e_{m-1} 0]$. Using (28) and (29), we obtain

$$[\exp t_{m-1} \text{ ad}_{B_{m-1}}](C_m) = C_m + t_{m-1}(C_{m-1} - B_m)$$

and

$$\begin{aligned} & [\exp t_{m-1} \text{ ad}_{B_{m-1}}] \circ [\exp t_m \text{ ad}_{B_m}](A) \\ &= [(\exp t_m) - 1] [\exp t_{m-1} \text{ ad}_{B_{m-1}}](C_m) + [\exp t_{m-1} \text{ ad}_{B_{m-1}}](A) \\ &= [(\exp t_m) - 1] [C_m + t_{m-1}(C_{m-1} - B_m)] + [\exp t_{m-1} \text{ ad}_{B_{m-1}}](A). \end{aligned} \quad (30)$$

On the other hand, compute

$$[\exp t_{m-1} \text{ ad}_{B_{m-1}}](A) = A + t_{m-1}(C_{m-1} - B_m) - \frac{t_{m-1}^2}{2!} B_{m-1}, \quad (31)$$

where $C_{m-1} = [0 \dots 0 e_{m-1} 0]$, and inserting (31) into (30), we get

$$\begin{aligned} & [\exp t_{m-1} \text{ ad}_{B_{m-1}}] \circ [\exp t_m \text{ ad}_{B_m}](A) \\ &= A + (\exp t_m) [C_m + t_{m-1}(C_{m-1} - B_m)] - C_m - \frac{t_{m-1}^2}{2!} B_{m-1}. \end{aligned} \quad (32)$$

Notice that (see (31))

$$[\exp t_{m-2} \text{ ad}_{B_{m-2}}](A) = A + t_{m-2}(C_{m-2} - B_{m-1}), \quad (33)$$

and applying $[\exp t_{m-2} \text{ ad}_{B_{m-2}}]$ to (32), we obtain

$$\begin{aligned} & [\exp t_{m-2} \text{ ad}_{B_{m-2}}] \circ [\exp t_{m-1} \text{ ad}_{B_{m-1}}] \circ [\exp t_m \text{ ad}_{B_m}](A) \\ &= A + t_{m-2}(C_{m-2} - B_{m-1}) + (\exp t_m) [C_m + t_{m-1}(C_{m-1} - B_m)] \\ &\quad - C_m - \frac{t_{m-1}^2}{2!} B_{m-1} + t_{m-2} [(\exp t_m) - 1] C_{m-2}. \end{aligned} \quad (34)$$

This computation shows that

$$\hat{A}(t_1, \dots, t_m) := [\exp t_1 \operatorname{ad}_{B_1}] \circ \dots \circ [\exp t_m \operatorname{ad}_{B_m}](A)$$

and, in particular, $A(t) = \hat{A}(t_1(t), \dots, t_m(t))$ (see (25)), can be written as

$$\hat{A}(t_1, \dots, t_m) = A + \sum_{k=1}^m \alpha_k(t_1, \dots, t_m) B_k + \sum_{k=1}^m \beta_k(t_1, \dots, t_m) C_k, \quad (35)$$

for $\alpha_k, \beta_k \in C^\omega(R^m)$.

Remark 2. Any solution of ODE (4) can be represented as in (23) (see Remark 1) and it allows us to extend a solution considering Radon measures and their bounded variation functions

$$p(t) := (t_1(t), \dots, t_m(t)), \quad t_j(t) = \mu_j(t) = \int_0^t d\mu_j(s), \quad j \in \{1, 2, \dots, m\}, \quad (36)$$

with $t \in [0, T] \subseteq I$, replacing $p \in C([0, T]; R^m)$ defined in (20).

In addition, any solution of ODE (4) can be represented as in (23) using Wiener, $W(t, w): [0, T] \times \Omega \rightarrow R^m$, or Levy processes, $p(t, w): [0, T] \times \Omega \rightarrow R^m$, replacing $p \in C([0, T]; R^m)$ in (20). If it is the case, then any solution of ODE (4) and, as a consequence, $\operatorname{Ker}(P_m)$ defined in (3), changes accordingly. We get (see (23))

$$\begin{aligned} y(t, w; y_0) &= G(W(t, w))\hat{y}(t, w; y_0), \quad t \in [0, T], \\ y(t, w; y_0) &= G(p(t, w))\hat{y}(t, w; y_0), \quad t \in [0, T], \end{aligned} \quad (37)$$

where $\{\hat{y}(t, w; y_0): t \in [0, T], w \in \Omega\}$ is a continuous process satisfying a system of ODEs

$$\frac{dy}{dt} = G(-\hat{p}(t, w)) A G(\hat{p}(t, w)) y, \quad y(0) = y_0, \quad t \in [0, T], \quad (38)$$

with $\hat{p}(t, w) = W(t, w)$, $t \in [0, T]$, $w \in \Omega$, or $\hat{p}(t, w) = p(t, w)$, $t \in [0, T]$, $w \in \Omega$.

3. Main Results

This section contains the main result associated with the present paper. In order to formulate and prove it, we start with the following mathematical tools and hypotheses.

Let $D \subseteq R^n$ be an open and convex set. Consider a derivation mapping $\vec{X}: C^\infty(D) \rightarrow C^m(D)$ given by $\vec{X}(\varphi)(x) := \sum_{i=1}^n X_i(x) \partial_i \varphi(x)$, where $X_i \in C^m(D)$, $1 \leq i \leq n$, $\varphi \in C^\infty(D)$. Associate a polynomial differential equation

$$P_m(x; \vec{X})(\varphi)(x) := f(x) + \sum_{j=1}^m a_j(x) y_j(x) - y_{m+1}(x) = 0, \quad (39)$$

where $f, a_j \in C^1(D)$, $x \in \hat{B} := B(x_0, \rho) \subseteq D$, $y_1 = \varphi$, $\vec{X}(y_j) = y_{j+1}$, and $(X_1(x_0), \dots, X_n(x_0)) \neq 0_{R^n}$. Assume that

$$X_n(x) \neq 0, \quad \forall x \in B(x_0, \rho) \subseteq D, \quad (40)$$

where $X = (X_1, \dots, X_n)$. Define a restricted kernel of P_m to $\hat{B}$ by

$$\text{Ker } P_m|_{\hat{B}} := \left\{ \varphi \in C^m(\hat{B} \subseteq D) : P_m(x; \vec{X})(\varphi)(x) = 0, x \in \hat{B} \right\}. \quad (41)$$

A procedure for describing $\{\text{Ker } P_m|_{\hat{B}}\}$ is to associate a first order system of PDEs involving a vector state variable $y = (y_1, y_2, \dots, y_m)$, $y_i(x) := (\vec{X})^{(i-1)}(\varphi)(x)$, $1 \leq i \leq m$. More precisely, consider the following first order system of PDEs

$$\vec{X}(y_1)(x) = y_2(x), \dots, \vec{X}(y_{m-1})(x) = y_m(x) \quad (42)$$

$$\vec{X}(y_m)(x) = f(x) + \sum_{i=1}^m a_i(x)y_i(x).$$

A Cauchy Problem (CP) for (42) is defined considering a hyperplane $H := \{(\hat{x}, x_{0n}) \in R^n : \hat{x} \in R^{n-1}\}$ (see $x_0 := (\hat{x}_0, x_{0n})$) and some fixed Cauchy conditions on $H \cap D$

$$y_1(\hat{x}, x_{0n}) = y_1^0(\hat{x}), \dots, y_m(\hat{x}, x_{0n}) = y_m^0(\hat{x}), \hat{x} \in B(\hat{x}_0, 2\rho) \subseteq R^{n-1}, \quad (43)$$

where $y_{i+1}^0 \in C^{m-i}(B(\hat{x}_0, 2\rho))$ satisfies $y_{i+1}^0 = (\vec{X})^{(i)}(y_1^0)$, $1 \leq i \leq m-1$, and $\{(\hat{x}, x_{0n}) : \hat{x} \in B(\hat{x}_0, 2\rho) \subseteq R^{n-1}\} \subseteq H \cap D$.

A solution for (CP) (see (42) and (43)) is found by solving the corresponding characteristic system

$$\frac{d\hat{y}_1}{d\sigma}(\sigma, \lambda) = \hat{y}_2(\sigma, \lambda), \dots, \frac{d\hat{y}_{m-1}}{d\sigma}(\sigma, \lambda) = \hat{y}_m(\sigma, \lambda), \quad (44)$$

$$\frac{d\hat{y}_m}{d\sigma}(\sigma, \lambda) = \hat{f}(\sigma, \lambda) + \sum_{i=1}^m \hat{a}_i(\sigma, \lambda)\hat{y}_i(\sigma, \lambda), \sigma \in [-\alpha, \alpha],$$

with Cauchy conditions $\hat{y}_i(0, \lambda) = y_i^0(\lambda)$, $\lambda \in B(\hat{x}_0, 2\rho) \subseteq R^{n-1}$, $1 \leq i \leq m$. Here, the smooth functions y_i^0 are fixed in (43) and $\hat{f}, \hat{a}_i$ are given by

$$\hat{f}(\sigma, \lambda) = f(x(\sigma, \lambda)), \hat{a}_i(\sigma, \lambda) = a_i(x(\sigma, \lambda)), 1 \leq i \leq m, \quad (45)$$

for $\sigma \in [-\alpha, \alpha]$, $\lambda \in B(\hat{x}_0, 2\rho) \subseteq R^{n-1}$, where the local flow $\{x(\sigma, \lambda)\}$ satisfies the following system of ODEs

$$\frac{dx}{d\sigma}(\sigma, \lambda) = X(x(\sigma, \lambda)), x(0, \lambda) = (\lambda, x_{0n}), \lambda \in B(\hat{x}_0, 2\rho) \subseteq R^{n-1}. \quad (46)$$

The analysis of ODEs (44) relies on the results derived in Section 2 (see Lemmas 1 and 2 and Remarks 1 and 2), for each $\lambda \in B(\hat{x}_0, 2\rho)$. This will lead us to the corresponding results for (44) which can be transferred to (42) and (43) provided the local flow $\{x(\sigma, \lambda)\}$ is written as follows

$$x(\sigma, \lambda) = G(\sigma)[\lambda, x_{0n}], \sigma \in [-\alpha, \alpha], \lambda \in B(\hat{x}_0, 2\rho) \subseteq R^{n-1}, \quad (47)$$

where $\{G(\sigma)[z] \in B(x_0, 3\rho) \subseteq D : \sigma \in [-2\alpha, 2\alpha], z \in B(x_0, 2\rho)\}$ is the local flow generated by $X \in C^m(D \subseteq R^n; R^n)$, with initial condition $G(0)[z] = z$. In addition, the following equation must be solved

$$x(\sigma, \lambda) = x \in B(x_0, \rho) \subseteq D, x_0 := (\hat{x}_0, x_{0n}), \quad (48)$$

for $\lambda = (\lambda_1, \dots, \lambda_{m-1}) \in B(\hat{x}_0, 2\rho) \subseteq R^{n-1}$ and $\sigma \in [-\alpha, \alpha]$. The unique solution $\lambda = \hat{\psi}(x)$ and $\sigma = \hat{\sigma}(x)$ of (48) satisfies

$$\hat{\psi}(\hat{x}, x_{0n}) = \hat{x}, \hat{\sigma}(\hat{x}, x_{0n}) = 0, \quad (49)$$

for $\hat{x} \in B(\hat{x}_0, 2\rho) \subseteq R^{n-1}$.

For each $x \in \text{Int } B(x_0, \rho)$, let $\{G(\sigma)[x] \in B(x_0, \rho) : \sigma \in [-\varepsilon, \varepsilon]\}$ be the corresponding solution of (46) and notice that (see (47) and (48))

$$G(\sigma)[x] = G(\sigma) \circ G(\hat{\sigma}(x))[\hat{\psi}(x), x_{0n}] = G(\sigma + \hat{\sigma}(x))[\hat{\psi}(x), x_{0n}], \quad (50)$$

for any $\sigma \in [-\varepsilon, \varepsilon]$. Using (50) and the unique solution of (48), we get

$$\hat{\sigma}(G(\sigma)[x]) = \sigma + \hat{\sigma}(x) \in [-\alpha, \alpha], \quad \hat{\psi}(G(\sigma)[x]) = \hat{\psi}(x) \in B(\hat{x}_0, 2\rho), \quad (51)$$

for any $\sigma \in [0, \varepsilon]$ (or $\sigma \in [-\varepsilon, 0]$).

Define a solution for (CP) (see (42) and (43)) by

$$y_i(x) = \hat{y}_i(\hat{\sigma}(x); \hat{\psi}(x)), \quad i \in \{1, \dots, m\}, \quad x \in B(x_0, \rho) \subseteq D, \quad (52)$$

where $\hat{y}_i(\sigma, \lambda)$, $i \in \{1, \dots, m\}$, fulfil (44) and $\{(\hat{\sigma}(x), \hat{\psi}(x)) : x \in B(x_0, \rho) \subseteq D\}$ verifies (49) and (50).

By a direct inspection, the Cauchy conditions (43) are verified (see (49) and (44)). We get a solution of (42) relying on (51) and

$$y_i(G(\sigma)[x]) = \hat{y}_i(\sigma + \hat{\sigma}(x); \hat{\psi}(x)), \quad i = \overline{1, m}, \quad \sigma \in [0, \varepsilon]. \quad (53)$$

Applying a direct derivation $\frac{d}{d\sigma}|_{\sigma=0}$, from (53), we obtain

$$\langle \partial_x y_i(x), X(x) \rangle = y_{i+1}(x), \quad i \in \{1, \dots, m-1\}, \quad (54)$$

$$\langle \partial_x y_m(x), X(x) \rangle = f(x) + \sum_{i=1}^m a_i(x) y_i(x), \quad x \in \text{Int } B(x_0, \rho) \subseteq D,$$

which stands for the first order system (42).

We are able now to provide the main result of this paper. The above given computation will be stated as follows.

Theorem 1. Let the smooth vector field $X \in C^m(D \subseteq R^n; R^n)$ be given such that $X = (X_1, \dots, X_n)$ satisfies $X_n(x_0) \neq 0$ for some $x_0 = (\hat{x}_0, x_{0n}) \in D$. Consider (44) and its solution

$$\{\hat{y}_1(\sigma, \lambda), \dots, \hat{y}_m(\sigma, \lambda) : \sigma \in [-\alpha, \alpha], \lambda \in B(\hat{x}_0, 2\rho) \subseteq R^{n-1}\}.$$

Define $y_i(x) = \hat{y}_i(\hat{\sigma}(x); \hat{\psi}(x))$, $i \in \{1, \dots, m\}$, $x \in B(x_0, \rho) \subseteq D$, where $\lambda = \hat{\psi}(x) \in B(\hat{x}_0, 2\rho) \subseteq R^{n-1}$ and $\hat{\sigma}(x) \in [-\alpha, \alpha]$ is the unique solution of (48) fulfilling (49) and (51). Then $\{(y_1(x), \dots, y_m(x)) \in R^m : x \in B(x_0, \rho) \subseteq D\}$ satisfies the first order system of PDEs given in (42), with Cauchy conditions (43) and $y_1 \in \text{Ker } P_m|_{\hat{B}}$ (see (41)).

Proof. Taking into account the aforementioned computations (see relations (53) and (54)), the proof is immediate and complete. $\square$

Remark 3. Consider a second order PDE of the form

$$\langle \partial_x^2 h(x) X(x), X(x) \rangle + \langle \partial_x h(x), [\partial_x X(x)] X(x) \rangle = a_1(x) h(x), \quad (55)$$

where $a_1 \in C^1(D \subseteq \mathbb{R}^n)$ and $X \in C^2(D \subseteq \mathbb{R}^n; \mathbb{R}^n)$ is a smooth vector field satisfying $X(x_0) \neq 0_{\mathbb{R}^n}$ for some $x_0 \in D$. Then, there exists a nontrivial solution $h(\cdot) \in C^2(B(x_0, \rho) \subseteq D)$ satisfying (55), for any $x \in \text{Int } B(x_0, \rho)$. It relies on Lemma 1, noticing that (55) can be rewritten as

$$(\vec{X})^2(h)(x) = a_1(x) h(x), \quad x \in B(x_0, \rho) \subseteq D, \quad a_1 \in C^1(D), \quad (56)$$

where $\vec{X}(h)(x) := \langle \partial_x h(x), X(x) \rangle$. It shows that any $a_1(\cdot) = \mu \in \mathbb{R}$ is an eigenvalue for the linear differential operator $(\vec{X})^2$ acting on the space $C^2(B(x_0, \rho) \subseteq D)$. In particular, for $a_1(\cdot) = 0 \in \mathbb{R}$ and $X(x_0) \neq 0_{\mathbb{R}^n}$, there exists a nontrivial solution $(\vec{X}(h) \neq 0, h \in C^2(B(x_0, \rho) \subseteq D))$ fulfilling a second order PDE

$$(\vec{X})^2(h)(x) = 0, \quad x \in B(x_0, \rho) \subseteq D. \quad (57)$$

For the case $n \geq 2$, a nontrivial solution of (57) can be found such that $\vec{X}(h)(x) \neq \text{const.}$, for $x \in B(x_0, \rho) \subseteq D$. It comes from the existence of a nontrivial first integral associated with ODE

$$\frac{dG}{d\sigma}(\sigma, \lambda) = X(G(\sigma, \lambda)), \quad G(0, \lambda) = \lambda \in B(0, \rho) \subseteq D, \quad \sigma \in [-a, a]. \quad (58)$$

4. Conclusions and Further Developments

In this paper, we have investigated the kernel of a polynomial differential equation involving a derivation in $\mathbb{R}^n$ by solving the Cauchy Problem for the corresponding first order system of PDEs. Moreover, we have proved that the kernel representation has a special significance on the space of solutions to corresponding system of PDEs.

The mathematical framework developed in this work can be easily extended for the study of some higher-order hyperbolic, parabolic or Hamilton–Jacobi equations involving a finite set of derivations.

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