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Robustness of Interval Monge Matrices in Fuzzy Algebra

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Abstract: Max–min algebra (called also fuzzy algebra) is an extremal algebra with operations maximum and minimum. In this paper, we study the robustness of Monge matrices with inexact data over max–min algebra. A matrix with inexact data (also called interval matrix) is a set of matrices given by a lower bound matrix and an upper bound matrix. An interval Monge matrix is the set of all Monge matrices from an interval matrix with Monge lower and upper bound matrices. There are two possibilities to define the robustness of an interval matrix. First, the possible robustness, if there is at least one robust matrix. Second, universal robustness, if all matrices are robust in the considered set of matrices. We found necessary and sufficient conditions for universal robustness in cases when the lower bound matrix is trivial. Moreover, we proved necessary conditions for possible robustness and equivalent conditions for universal robustness in cases where the lower bound matrix is non-trivial.

Keywords: robustness; fuzzy algebra; interval matrices; Monge matrices; inexact data

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1. Introduction

In many optimization problems maximum and minimum operations are involved. Both of these operations have the property that they create no new elements. An extremal algebra is defined as an algebra in which at least one of the operations possesses the above-mentioned property. The notion extremal algebra was introduced in [1]. Most frequented are the max–min algebra (called also fuzzy algebra) with operations maximum and minimum and max-plus algebra (called also max algebra) with operations maximum and addition. The role of max-plus algebra in the field of discrete event systems and wide variety of topics in this algebra were presented in [2]. How the concept of max-plus algebra can be used in modeling discrete event systems was described in [3]. Theory regarding fuzzy sets and their applications was published in [4,5]. A generalized approach to max–min algebra was introduced in [6].

Matrices in fuzzy algebra are helpful for expressing applications in graph theory, knowledge engineering, cluster analysis, and also for modeling discrete dynamic systems, for describing diagnosis of technical devices [7], fuzzy logic programs [8] or medical diagnosis [9].

If the considered matrices possess special properties, e.g., Monge matrices, the algorithms for matrix computations can be more efficient. We have studied Monge matrices, their structural properties and algorithms solving many problems related to Monge matrices in [10–15]. Robustness of matrices is a special case of matrix periodicity. Periodicity in max–min algebra was studied in [16]. The formula for computing matrix period in max–min algebra was proved in [16], namely the period is equal to least common multiple of the periods of all non-trivial strongly connected components from all threshold digraphs of the matrix. The steady state of a discrete dynamic system is important for the stability of the system. It is connected with the eigenvectors of the system matrix [17,18]. Exactly a robust matrix has the property that an eigenvector is reached by arbitrary initial vector. Robustness of matrices in max-plus algebra was studied in [19–21] and in max–min algebra in [22]. Robustness of Monge matrices was presented in [23], where the sufficient and necessary condition for a Monge matrix to be robust was proved using similarly to [16] the method of investigating relevant properties of the threshold digraphs corresponding to given matrix. The proof of the equivalent condition is based on the fact that a threshold digraph of a fuzzy Monge matrix cannot contain loops in two different strongly connected components.

Matrix inputs are in practice rather inexact values, i.e., contained in an interval, than exact values. Hence it is useful to investigate interval matrices, i.e., matrices with interval coefficients [9,24–27]. The question of matrix robustness turns to questions of possible robustness (there exists at least one robust matrix in an interval matrix) and universal robustness (all matrices are robust in an interval matrix) in case of interval matrices. Robustness of interval matrices in max-plus algebra was studied in [28]. Necessary and sufficient conditions together with polynomial algorithms for verifying the robustness of interval circulant matrices and matrices in binary max-plus algebra were presented in [29]. Equivalent conditions for an interval matrix to be possibly robust in max–min algebra were proved in [30]. However, the resulting robust matrix of the described algorithm needs not to possess the Monge property. In addition, there is no polynomial algorithm to check the universal robustness in max–min algebra. The computational complexity of the algorithm presented in [30] can be exponentially large. In contrast to general case of an interval fuzzy matrix equivalent conditions for possible robustness and equivalent conditions for universal robustness of circulant matrices with corresponding polynomial algorithms to verify the robustness were found in [30]. Results dealing with another types of robustness of matrices in fuzzy algebra were presented in [31]. These results encouraged us to turn our attention to another class of special matrices, namely to matrices with Monge property in max–min algebra. Equivalent conditions for possible robustness as well as universal robustness of special classes of interval Monge matrices were presented in [32], i.e., interval Monge matrices with additional restrictions were considered.

The aim of this paper is to prove equivalent conditions for universal robustness and necessary condition for possible robustness of interval Monge matrices in binary case of max–min algebra.

We briefly outline the content and main results of the paper. Section 2 provides the necessary preliminaries on max–min algebra, periodicity and robustness in max–min algebra. In Section 3, the notion of a Monge matrix is introduced. Important properties in regard of digraph structure of Monge matrices are proved and results concerning robustness of Monge matrices with exact data are summarized. In Section 4, the notions of an interval matrix, interval Monge matrix and universal robustness are introduced. Results concerning possible and universal robustness are presented. The key results of this section are Theorem 10, which formulates a necessary condition for an interval Monge matrix to be possibly robust, and the Theorem 11, which formulates a necessary and sufficient condition for an interval Monge matrix to be universally robust.

2. Background

The *fuzzy algebra* $\mathcal{B}$ is a triple $(B, \oplus, \otimes)$, where $(B, \leq)$ is a bounded linearly ordered set with binary operations *maximum* and *minimum*, denoted by $\oplus$, $\otimes$. The least element in B is denoted by O, the greatest one by I. For a given natural $n \in \mathbb{N}$, we use the notation N for the set of all smaller or equal positive natural numbers, i.e., $N = \{1, 2, \ldots, n\}$.

A *digraph* is an ordered pair of sets $G = (V, E)$, where V is a finite set of vertices, and E, the set of edges, is a subset of $V \times V$. A path in the digraph $G = (V, E)$ is a sequence of vertices $p = (i_1, \ldots, i_{k+1})$ such that $(i_j, i_{j+1}) \in E$ for $j = 1, \ldots, k$. The number k is the length of the path p and is denoted by $\ell(p)$. If $i_1 = i_{k+1}$, then p is called a *cycle*. For any $m, n \in \mathbb{N}$, $B(m, n)$ denotes the set of all matrices of type $m \times n$ and $B(n)$ the set of all n–dimensional column vectors over $\mathcal{B}$. For a given matrix $A \in B(n, n)$ the symbol $G(A) = (N, E)$ stands for the complete, edge-weighted digraph associated with A. The vertex set of $G(A)$ is N, and the capacity of any edge $(i,j) \in E$ is a_{ij}. In addition, for given $h \in B$ the *threshold digraph* $G(A,h)$ is the digraph $G = (N, E')$ with the vertex set N and the edge set $E' = \{(i,j); i,j \in N, a_{ij} \geq h\}$. We define the *period* of a strongly connected component $\mathcal{K}$ as per $\mathcal{K} = \gcd\{\,\ell(c);\ c$ is a cycle in $\mathcal{K}$ with length $\ell(c) > 0\,\}$. If $\mathcal{K}$ is trivial, then per $\mathcal{K} = 1$.

Let $A \in \mathbb{B}(n,n)$ and $x \in \mathbb{B}(n)$. The sequence $O(A,x) = \{x^{(0)}, x^{(1)}, x^{(2)}, \ldots, x^{(n)}, \ldots\}$ is the orbit of $x = x^{(0)}$ generated by A, where $x^{(r)} = A^r \otimes x^{(0)}$ for each $r \in \mathbb{N}$. For a given matrix $A \in \mathbb{B}(n,n)$, the number $\lambda \in \mathbb{B}(n)$ and the n–tuple $x \in B(n)$ are the so-called *eigenvalue* of A and *eigenvector* of A, respectively, if they satisfy the equation $A \otimes x = \lambda \otimes x$. We define the corresponding *eigenspace* $V(A,\lambda)$ as the set $V(A,\lambda) = \{x \in B(n);\ A \otimes x = \lambda \otimes x\}$. Let $A = (a_{ij}) \in B(n,n)$, $\lambda \in B$. Let $T(A,\lambda) = \{x \in B(n);\ O(A,x) \cap V(A,\lambda) \neq \emptyset\}$. A is called *λ–robust* if $T(A,\lambda) = B(n)$. A λ–robust matrix with $\lambda = I$ is called *a robust matrix*.

According to [16] we define

$$\mathrm{SCC}^*(A) = \cup\,\{\mathrm{SCC}^*(G(A,h));\ h \in \{a_{ij};\ i,j \in N\}\}.$$

The following theorems are crucial investigating robustness of matrices.

Theorem 1 ([16]). *Let $A \in B(n,n)$. Then per $A = lcm\{per\,\mathcal{K};\ \mathcal{K} \in SCC^*(A)\}$.*

Theorem 2 ([22]). *Let $A = (a_{ij}) \in B(n,n)$. Then A is robust if and only if* per$A = 1$.

In this paper, we consider binary max–min algebra $\mathcal{B}$, i.e., $\mathcal{B}$ is an ordered set $(\mathbb{B}, \oplus, \otimes)$, where $\mathbb{B} = \{0, 1\}$.

3. Robustness of Monge Matrices

In this section, we introduce the notion of a convex Monge matrix. We prove important properties in regard of digraph structure of Monge matrices which will be useful to investigate robustness of Monge matrices with inexact data. Moreover, we recall results concerning robustness of Monge matrices with exact data.

Definition 1. *A matrix $A = (a_{ij}) \in B(m,n)$ is a convex Monge matrix if and only if*

$$a_{ij} \otimes a_{kl} \leq a_{il} \otimes a_{kj} \qquad \textit{for all} \quad i < k,\ j < l.$$

According to robustness we consider square matrices, only, in this paper, i.e., $A \in B(n,n)$.

Definition 2. *A matrix A is a non-trivial matrix, if the threshold digraph $G(A,1)$ contains a cycle of positive length. Otherwise the matrix is trivial.*

Theorem 3. *Let A be a trivial matrix. Then A is robust.*

Proof. By Definition 2 the digraph $G(A,1)$ contains only trivial strongly connected components. The period of a trivial strongly connected component is 1. Hence, by Theorem 1 *per* $A = 1$. By Theorem 2, if the period of a matrix is 1, then this matrix is robust. □

Lemma 1. *Let A be a Monge matrix. Let there exist cycles (i,j,i) and (k,l,k) for $i \leq j$ and $k \leq l$ in the digraph $G(A,1)$. If one of the statements holds:*

(i) $k \leq l \leq i \leq j$,
(ii) $k \leq i \leq l \leq j$,

then the cycles (i,j,i) and (k,l,k) lie in a common non-trivial strongly connected component $\mathcal{K} \subseteq G(A,1)$.

Proof.

(i) Let there exist cycles $c_1 = (i, j, i)$ and $c_2 = (k, l, k)$ in the digraph $G(A, 1)$ and let $k \leq l \leq i \leq j$. We prove that the cycles c_1 and c_2 lie in a common non-trivial strongly connected component $\mathcal{K}$ of $G(A, 1)$. The following cases may occur:

(a.) Let $k = l = i \leq j$. Then $(k, l, k) = (i, i)$. It means that the cycle c_2 is a loop lying on the cycle c_1 in $G(A, 1)$, or if $i = j$, c_1 and c_2 are identical loops, respectively. Thus, nodes i, j, k, l lie in a common non-trivial strongly connected component.

(b.) Let $k < l = i \leq j$. Then $(k, l, k) = (k, i, k)$. It means that there exists a cycle $c = (k, i, j, i, k)$ in $G(A, 1)$. Thus, nodes i, j, k, l lie in a common non-trivial strongly connected component.

(c.) Let $k \leq l < i \leq j$. Based on Definition 1 for $k < i$, $l < j$ holds:

$$1 = a_{kl} \otimes a_{ij} \leq a_{kj} \otimes a_{il}.$$

Since $a_{kl} = a_{ij} = 1$, a_{kj} and a_{il} must equal to 1. Thus, there exist edges (k, j) and (i, l) in $G(A, 1)$, i.e., there is a cycle $c = (j, i, l, k, j)$ in $G(A, 1)$. Hence, the nodes i, j, k, l lie in a common non-trivial strongly connected component.

(ii) Let there exist cycles $c_1 = (i, j, i)$ and $c_2 = (k, l, k)$ in the digraph $G(A, 1)$ and let $k \leq i \leq l \leq j$. We prove that the cycles c_1 and c_2 lie in a common non-trivial strongly connected component $\mathcal{K}$ of $G(A, 1)$. The following cases may occur:

(a.) Let $k = i = l$. Then $(k, l, k) = (i, i)$. It means that the cycle c_2 is a loop lying on cycle c_1 in $G(A, 1)$, or if $i = j$, c_1 and c_2 are identical loops, respectively. Thus, the nodes i, j, k, l lie in a common non-trivial strongly connected component.

(b.) Let $k = i < l$. This means that $(k, l, k) = (i, l, i)$, thus there exists a cycle $c = (j, i, l, i, j)$ in $G(A, 1)$. Hence, the nodes i, j, k, l lie in a common non-trivial strongly connected component.

(c.) Let $k < i = l$. This means that $(k, l, k) = (k, i, k)$, thus there exists a cycle $c = (k, i, j, i, k)$ in $G(A, 1)$. Hence, the nodes i, j, k, l lie in a common non-trivial strongly connected component.

(d.) Let $k < i < l \leq j$. Based on Definition 1 for $k < i$, $l < j$ holds:

$$1 = a_{kl} \otimes a_{ij} \leq a_{kj} \otimes a_{il}.$$

Since $a_{kl} = a_{ij} = 1$, a_{kj} and a_{il} must equal to 1. Then there exist edges (k, j) and (i, l) in $G(A, 1)$, i.e., there is a cycle $c = (j, i, l, k, j)$ in $G(A, 1)$. Hence, the nodes i, j, k, l lie in a common non-trivial strongly connected component. In case, if $l = j$, the nodes i, j, k, l lie in a common non-trivial strongly connected component trivially.

□

Lemma 2. *Let A be a Monge matrix. Let $G(A, 1)$ contain a non-trivial strongly connected component $\mathcal{K} = (N_\mathcal{K}, E_\mathcal{K})$. Then there is a cycle (t, u, t) for $t = \min\{N_\mathcal{K}\}$, $u = \max\{N_\mathcal{K}\}$ in the digraph $G(A, 1)$.*

Proof. Let $G(A, 1)$ contains a non-trivial strongly connected component $\mathcal{K} = (N_\mathcal{K}, E_\mathcal{K})$ with a loop, for which holds $t = \min\{N_\mathcal{K}\}$, $u = \max\{N_\mathcal{K}\}$. If $t = u$, then the statement holds trivially. Hence, in the following text we consider $t < u$.

▷ Let $l \in N_\mathcal{K}$ be the maximal index for which holds $a_{tl} = 1$ and let $l < u$, i.e., $a_{tu} = 0$. The node u lies in $\mathcal{K}$. This implies that there exists $k \in N_\mathcal{K}$ with $a_{ku} = 1$. By the Monge property of A we get:

$$1 = a_{tl} \otimes a_{ku} \leq a_{tu} \otimes a_{kl}.$$

Since $a_{tl} = a_{ku} = 1$, then also $a_{tu} = 1$ (and $a_{kl} = 1$ as well). This is a contradiction with the assumption $a_{tu} = 0$.

- ▷ Let $k \in N_{\mathcal{K}}$ be the minimal index for which holds $a_{uk} = 1$ and let $k > t$, i.e., $a_{ut} = 0$. The node t lies in $\mathcal{K}$. This implies that there exists $l \in N_{\mathcal{K}}$ with $a_{lt} = 1$. By the Monge property of A we get:

$$1 = a_{lt} \otimes a_{uk} \leq a_{lk} \otimes a_{ut}.$$

Since $a_{lt} = a_{uk} = 1$, then also $a_{tu} = 1$ (and $a_{lk} = 1$ as well). This is a contradiction with the assumption $a_{tu} = 0$.

□

Lemma 3. *Let A be a Monge matrix. Let $G(A,1)$ contain a non-trivial strongly connected component $\mathcal{K} = (N_{\mathcal{K}}, E_{\mathcal{K}})$ with a loop. Let $t = \min\{N_{\mathcal{K}}\}$, $u = \max\{N_{\mathcal{K}}\}$. If there is a cycle (k,l,k) for $t < k \leq l < u$ in the digraph $G(A,1)$, then $k, l \in N_{\mathcal{K}}$.*

Proof. Let $G(A,1)$ contain strongly connected component $\mathcal{K} = (N_{\mathcal{K}}, E_{\mathcal{K}})$ with a loop, for which holds $t = \min\{N_{\mathcal{K}}\}$, $u = \max\{N_{\mathcal{K}}\}$. If $t = u$, then the statement holds trivially. Hence, we consider $t < u$ in the following text.

(i) Let $t < k = l < u$. $\mathcal{K}$ contains a loop, i.e., by Lemma 4 it follows that $k \in N_{\mathcal{K}}$.

(ii) Let $t < k < l < u$. Since $\mathcal{K}$ contains a loop, there exists $m \in N_{\mathcal{K}}$, for which $a_{mm} = 1$. The following cases may occur:

- ▷ Let $t = m < u$. Thus, $a_{tt} = a_{mm} = 1$. According to Definition 1 the following statement holds:

$$1 = a_{mm} \otimes a_{kl} \leq a_{ml} \otimes a_{km}.$$

Since $a_{mm} = a_{kl} = 1$, $a_{ml} = 1$, $a_{km} = 1$ as well. This implies that there exist edges (t,l) and (k,t) in $G(A,1)$. Thus, the nodes t, u, k, l lie in a common non-trivial strongly connected component $\mathcal{K}$.

- ▷ We prove the cases $t < k < l < u = m$ and $t < m < k < l < u$ by analogy with the above case.
- ▷ Let $t < k < m < l < u$. Since $m \in N_{\mathcal{K}}$, there exists a path $(t, \ldots, o, p, \ldots, m)$, which connects the node t with the node m in the strongly connected component $\mathcal{K}$ while $t \leq o < k$ or $l < o \leq u$, respectively, and $k < p < l$ at the same time. The same applies that there exists a path $(u, \ldots, o, p, \ldots, m)$, which connects the node u with the node m in the strongly connected component $\mathcal{K}$ while the same inequalities as above hold for the given indices.

  - ► Let $t \leq o < k$. According to Definition 1 the following statement holds:

$$1 = a_{op} \otimes a_{kl} \leq a_{ol} \otimes a_{kp}.$$

Since $a_{op} = a_{kl} = 1$, $a_{ol} = 1$, $a_{kp} = 1$ as well. This implies that there exist edges (o,l) and (k,p) in $G(A,1)$. Thus, the nodes t, u, k, l lie in a common non-trivial strongly connected component $\mathcal{K}$.

  - ► We prove the case $l < o \leq u$ by analogy with the above case, when $t \leq o < k$.
- ▷ We prove the case $t < k < l < m < u$ by analogy with the case, when $t < k < m < l < u$.

□

It is enough to consider $h \in H = \{a_{ij};\ i, j \in N\}$ while investigating robustness of matrices using threshold digraphs. Since $G(A,0)$ is a complete digraph, a crucial role plays the digraph $G(A,1)$ in binary case.

Lemma 4 ([23]). *Let $A \in B(n,n)$ be a Monge matrix. Let $h \in H$. Let for $i,k \in N$ be the loops (i,i) and (k,k) in the digraph $G(A,h)$. Then the nodes i and k are in the same non-trivial strongly connected component $\mathcal{K}$ of $G(A,h)$.*

Lemma 5. *Let A be a Monge matrix. Let the digraph $G(A,1)$ contain a cycle (c,d,c) for $c < d$. Then the following statements hold:*

(i) *If $\exists\, k,l$: $k < l \leq c$ such that whether $a_{kl} = 1$ or $a_{lk} = 1$ then $G(A,1)$ contains a loop on the node c.*
(ii) *If $\exists\, k,l$: $d \leq k < l$ such that whether $a_{kl} = 1$ or $a_{lk} = 1$ then $G(A,1)$ contains a loop on the node d.*

Proof.

(i) Let there exist k,l: $k < l \leq c$. The following cases may occur:

(a.) Let $a_{kl} = 1$.
According to Definition 1 the following statements hold:

$$1 = a_{kl} \otimes a_{cd} \leq a_{kd} \otimes a_{cl},$$

$$1 = a_{cl} \otimes a_{dc} \leq a_{cc} \otimes a_{dl}.$$

Since $a_{kl} = a_{cd} = 1$ and the first inequality holds, then $a_{kd} = a_{cl} = 1$ as well. Moreover, $a_{cl} = a_{dc} = 1$ and the second inequality holds, i.e., also $a_{cc} = a_{dl} = 1$. Thus, $G(A,1)$ contains a loop on the node c along with the cycle (c,d,c).
If $l = c$, the claim follows from the first inequality.

(b.) We prove the case $a_{lk} = 1$ by analogy with the above case.

(ii) Let there exist k,l: $d \leq k < l$. We prove the cases $a_{kl} = 1$ and $a_{lk} = 1$ by analogy with the case (i).

□

In case of the existence of two cycles with loops, Lemma 4 guaranties that the cycles lie in a common non-trivial strongly connected component.

Remark 1 ([26]). *In our considerations we use an important property of a threshold digraph corresponding to a Monge matrix. Specifically, each cycle can be spread into cycles of length 1 and 2.*

The equivalent condition for a Monge matrix with exact data to be robust was proved in [23]. It will be helpful investigating robustness of interval Monge matrices in the following section.

Theorem 4 ([23]). *Let $A \in B(n,n)$ be a Monge matrix. Then A is robust if and only if for each $h \in H$ the digraph $G(A,h)$ contains at most one non-trivial strongly connected component and this has a loop.*

For the binary case the above theorem has the following form.

Theorem 5. *Let A be a Monge matrix. Then A is robust if and only if the digraph $G(A,1)$ contains at most one non-trivial strongly connected component and this has a loop.*

4. Robustness of Binary Interval Monge Matrices

In this section, we deal with robustness of matrices with inexact data, which are represented as matrices with interval coefficients. We prove some necessary or sufficient conditions for possible or universal robustness, respectively and equivalent conditions for universal robustness in cases where the lower bound matrix is non-trivial using results from the previous section.

As with [24,25], we define an interval matrix.

Definition 3. *Let $\underline{A}, \overline{A} \in B(n,n)$, $\underline{A} \leq \overline{A}$. An interval matrix* $\mathbf{A}$ *with bounds* $\underline{A}$ *and* $\overline{A}$ *is defined as follows:*

$$\mathbf{A} = [\underline{A}, \overline{A}] = \left\{ A \in B(n,n);\ \underline{A} \leq A \leq \overline{A} \right\}.$$

Definition 4. *An interval matrix* $\mathbf{A}^M$ *for* $\mathbf{A} = [\underline{A}, \overline{A}]$ *is called interval Monge, if* $\underline{A}$, $\overline{A} \in B(n,n)$ *are Monge matrices and* $\mathbf{A}^M = \{A \in \mathbf{A};\ A \text{ is Monge}\}$.

Definition 5. *An interval matrix* $\mathbf{A}$ *is called:*

- *possibly robust if there exists a matrix* $A \in \mathbf{A}$ *such that* A *is robust,*
- *universally robust if each matrix* $A \in \mathbf{A}$ *is robust.*

Definition 6. *An interval Monge matrix* $\mathbf{A^M} = [\underline{\mathbf{A}}, \overline{\mathbf{A}}]$ *is called:*

- *possibly robust if there exists a matrix* $A \in \mathbf{A^M}$ *such that* A *is robust,*
- *universally robust if each matrix* $A \in \mathbf{A^M}$ *is robust.*

In this paper, we use the following property in our considerations:

$$\text{For } \underline{A} \leq A \leq \overline{A} \text{ holds } G(\underline{A}, 1) \subseteq G(A, 1) \subseteq G(\overline{A}, 1).$$

Theorem 6. *Let* $\mathbf{A} = [\underline{A}, \overline{A}]$. *Let the matrix* $\underline{A} \in B(n,n)$ *be trivial. Let* $n \geq 2$. *Let* $G(\overline{A}, 1)$ *contain the cycle* $(1, n, 1)$. *Then* $\mathbf{A}^M$ *is not universally robust.*

Proof. Based on Definition 3 there exists a matrix $A \in \mathbf{A}^M$, $A > \underline{A}$, which is constructed as follows:

$$a_{ij} = \begin{cases} 1 & \text{for } i = 1, j = n, \\ 1 & \text{for } i = n, j = 1, \\ \underline{a}_{ij} & \text{otherwise.} \end{cases}$$

We show that A is a Monge matrix.

$\underline{A}$ is a Monge matrix, thus by setting $a_{1n} = a_{n1} = 1$ we do not break the Monge property defined in Definition 1, since for all $1 < k$ and $j < n$ holds (Without loss of generality we assume that $k \leq j$.):

$$a_{1j} \otimes a_{kn} = \underline{a}_{1j} \otimes \underline{a}_{kn} \leq \underline{a}_{1n} \otimes \underline{a}_{kj} \leq 1 \otimes a_{kj} = a_{1n} \otimes a_{kj},$$
$$a_{j1} \otimes a_{nk} = \underline{a}_{j1} \otimes \underline{a}_{nk} \leq \underline{a}_{jk} \otimes \underline{a}_{n1} \leq a_{jk} \otimes 1 = a_{jk} \otimes a_{n1}.$$

Consequently, the matrix A is a Monge matrix and the corresponding threshold digraph $G(A, 1)$ contains exactly one cycle, namely $(1, n, 1)$. Hence, by Theorem 5 the matrix A is not robust, thus according to Definition 6 $\mathbf{A}^M$ is not universally robust. □

Theorem 7. *Let* $\mathbf{A} = [\underline{A}, \overline{A}]$. *Let the matrix* $\overline{A} \in B(n,n)$ *be trivial. Then* $\mathbf{A}^M$ *is universally robust.*

Proof. Since $\overline{A}$ is trivial then by Definition 5 each matrix $A \in \mathbf{A}$ is trivial. Thus, the digraph $G(A, 1)$ for $A \in \mathbf{A}$ does not contain any non-trivial strongly connected component. Moreover, by Theorem 3 each matrix $A \in \mathbf{A}$ is robust, thus the matrix $\mathbf{A}^M$ is universally and possibly robust. □

Corollary 1. *Let* $\mathbf{A} = [\underline{A}, \overline{A}]$. *Let the matrix* $\overline{A} \in B(n,n)$ *be trivial. Then* $\mathbf{A}^M$ *is possibly robust.*

Proof. The assertion follows by Definition 6. □

Theorem 8. *Let* $\mathbf{A} = [\underline{A}, \overline{A}]$. *Let the matrix* $\underline{A} \in B(n,n)$ *be trivial. Then* $\mathbf{A}^M$ *is possibly robust.*

Proof. By Theorem 3 the matrix $\underline{A}$ is robust. Using Definition 6, if at least one matrix from the interval is robust then the interval Monge matrix $\mathbf{A}^M$ is possibly robust. □

Theorem 9. *Let $\mathbf{A} = [\underline{A}, \overline{A}]$. Let the matrix $\underline{A} \in B(n,n)$ be trivial and the matrix $\overline{A} \in B(n,n)$ be non-trivial. Then, $\mathbf{A}^M$ is universally robust if the following statements hold:*

(i) $G(\overline{A}, 1)$ *contains exactly one non-trivial strongly connected component $\overline{\mathcal{K}}$ and this has a loop,*
(ii) $\overline{\mathcal{K}}$ *does not contain the cycle $(1, n, 1)$,*
(iii) *for each cycle (c, d, c), $c < d$ in $\overline{\mathcal{K}}$ at least one of the statements hold:*

- $\exists\, k, l \leq c, k \neq l$ *such that either* $\underline{a}_{kl} = 1$ *or* $\underline{a}_{lk} = 1$.
- $\exists\, k, l \geq d, k \neq l$ *such that either* $\underline{a}_{kl} = 1$ *or* $\underline{a}_{lk} = 1$.

Proof. Assume 9.(i), 9.(ii) and 9.(iii) hold. ($k \neq l$, because $\underline{A}$ is trivial.) We prove that $\mathbf{A}^M$ is universally robust. The following three cases may occur:

(1) If the only non-trivial strongly connected component $\overline{\mathcal{K}}$ of $G(\overline{A}, 1)$ is generated by one node, then for all matrices $A \in \mathbf{A}^M$ holds that the digraph $G(A, 1)$ contains either a cycle of length 1, or does not contain any cycle (trivial matrix). Using Theorem 3 and Theorem 5 each matrix $A \in \mathbf{A}^M$ is robust, thus $\mathbf{A}^M$ is universally robust.

(2) Let the non-trivial strongly connected component $\overline{\mathcal{K}}$ of the digraph $G(\overline{A}, 1)$ be generated by two nodes, thus the strongly connected component $\overline{\mathcal{K}}$ contains cycle (c, d, c) for $1 < c < d < n$. Since the statement 9.(iii) holds, by Lemma 5 there exists a loop on the node c or d, thus the digraph $G(A, 1)$ of each matrix $A \in \mathbf{A}^M$ which contains a cycle of length 2, also contains a loop on one of the nodes. This implies that the digraph $G(A, 1)$ of each non-trivial matrix $A \in \mathbf{A}^M$ contains a non-trivial strongly connected component $\mathcal{K}$ with a loop. Moreover, since each cycle of length 2 is connected to a loop, by Lemma 4 all such cycles are in a common non-trivial strongly connected component. Using Theorem 5 each matrix $A \in \mathbf{A}^M$ is robust, thus $\mathbf{A}^M$ is universally robust.

(3) Let the non-trivial strongly connected component of $G(\overline{A}, 1)$ be generated by more than two nodes. According to Remark 1, for each non-trivial matrix $A \in \mathbf{A}^M$ holds that each cycle of length $l(c) \geq 3$ in $G(A, 1)$ consists of cycles of length 1 and 2. Since each cycle of length $l(c) = 2$ based on 9.(iii) contains a loop as well (see the previous case) and by Lemma 4 all loops in $G(A, 1)$ lie in the same non-trivial strongly connected component $\mathcal{K}$, the digraph $G(A, 1)$ of each non-trivial matrix $A \in \mathbf{A}^M$ contains exactly one non-trivial strongly connected component, which has a loop. By Theorem 5 follows that the interval matrix $\mathbf{A}^M$ is universally robust.

□

Theorem 10. *Let $\mathbf{A} = [\underline{A}, \overline{A}]$. Let the matrix $\underline{A} \in B(n,n)$ be non-trivial. If $\mathbf{A}^M$ is possibly robust, then the following statements hold:*

(i) $G(\overline{A}, 1)$ *contains exactly one non-trivial strongly connected component $\overline{\mathcal{K}} = (N_{\overline{\mathcal{K}}}, E_{\overline{\mathcal{K}}})$ with a loop,*
(ii) $G(\underline{A}, 1)$ *contains only such non-trivial strongly connected components, which are generated by nodes from $N_{\overline{\mathcal{K}}}$ exclusively.*

Proof. Let $\mathbf{A}^M$ be possibly robust. We show that the statements 10.(i) and 10.(ii) hold.

(i) According to Lemma 4 $G(\overline{A}, 1)$ contains at most one non-trivial strongly connected component with a loop.
Let $G(\overline{A}, 1)$ do not contain any non-trivial strongly connected component with a loop, i.e., every component is without any loop. Hence for each matrix $A \leq \overline{A}$ the digraph $G(A, 1)$ does not contain any non-trivial strongly connected component with a loop. Moreover, since $\underline{A}$

is non-trivial, $G(A,1)$ contains at least one non-trivial component. According to Theorem 5 there is no robust matrix $A \in \mathbf{A}^M$, what is contrary to the assumption that $\mathbf{A}^M$ is possibly robust.

(ii) Let $G(\underline{A},1)$ contain a non-trivial strongly connected component $\underline{\mathcal{K}} = (N_{\underline{\mathcal{K}}}, E_{\underline{\mathcal{K}}})$ such that $N_{\underline{\mathcal{K}}} \nsubseteq N_{\overline{\mathcal{K}}}$.

We want to reach a dispute with the possible robustness; thus we show that there is no matrix $A \in \mathbf{A}^M$, which is robust.

The period of the component $\underline{\mathcal{K}}$ is $\mathrm{per}(\underline{\mathcal{K}}) > 1$, because it does not contain any loop, since $N_{\underline{\mathcal{K}}} \nsubseteq N_{\overline{\mathcal{K}}}$. For every matrix A, $\underline{A} \leq A \leq \overline{A}$ the digraph $G(A,1)$ must contain the component $\underline{\mathcal{K}}$, thus for every matrix $A \in \mathbf{A}^M$ the digraph $G(A,1)$ must contain a non-trivial strongly connected component without any loop. By Theorem 5 none of the matrices $A \in \mathbf{A}^M$ is robust, i.e., $\mathbf{A}^M$ is not possibly robust.

□

Theorem 11. *Let $\mathbf{A} = [\underline{A}, \overline{A}]$. Let the matrix $\underline{A} \in B(n,n)$ be non-trivial. Then $\mathbf{A}^M$ is universally robust if and only if the following statements hold:*

(i) $G(\overline{A},1)$ *contains exactly one non-trivial strongly connected component* $\overline{\mathcal{K}} = (N_{\overline{\mathcal{K}}}, E_{\overline{\mathcal{K}}})$ *and this has a loop,*

(ii) $G(\underline{A},1)$ *contains exactly one non-trivial strongly connected component* $\underline{\mathcal{K}} = (N_{\underline{\mathcal{K}}}, E_{\underline{\mathcal{K}}})$ *and this has a loop,*

(iii) *Let* $t = \min\{N_{\underline{\mathcal{K}}}\}$, $u = \max\{N_{\underline{\mathcal{K}}}\}$. *For each cycle* (m,q,m) *in* $\overline{\mathcal{K}}$, $m < t, q > u$ *there exists at least one of the indices* b,c *and at least one of the indices* d,e *for* $b,d \leq m; c,e \geq q$ *such that both conditions hold:*

$$\underline{a}_{bu} = 1 \quad or \quad \underline{a}_{ct} = 1, \tag{1}$$

$$\underline{a}_{te} = 1 \quad or \quad \underline{a}_{ud} = 1. \tag{2}$$

Proof. We prove the sufficient condition.

Assume 11.(i), 11.(ii), 11.(iii) hold. $G(\overline{A},1)$ and also $G(\underline{A},1)$ contains exactly one non-trivial strongly connected component, and this has a loop, thus matrices $\underline{A}$ and $\overline{A}$ are robust. We prove that each matrix $A \in \mathbf{A}^M$, $\underline{A} < A < \overline{A}$ is robust, i.e., $G(A,1)$ contains exactly one non-trivial strongly connected component and this has a loop. According to 1, 4 and assumption 11.(i) it is enough to show that every cycle of length two in $G(A,1)$ lies in the same non-trivial strongly connected component with a loop and hence this component is unique. The following cases may occur:

Case 1. Let the non-trivial strongly connected component $\overline{\mathcal{K}}$ of the digraph $G(\overline{A},1)$ with a loop be generated by 1 node. Then also the digraph $G(\underline{A},1)$ contains this component with a loop, thus $\underline{\mathcal{K}} = \overline{\mathcal{K}}$. In this case the digraph $G(A,1)$ of each matrix $A \in \mathbf{A}^M$ contains exactly one non-trivial strongly connected component, which has a loop and this component is generated by one node. Thus, for the non-trivial strongly connected component of the digraph $G(A,1)$ of each matrix $A \in \mathbf{A}^M$ holds $\mathcal{K} = \overline{\mathcal{K}}$. According to Theorem 5 and the Definition 6 the matrix $\mathbf{A}^M$ is universally robust.

Case 2. Let the non-trivial strongly connected component $\overline{\mathcal{K}}$ of the digraph $G(\overline{A},1)$ with a loop be generated by two nodes. The digraph $G(\underline{A},1)$ of the matrix $\underline{A}$ contains exactly one non-trivial strongly connected component $\underline{\mathcal{K}}$ with a loop. Since, for a Monge matrix according to Lemma 4 all loops lie in the same non-trivial strongly connected component, the digraph $G(A,1)$ of each matrix $A \in \mathbf{A}^M$, $\underline{A} < A$ contains exactly one non-trivial strongly connected component which has a loop and is generated by one or two nodes. Based on Theorem 5 is each matrix $A \in \mathbf{A}^M$ robust, thus $\mathbf{A}^M$ is universally robust.

Case 3. Let $t = \min\{N_{\underline{\mathcal{K}}}\}$, $u = \max\{N_{\underline{\mathcal{K}}}\}$. Let the non-trivial strongly connected component $\overline{\mathcal{K}}$ of the digraph $G(\overline{A},1)$ be generated by more than two nodes. Let there exist a cycle (m,q,m) in $\overline{\mathcal{K}}$ for $m < t, q > u$ for which there exists at least one of the indices b,c and at least one of the indices d,e; $b,d \leq m; c,e \geq q$ such that the conditions (1) and (2) hold.

Let $\underline{A} < A < \overline{A}$. Let there exist a non-trivial strongly connected component $\mathcal{K} \subset G(A,1)$ such that $\underline{\mathcal{K}} \subset \mathcal{K}$. We show that if there exists a cycle (m,q,m) in $G(A,1)$ for $m < t$, $q > u$, then $m,q \in N_{\mathcal{K}}$, i.e., the cycle (m,q,m) lies in the non-trivial strongly connected component $\mathcal{K}$.

We shall consider all possibilities satisfying conditions (1) and (2) in 11.(iii), i.e.,

- $\underline{a}_{bu} = \underline{a}_{te} = 1$,
- $\underline{a}_{ct} = \underline{a}_{ud} = 1$,
- $\underline{a}_{bu} = \underline{a}_{ud} = 1$,
- $\underline{a}_{ct} = \underline{a}_{te} = 1$.

Hereinafter we investigate all the possible cases that can occur.

In cases, if in $b,d \leq t$ or $c,e \geq u$, equality occurs for at least one index from each pair b,c or d,e respectively, or if the pairs b,d or c,e respectively, are equal to each other, then the cycle (m,q,m) is part of the component $\underline{\mathcal{K}}$. Thus, it does not satisfy the assumptions $t = \min\{N_{\underline{\mathcal{K}}}\}$, $u = \max\{N_{\underline{\mathcal{K}}}\}$. These cases are irrelevant for our calculations.

Let us suppose $\underline{a}_{bu} = \underline{a}_{te} = 1$, i.e., there exist edges (b,u) and (t,e) in $G(\underline{A},1)$. Based on our considerations that $b,d \leq m$; $c,e \geq q$, while the above-mentioned cases are irrelevant for our calculations, the following three cases with respect to cycle position may occur:

(1.) Let $b = m$ and $e > q$ at the same time.
According to Definition 1, for the digraph $G(A,1)$ of each matrix $A \in \mathbf{A}^M$, which contains the cycle (m,q,m) the statement

$$1 = a_{mq} \otimes a_{te} \leq a_{me} \otimes a_{tq}$$

holds only in case, if $a_{me} = a_{tq} = 1$. Thus, the digraph $G(A,1)$ of each matrix $A \in \mathbf{A}^M$ which contains the non-trivial strongly connected component $\underline{\mathcal{K}}$ and the cycle (m,q,m), contains also the edge (t,q). Moreover, since $b = m$, there exists an edge (m,u) (based on the assumption $\underline{a}_{bu} = 1$) in the digraph $G(A,1)$ of each matrix $A \in \mathbf{A}^M$. This implies that the digraph $G(A,1)$ of each matrix $A \in \mathbf{A}^M$ which contains the non-trivial strongly connected component $\underline{\mathcal{K}}$ and the cycle (m,q,m), contains also the edges (t,q) and (m,u). These edges connect the cycle (m,q,m) with $\underline{\mathcal{K}}$ to a common non-trivial strongly connected component $\mathcal{K}$.

(2.) The case $b < m$ and $e = q$ at the same time can be proved by analogy with the above case.

(3.) Let $b < m$ and $e > q$ at the same time.
According to Definition 1, for the digraph $G(A,1)$ of each matrix $A \in \mathbf{A}^M$, which contains the cycle (m,q,m) the statements

$$1 = a_{bu} \otimes a_{mq} \leq a_{bq} \otimes a_{mu},$$

$$1 = a_{mq} \otimes a_{te} \leq a_{me} \otimes a_{tq}$$

hold only in case, if $a_{bq} = a_{mu} = 1$ and $a_{me} = a_{tq} = 1$. Thus, the digraph $G(A,1)$ of each matrix $A \in \mathbf{A}^M$ which contains the non-trivial strongly connected component $\underline{\mathcal{K}}$ and the cycle (m,q,m), contains also the edges (m,u) and (t,q). These edges connect the cycle (m,q,m) with $\underline{\mathcal{K}}$ to a common non-trivial strongly connected component $\mathcal{K}$.

Since $\underline{\mathcal{K}} \subseteq \mathcal{K}$, the non-trivial strongly connected component $\mathcal{K}$ contains a loop as well, then according to Theorem 5 each matrix $A \in \mathbf{A}^M$ is robust, i.e., $\mathbf{A}^M$ is universally robust.

All the remaining cases can be proved by analogy with the above case.

We prove the necessary condition.

We assume that the matrix $\mathbf{A}^M$ is universally robust, i.e., every matrix $A \in \mathbf{A}^M$ is robust, in other words $G(A,1)$ contains exactly one non-trivial strongly connected component and this has a loop. We prove that the conditions 11.(i), 11.(ii), 11.(iii) hold.

$(i\text{–}ii)$ Since the matrix $\underline{A}$ is non-trivial, according to Theorem 5 the digraph $G(\underline{A}, 1)$ contains exactly one non-trivial strongly connected component $\underline{\mathcal{K}}$, which has a loop and the digraph $G(\overline{A}, 1)$ also contains exactly one non-trivial strongly connected component $\overline{\mathcal{K}}$, which has a loop.

(iii) $\mathbf{A}^M$ is universally robust, thus the digraph $G(A, 1)$ of each matrix $A \in \mathbf{A}^M$ contains exactly one non-trivial strongly connected component, which has a loop.

Let $\overline{\mathcal{K}} = \underline{\mathcal{K}}$. Then the digraph $G(A, 1)$ of each matrix $A \in \mathbf{A}^M$ contains exactly one non-trivial strongly connected component which has a loop. Thus, $\mathcal{K} = \underline{\mathcal{K}} = \overline{\mathcal{K}}$ and the condition 11.(iii) holds trivially (does not exist any cycle $(m, q, m) \in \overline{\mathcal{K}}$ for which holds $m, q \notin N_{\underline{\mathcal{K}}}$).

Let $\underline{\mathcal{K}} \subset \overline{\mathcal{K}}$ and let $\overline{\mathcal{K}}$ contains a cycle (m, q, m) for $m < t, q > u$.

Let $t = \min\{N_{\underline{\mathcal{K}}}\}$, $u = \max\{N_{\underline{\mathcal{K}}}\}$. According to Lemma 2 there is a cycle (t, u, t) in $\underline{\mathcal{K}}$. Let $b, d \leq m$, $c, e \geq q$ and let the condition 11.(iii) does not hold. We show that there exists a matrix $A \in \mathbf{A}^M$, which is not robust.

Since $m < t$, $q > u$, it holds that $(m, q, m) \in \overline{\mathcal{K}} \setminus \underline{\mathcal{K}}$. Thus, there is a matrix $A \in \mathbf{A}$, whose digraph $G(A, 1)$ contains at least two non-trivial strongly connected components. According to Theorem 5 this matrix is not robust. We find $A \in \mathbf{A}^M$ which is not robust.

Based on the contradiction of the consideration 11.(iii), either (1) is false, (2) is false, or both conditions (1) and (2) are false.

Case 1. Let us suppose (1) in 11.(iii) is false, i.e., none of the indices b, c exists, for which $\underline{a}_{bu} = 1$ or $\underline{a}_{ct} = 1$, i.e. $\underline{a}_{bu} = \underline{a}_{ct} = 0$ for all $b \leq m$, $c \geq q$. Let us define the matrix A as follows:

$$a_{ij} = \begin{cases} \underline{a}_{ij} & \text{if } i \leq m,\ j \leq u, \\ \underline{a}_{ij} & \text{if } i \geq q,\ j \geq t, \\ \overline{a}_{ij} & \text{otherwise.} \end{cases} \tag{3}$$

We prove that A is a Monge matrix. Based on the Monge property of matrix $\underline{A}$ and the assumption $\underline{a}_{bu} = \underline{a}_{ct} = 0$ for all $b \leq m$, $c \geq q$ it follows that the values $\underline{a}_{ij}$ used in the matrix A are equal to 0.

The matrix A consists of three blocks. Block A_1 contains values a_{ij} for $i \leq m$, $j \leq u$, block A_2 contains values a_{ij} for $i \geq q$, $j \geq t$ and block A_3 includes the remaining cases.

To verify the Monge property of the matrix A we must consider all the possibilities of the selection of the indices i, j, k, l for $i < k$, $j < l$, and verify the following condition:

$$a_{ij} \otimes a_{kl} \leq a_{il} \otimes a_{kj}.$$

Let a_{ij} be in the block A_1. Hence, $a_{ij} = \underline{a}_{ij} = 0$ and it holds that $a_{ij} \otimes a_{kl} = 0$. Let a_{kl} be in the block A_2. Hence, $a_{kl} = \underline{a}_{kl} = 0$ and it holds that $a_{ij} \otimes a_{kl} = 0$. Let a_{ij}, a_{kl}, a_{il}, a_{kj} be in the block A_3. Based on the definition of the matrix A these values are the same as in the matrix $\overline{A}$ (see (3)), which is a Monge matrix.

It follows that A is a Monge matrix, and since it is not robust, $\mathbf{A}^M$ is not universally robust and this is a contrary to the assumption.

Case 2. Let us suppose (2) in 11.(iii) is false, i.e., none of the indices d, e exists, for which $\underline{a}_{te} = 1$ or $\underline{a}_{ud} = 1$, i.e. $\underline{a}_{te} = \underline{a}_{ud} = 0$ for all $d \leq m$, $e \geq q$. We define the matrix A as follows:

$$a_{ij} = \begin{cases} \underline{a}_{ij} & \text{if } i \leq u,\ j \leq m, \\ \underline{a}_{ij} & \text{if } i \geq t,\ j \geq q, \\ \overline{a}_{ij} & \text{otherwise.} \end{cases} \tag{4}$$

We prove the Monge property by analogy with the above case.

Case 3. Let us suppose (1) and (2) in 11.(*iii*) are false, i.e., none of the indices b, c exists, for which $\underline{a}_{bu} = 1$ or $\underline{a}_{ct} = 1$, i.e. $\underline{a}_{bu} = \underline{a}_{ct} = 0$ for all $b \leq m, c \geq q$ and at the same time, does not exist any of the indices d, e such that it holds that $\underline{a}_{te} = 1$ or $\underline{a}_{ud} = 1$, i.e., $\underline{a}_{te} = \underline{a}_{ud} = 0$ for all $d \leq m, e \geq q$. We define the matrix A as follows:

$$a_{ij} = \begin{cases} \underline{a}_{ij} & \text{if } i \leq m,\, j \leq u, \\ \underline{a}_{ij} & \text{if } i \geq q,\, j \geq t, \\ \underline{a}_{ij} & \text{if } i \leq u,\, j \leq m, \\ \underline{a}_{ij} & \text{if } i \geq t,\, j \geq q, \\ \overline{a}_{ij} & \text{otherwise.} \end{cases} \tag{5}$$

We prove that A is a Monge matrix. Based on the Monge property of matrix $\underline{A}$ and the assumption $\underline{a}_{bu} = \underline{a}_{ct} = 0$ for all $b \leq m, c \geq q$ it follows that the values $\underline{a}_{ij}$ used in the matrix A are equal to 0.

The matrix A consists of three blocks. Block A_1 contains values a_{ij} for $i \leq m, j \leq u$ or $i \leq u$, $j \leq m$, block A_2 contains values a_{ij} for $i \leq m, j \leq u$ or $i \geq t, j \geq q$ and block A_3 includes the remaining cases.

To verify the Monge property of the matrix A we must consider all the possibilities of the selection of indices i, j, k, l for $i < k, j < l$, and verify the following condition:

$$a_{ij} \otimes a_{kl} \leq a_{il} \otimes a_{kj}.$$

The verification is analogical to the previous two cases. It follows that A is a Monge matrix, and since it is not robust, $\mathbf{A}^M$ is not universally robust and this is a contrary to the assumption.

□

We demonstrate the necessary and sufficient condition for universal robustness in following examples.

Example 1. *Let* $\mathbf{A} = [\underline{A}, \overline{A}]$ *is given by matrices* $\underline{A} \in B(5,5)$ *and* $\overline{A} \in B(5,5)$*:*

$$\underline{A} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \overline{A} = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{pmatrix}.$$

Since $\underline{A}$ *and* $\overline{A}$ *are Monge matrices, they define an interval Monge matrix* $\mathbf{A}^M$*. It is easy to verify that* $\underline{A}$ *is a non-trivial matrix (see Figure 1).*

We verify, the universal robustness of $\mathbf{A}^M$ *using Theorem 11:*

(*i*) $G(\overline{A}, 1)$ *contains exactly one non-trivial strongly connected component* $\overline{\mathcal{K}} = (N_{\overline{\mathcal{K}}}, E_{\overline{\mathcal{K}}})$ *with* $N_{\overline{\mathcal{K}}} = \{2, 3, 4, 5\}$*, which has a loop, see Figure 2.* ✓

(*ii*) $G(\underline{A}, 1)$ *contains exactly one non-trivial strongly connected component* $\underline{\mathcal{K}} = (N_{\underline{\mathcal{K}}}, E_{\underline{\mathcal{K}}})$ *with* $N_{\underline{\mathcal{K}}} = \{3, 4\}$*, which has a loop, see Figure 1.* ✓

(*iii*) *For* $t = \min\{N_{\underline{\mathcal{K}}}\} = 3$*,* $u = \max\{N_{\underline{\mathcal{K}}}\} = 4$ *there exists exactly one cycle* $s = (m, q, m) = (2, 5, 2)$ *in* $\overline{\mathcal{K}}$*, for which* $m < \min\{N_{\underline{\mathcal{K}}}\} = 3$*,* $q > \max\{N_{\underline{\mathcal{K}}}\} = 4$ *and* $m, q \notin N_{\underline{\mathcal{K}}}$*. There exist* $b = 1 < m$ *and* $d = 2 = m$ *such that the conditions (1), (2) hold:*

- $\underline{a}_{bu} = \underline{a}_{14} = 1,$ ✓
- $\underline{a}_{ud} = \underline{a}_{42} = 1.$ ✓

Let $A \in \boldsymbol{A}^M$. We show that if $G(A,1)$ contains the cycle $s = (m,q,m)$, then it is included in the unique non-trivial strongly connected component $\mathcal{K}$ for which $\underline{\mathcal{K}} \subset \mathcal{K}$. Based on the Monge property of the matrix A:

$$1 = \underline{a}_{14} \otimes \underline{a}_{25} \leq a_{14} \otimes a_{25} \leq a_{15} \otimes a_{24}.$$

Hence, $a_{24} = 1$. In addition, $1 = \underline{a}_{42} \leq a_{42}$ holds, thus $a_{42} = 1$. Hence, the corresponding edges $(2,4)$ and $(4,2)$ connect the cycle s to the component $\underline{\mathcal{K}}$.

Consequently, the digraph $G(A,1)$ contains exactly one non-trivial strongly connected component and this has a loop. Hence, by Theorem 5 the matrix A is robust, i.e., $\boldsymbol{A}^M$ is universally robust.

(✓ means that the condition is met.)

A slight modification of the previous example gives a negative answer.

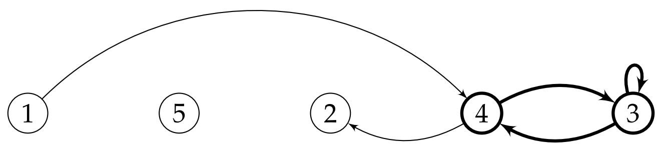

Figure 1. The threshold digraph $G(\underline{A},1)$.

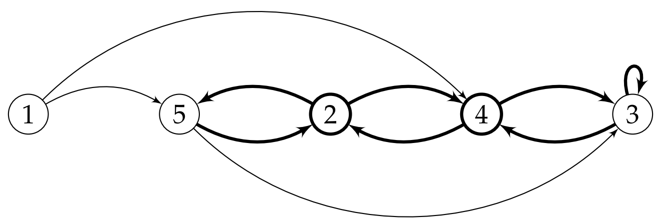

Figure 2. The threshold digraph $G(\overline{A},1)$.

Example 2. *Let $\mathbf{A} = [\underline{A}, \overline{A}]$ is given by matrices $\underline{A} \in B(5,5)$ and $\overline{A} \in B(5,5)$:*

$$\underline{A} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \overline{A} = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{pmatrix}.$$

In contrast to the previous example $\underline{a}_{14} = 0$, $\underline{a}_{15} = 1$ and $\underline{a}_{42} = 0$ while the rest of the entries in $\underline{A}$ is the same. Since the matrix $\overline{A}$ is without any changes, $G(\overline{A},1)$ is the same as in the Example 1, see Figure 2.
We verify, whether the matrix $\boldsymbol{A}^M$ is universally robust:

- (i) *$G(\overline{A},1)$ contains exactly one non-trivial strongly connected component $\overline{\mathcal{K}} = (N_{\overline{\mathcal{K}}}, E_{\overline{\mathcal{K}}})$ with $N_{\overline{\mathcal{K}}} = \{2,3,4,5\}$, which has a loop, see Figure 2* ✓
- (ii) *$G(\underline{A},1)$ contains exactly one non-trivial strongly connected component $\underline{\mathcal{K}} = (N_{\underline{\mathcal{K}}}, E_{\underline{\mathcal{K}}})$ with $N_{\underline{\mathcal{K}}} = \{3,4\}$, which has a loop, see Figure 3.* ✓
- (iii) *For $t = \min\{N_{\underline{\mathcal{K}}}\} = 3$, $u = \max\{N_{\underline{\mathcal{K}}}\} = 4$ there exists exactly one cycle $s = (m,q,m) = (2,5,2)$ in $\overline{\mathcal{K}}$, for which holds $m < \min\{N_{\underline{\mathcal{K}}}\} = 3$, $q > \max\{N_{\underline{\mathcal{K}}}\} = 4$ and $m, q \notin N_{\overline{\mathcal{K}}}$.*

Since for all indices $b \leq 2$ and $c \geq 5$ holds $\underline{a}_{bu} = \underline{a}_{ct} = 0$, the condition (iii) of the Theorem 11 does not hold. Thus, there exists $A \in \boldsymbol{A}^M$, whose digraph $G(A,1)$ contains along with the non-trivial strongly *connected component* $\underline{K}$ *also the cycle* $(2,5,2)$, *i.e.,* $G(A,1)$ *contains two non-trivial strongly connected components. We construct the matrix A according to proof of the Theorem 11 (see (3)):*

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

As shown on the Figure 4 there exists the edge $(2,4)$, *which connects the cycle* $(2,5,2)$ *with the non-trivial strongly connected component* $\underline{K}$; *however there is no edge, which could connect* $\underline{K}$ *with the cycle* $(2,5,2)$. *According to Theorem 5 the matrix A is not robust, thus by Definition 6 the matrix* $\mathbf{A}^M$ *is not universally robust. (✓ means that the condition is met, ✗ means that the condition is not met.)*

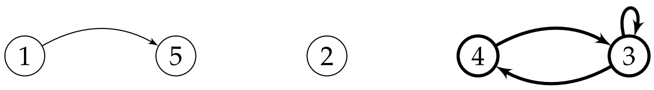

Figure 3. The threshold digraph $G(\underline{A},1)$.

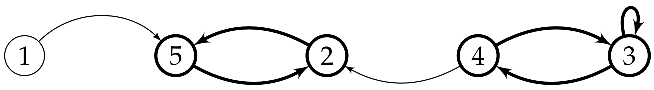

Figure 4. The threshold digraph $G(A,1)$—contains two strongly connected components.

5. Conclusions

The aim of this paper is to investigate the possible and universal robustness of Monge matrices with inexact data in binary case of max–min algebra. We proved necessary condition for universal robustness, sufficient condition for possible robustness and sufficient condition for universal robustness under assumption that $\underline{A}$ is trivial. Moreover, we found necessary conditions for possible robustness and equivalent conditions for universal robustness under assumption that $\underline{A}$ is non-trivial. A future work will focus on finding equivalent conditions for possible and universal robustness of interval Monge matrices in case, if the lower bound matrix $\underline{A}$ is trivial and the upper bound matrix $\overline{A}$ is non-trivial. Another possible extension is to find sufficient conditions for possible robustness in case, if both matrices $\underline{A}$ and $\overline{A}$ are non-trivial. It will be necessary to find further properties of the structure of threshold digraphs corresponding to a Monge matrix to be able to solve the problems connected especially with possible robustness. To find a Monge matrix in an interval Monge matrix which is robust is namely the crucial point to find necessary and sufficient condition for such a matrix with inexact data to be possibly robust. Afterwards, equivalent conditions for universal robustness and equivalent conditions for possible robustness of an interval Monge matrix could be generalized in max–min algebra without restriction to binary case.

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Abbreviations

The following symbols and abbreviations are used in this manuscript:

gcd	greatest common divisor
lcm	least common multiple
$\oplus$	operation of maximum
$\otimes$	operation of minimum
$\mathbb{R}$	set of all finite real numbers
$\mathbb{B}(n)$	set of all n-dimensional column vectors
$\mathbb{B}(m,n)$	set of all matrices of type $m \times n$
per	period
$G(A)$	digraph associated with matrix A
$G(A,h)$	threshold digraph associated with matrix A
$\text{SCC}^*(G(A,h))$	set of all non-trivial strongly connected components of $G(A,h)$

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