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Boundary Value Problems for a Class of First-Order Fuzzy Delay Differential Equations

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Abstract: In this paper, we study a class of fuzzy differential equations with variable boundary value conditions. Applying the upper and lower solutions method and the monotone iterative technique, we provide some sufficient conditions for the existence of solutions, which can be applied to discuss some dynamical models in biology and economics.

Keywords: fuzzy delay differential equations; boundary value problem; upper and lower solutions; monotone iterative technique

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1. Introduction

Delay differential equations are frequently used to study system models in biology, economics, physics, engineering and so on [1]. However, deterministic differential equations cannot be applied to systems with uncertainty. For example, if the dynamical system depends upon a subjective decision, state variables or parameters will be inaccuracies [2]. To consider these issues in mathematical models, we might need to utilize the aspect of fuzziness.

Fuzzy differential equations have been studied frequently during the last few years. We can find many papers concerned with the existence of solutions for fuzzy differential equations. With the strongly generalized differentiability concept introduced in [3,4], B. Bede et al. presented a variation of constants formulas for first-order, linear, fuzzy differential equations in [5]. Based on these results, some new theorems about the existence of fuzzy differential equations were obtained. We refer to [6–9] and the references therein.

The upper and lower solutions method is considered an important way to study fuzzy differential equations. For example, R. Rodríguez-López et al. discussed initial value problems of fuzzy differential equations with the upper and lower solutions method in [10,11], R. Alikhani and F. Bahrami [9] discussed a first-order, nonlinear, fuzzy integro-differential equation by using the upper and lower solutions method.

We can also find some new results about fuzzy delay differential equations. A. Khastan et al. studied the existence of solutions to an initial value problem of the fuzzy delay differential equation in [12]. Some other results can be found in [13–15]. The methods involved in these papers are fixed point theories, the variation of constants method and the upper and lower solutions method.

In [7,16,17], J. J. Nieto and R. Rodríguez-López studied boundary value problems of fuzzy differential equations. For example, the authors considered periodic boundary value conditions in [7]

$$\begin{cases} u'(t)+a(t)u(t)=\sigma(t), t\in[0,T]\setminus\{t_1,t_2,\cdots,t_m\}, \\ u(t_k^+)=(1-L_k)u(t_k)+d_k, k=1,2,\cdots,m, \\ \lambda u(0)=u(T)+b, \end{cases} \tag{1}$$

where $\{t_1, t_2, \cdots, t_m\} \in [0, T]$, a, σ are piecewise continuous on $[0, T]$. The authors changed the fuzzy differential Equation (1) into two crisp ordinary differential equations, which were deduced from the level set equations of Equation (1). A similar method also had been used to study periodic boundary value problems in [6,18,19].

In general, there are few papers discussing fuzzy delay differential equations, especially the boundary value problems of fuzzy delay differential equations. In our previous work [20], we studied boundary value problems $x'(t) = f(t, x(t))$, $x(0) = \alpha x(T)$, where $\alpha \in \mathbb{R} \setminus \{0, \pm 1\}$. From then on, we can find few papers on the topics of fuzzy differential equations with variable parameters in the boundary value conditions. Motivated by [12–15], we will extend the results in our previous work [20] to the fuzzy delay differential equation.

In the present paper, we consider

$$\begin{cases} x'(t) = f(t, x(\lambda(t))), t \in [0, T], \\ x(0) = \alpha x(T), \end{cases} \tag{2}$$

where $f \in C([0, T] \times R_F, R_F)$, $\lambda \in C([0, T], [0, T])$, $\alpha \in \mathbb{R} \setminus \{0, \pm 1\}$. Applying the upper and lower solutions method and the monotone iterative technique, we provide some sufficient conditions for the existence of maximal and minimal solutions to (2). The results can be applied to discuss some dynamical models in biology. At the end of this paper, we provide an example to verify our results.

2. Preliminaries

Let R_F be the class of fuzzy subsets of the real axis, $u : \mathbb{R} \to [0, 1]$, which satisfies:

(i) $\exists t_0 \in \mathbb{R}, u(t_0) = 1$ $(normality)$;
(ii) $\forall s \in [0, 1]$ and $t_1, t_2 \in \mathbb{R}, u(st_1 + (1 - s)t_2) \geq \min\{u(t_1), u(t_2)\}$ $(convex\ fuzzy)$;
(iii) u is upper semi-continuous on $\mathbb{R}$;
(iv) The closure of $\{t \in \mathbb{R} | u(t) > 0\}$ is compact.

Let $[u]^r = \{t \in \mathbb{R} | u(t) \geq r\}$ for $r \in (0, 1]$, $[u]^0 = \overline{\{t \in \mathbb{R} | u(t) > 0\}}$, where $\overline{A}$ means the closure of A. $[u]^r$ is also written as $[u^r_-, u^r_+]$.

For every $u, v \in R_F$, $D(u, v) = \sup_{r \in [0,1]} \max\{|u^r_- - v^r_-|, |u^r_+ - v^r_+|\}$ is known as a Hausdorff distance on R_F and (R_F, D) is a complete metric space.

Lemma 1 ([5]). *Let $u, v, w, e \in R_F$, $D(u + v, w + e) \leq D(u, w) + D(v, e)$.*

If $f : [a, b] \to R_F$ is continuous, then f is bounded; that is, there exists $M > 0$ such that $D(f(t), \tilde{0}) \leq M$ for every $t \in [a, b]$. Let $C([a, b], R_F)$ be the set of continuous functions on $[a, b]$. For every $f, g \in C([a, b], R_F)$, we set $d(f, g) = \max_{t \in [a,b]} D(f(t), g(t))$, then $(C([a, b], R_F), d)$ is a complete metric space.

Definition 1 ([21]). *For every $u, v \in R_F$, if there exists $z \in R_F$ such that $u = v + z$, then z is said to be the H-difference of u and v. We denote $z = u \ominus v$.*

Definition 2 ([3,4]). *A function $f : [a, b] \to R_F$ is a strongly generalized differentiable at $t \in (a, b)$; if there exists $f'(t) \in R_F$ such that $\forall h > 0$ sufficiently small, H-difference and limits in the following formulas exist with metric D:*

(i) $\lim_{h \to 0+} \frac{f(t+h) \ominus f(t)}{h} = \lim_{h \to 0+} \frac{f(t) \ominus f(t-h)}{h} = f'(t)$

or

(ii) $\lim_{h \to 0+} \frac{f(t) \ominus f(t+h)}{(-h)} = \lim_{h \to 0+} \frac{f(t-h) \ominus f(t)}{(-h)} = f'(t)$

or

(iii) $\lim\limits_{h\to 0+} \frac{f(t+h)\ominus f(t)}{h} = \lim\limits_{h\to 0+} \frac{f(t-h)\ominus f(t)}{(-h)} = f'(t)$

or

(iv) $\lim\limits_{h\to 0+} \frac{f(t)\ominus f(t+h)}{(-h)} = \lim\limits_{h\to 0+} \frac{f(t)\ominus f(t-h)}{h} = f'(t).$

Lemma 2 ([22,23]). *Suppose that $f, g \in C([a,b], R_F)$.*

(i) $[\int_a^b f(t)dt]^r = [\int_a^b [f(t)]_-^r dt, \int_a^b [f(t)]_+^r dt], r \in [0,1]$.

(ii) $F(t) = \int_a^t f(\theta)d\theta$ *is differentiable as in the Definition 2(i) and* $F'(t) = f(t)$.

(iii) $D\left(\int_a^b f(\theta)d\theta, \int_a^b g(\theta)d\theta\right) \leq \int_a^b D(f(\theta), g(\theta))d\theta$.

Lemma 3. *Let $f \in C([a,b], R_F)$, denote $[f(t)]^r = [f_{r-}(t), f_{r+}(t)]$.*

(i) *Suppose that f is differentiable as in the Definition 2(i); then f_{r-}, f_{r+} are differentiable and* $[f'(t)]^r = [f'_{r-}(t), f'_{r+}(t)]$.

(ii) *Suppose that f is differentiable as in the Definition 2(ii); then f_{r-}, f_{r+} are differentiable and* $[f'(t)]^r = [f'_{r+}(t), f'_{r-}(t)]$.

Proof. (i) is from Theorem 2.5.2 in [8]. The proof of (ii) is similar; here we omit it. □

Now, we define an ordering relation in R_F. $\forall u, v \in R_F$, we say $v \leq u$ if $v_-^r \leq u_-^r$ and $v_+^r \leq u_+^r$ for all $r \in [0,1]$. If $u, v \in R_F$ and $v \leq u$, we denote $[v,u] = \{z \in R_F | v \leq z \leq u\}$. Similarly, let $f, g \in C([a,b], R_F)$; we say $f \leq g$ if $f(t) \leq g(t), t \in [a,b]$. If $f \leq g$, we denote $[f,g] = \{x \in C([a,b], R_F) | \forall t \in [a,b], f(t) \leq x(t) \leq g(t)\}$.

Lemma 4 ([9,10,24]). *Let $u, v, z, w \in R_F$, $f, g \in C([a,b], R_F)$.*

(i) *If $v \leq u, z \leq w$, then $v + z \leq u + w$;*

(ii) *If $v \leq u$, then $cv \leq cu$ for $c \in (0, +\infty)$ and $cu \leq cv$ for $c \in (-\infty, 0)$;*

(iii) *If $f \leq g$, then $\int_a^t f(\theta)d\theta \leq \int_a^t g(\theta)d\theta, t \in [a,b]$;*

(iv) *If $f \leq g$ and $u \ominus (-1)\int_a^t f(\theta)d\theta$, $u \ominus (-1)\int_a^t g(\theta)d\theta$ exist for $t \in [a,b]$, then $u \ominus (-1)\int_a^t f(\theta)d\theta \leq u \ominus (-1)\int_a^t g(\theta)d\theta$;*

(v) *If $\{f_n\} \subset C([a,b], R_F)$, $f_n \leq g$ and $f_n \to f^*$, then $f^* \leq g$;*

(vi) *If $\{f_n\} \subset C([a,b], R_F)$, $f_n \leq f_{n+1}$ and there exists a subsequence $\{f_{n_k}\} \subseteq \{f_n\}$ satisfying $f_{n_k} \to f^*$, then $f_n \to f^*$.*

For every $f \in C([a,b], R_F)$ and $r \in [0,1]$, let $[m_f]^r = \left[\min\limits_{t\in[a,b]} [f(t)]_-^r, \min\limits_{t\in[a,b]} [f(t)]_-^1\right]$, $[M_f]^r = \left[\max\limits_{t\in[a,b]} [f(t)]_+^1, \max\limits_{t\in[a,b]} [f(t)]_+^r\right]$. By directly calculation, we can check that $m_f, M_f \in R_F$ and $m_f \leq M_f$.

Definition 3. *Let $x \in C([0,T], R_F)$. x is said to be (i) or (ii)-differentiable, if x is strongly generalized differentiable as in the Definition 2(i) or (ii).*

Lemma 5 ([20]). *Let $f \in C([0,T], R_F)$ be nontrivial; that is, $f_{r-}(t) \not\equiv f_{r+}(t)$ for $t \in [0,T]$ and $r \in [0,1]$.*

(i) *Boundary value problem $x'(t) = f(t)$, $x(0) = \alpha x(T)$ has (i)-differentiable solution only if $\alpha \in (-1,1)$ and the solution can be written as*

$$x(t) = \begin{cases} \frac{\alpha^2}{1-\alpha^2}\int_0^T f(\theta)d\theta + \frac{\alpha}{1-\alpha^2}\int_0^T f(\theta)d\theta + \int_0^t f(\theta)d\theta, \alpha \in (-1,0], \\ \frac{\alpha}{1-\alpha}\int_0^T f(\theta)d\theta + \int_0^t f(\theta)d\theta, \alpha \in (0,1). \end{cases} \tag{3}$$

(ii) *Boundary value problem $x'(t) = f(t)$, $x(0) = \alpha x(T)$ has (ii)-differentiable solution only if $\alpha \in (-\infty, -1) \cup (1, +\infty)$ and the solution can be written as*

$$x(t) = \begin{cases} \frac{\alpha^2}{1-\alpha^2} \int_0^T f(\theta)d\theta + \frac{\alpha}{1-\alpha^2} \int_0^T f(\theta)d\theta \ominus \int_0^t (-f(\theta))d\theta, \alpha \in (-\infty, -1), \\ \frac{\alpha}{1-\alpha} \int_0^T f(\theta)d\theta \ominus \int_0^t (-f(\theta))d\theta, \alpha \in (1, +\infty). \end{cases} \tag{4}$$

3. Existence of Solutions to $x'(t) = f(t, x(\lambda(t))), x(0) = \alpha x(T)$

We consider sufficient conditions for the existence of solutions to (2) in this section.

Definition 4. *Let $v, u \in C([0, T], R_F)$ and $v \leq u$. x is said to be a maximal(minimal) solution to (2) in $[v, u]$, if x satisfies (2) and any other solution y to (2) in $[v, u]$ satisfies $y \leq x (y \geq x)$.*

Lemma 6. *Suppose that $g \in C^1([0, T], \mathbb{R})$, $\alpha \in (0, 1)$.*

(i) *If $g(0) \geq \alpha g(T)$ and $\forall t \in (0, T), g'(t) \geq 0$, then $g(t) \geq 0$ for all $t \in [0, T]$;*
(ii) *If $g(0) \leq \alpha g(T)$ and $\forall t \in (0, T), g'(t) \leq 0$, then $g(t) \leq 0$ for all $t \in [0, T]$.*

Proof. To prove (i), we firstly assume that $g(t) < 0$ for all $t \in [0, T]$, then $g(0) \geq \alpha g(T) > g(T)$. This is contrary to $g'(t) \geq 0$.

Suppose that $g(T) < 0$ and there exists $t_1 \in [0, T)$ such that $g(t_1) \geq 0$. This is also contrary to $g'(t) \geq 0$. On the other hand, $g(T) \geq 0$ provides $g(0) \geq \alpha g(T) \geq 0$. Then by $g'(t) \geq 0$, we have $g(t) \geq g(0) \geq 0$ for all $t \in [0, T]$.

To prove (ii), let $G = -g$; then G satisfies all conditions in (i). As a result, $g(t) \geq 0$ for all $t \in [0, T]$, that is, $g(t) \leq 0, t \in [0, T]$. □

Lemma 7. *Suppose that $g \in C^1([0, T], \mathbb{R})$, $\alpha \in (1, +\infty)$.*

(i) *If $g(0) \leq \alpha g(T)$ and $\forall t \in (0, T), g'(t) \leq 0$, then $g(t) \geq 0$ for all $t \in [0, T]$;*
(ii) *If $g(0) \geq \alpha g(T)$ and $\forall t \in (0, T), g'(t) \geq 0$, then $g(t) \leq 0$ for all $t \in [0, T]$.*

Proof. (i) If $g(T) < 0$, then $g(0) \leq \alpha g(T) < g(T)$, it is contrary to $g'(t) \leq 0$ on $(0, T)$. By $g(T) \geq 0$ and $g'(t) \leq 0$ on $[0, T]$, we have $g(t) \geq g(T) \geq 0, t \in [0, T]$.

(ii) Let $G(t) = -g(t)$. (i) provides $G(t) \geq 0$ on $[0, T]$; that is, $g(t) \leq 0$ on $[0, T]$. Therefore, (ii) is also true. □

Theorem 1. *Suppose that $\alpha \in (0, 1)$, $f \in C([0, T] \times R_F, R_F)$.*

(i) *There exist (i)-differentiable functions $u, v \in C([0, T], R_F)$ satisfying $v \leq u$ and*

$$u'(t) \geq f(t, u(\lambda(t))), u(0) \geq \alpha u(T), \tag{5}$$

$$v'(t) \leq f(t, v(\lambda(t))), v(0) \leq \alpha v(T). \tag{6}$$

(ii) *$\forall t \in [0, T]$, $f(t, \cdot)$ is nondecreasing in $[m_v, M_u]$ and satisfies*

$$D(f(t, x), f(t, y)) \leq \varphi(t) \cdot D(x, y),\ x, y \in [m_v, M_u], \tag{7}$$

where $\varphi \in C([0, T], [0, \infty))$ and $\int_0^T \varphi(\theta)d\theta < 1 - \alpha$.

Then there exist maximal and minimal (i)-differentiable solutions to (2) in $[v, u]$.

Proof. Referring to (3), we denote

$$Ax(t) = \frac{\alpha}{1-\alpha}\int_0^T f(\theta, x(\lambda(\theta)))d\theta + \int_0^t f(\theta, x(\lambda(\theta)))d\theta.$$

Apparently, $\forall x \in C([0,T], R_F)$, Ax is (i)-differentiable and $Ax(0) = \alpha Ax(T)$. By Lemma 5(i), any $x \in C([0,T], R_F)$ satisfying $Ax = x$ is also a (i)-differentiable solution to (2).

Here we claim that A is nondecreasing in the interval $[v, u]$. In fact, let $x, y \in [v, u]$ and $x \leq y$, nondecreasing property of f provides that $f(t, x(\lambda(t))) \leq f(t, y(\lambda(t)))$, $\forall t \in [0, T]$. We can conclude from Lemma 6(v) that

$$\int_0^t f(\theta, x(\lambda(\theta)))d\theta \leq \int_0^t f(\theta, y(\lambda(\theta)))d\theta, t \in [0, T].$$

Together with Lemma 4(i), we obtain that $Ax(t) \leq Ay(t), t \in [0, T]$. Consequently, A is nondecreasing in $[v, u]$.

Now we demonstrate that $v \leq Av \leq Au \leq u$. $Av \leq Au$ can be deduced directly by nondecreasing property of A, we just need to prove $v \leq Av$ and $Au \leq u$.

By (6) and the definition of A, we have

$$v'(t) \leq f(t, v(\lambda(t))) = (Av)'(t), \tag{8}$$

$$v(0) \leq \alpha v(T), \ Av(0) = \alpha Av(T). \tag{9}$$

Lemma 3(i) and (8) imply that $\forall r \in [0, 1]$,

$$v'_{r-}(t) \leq (Av)'_{r-}(t), \ v'_{r+}(t) \leq (Av)'_{r+}(t). \tag{10}$$

On the other hand, (9) implies that

$$v_{r-}(0) \leq \alpha v_{r-}(T), \ v_{r+}(0) \leq \alpha v_{r+}(T), \tag{11}$$

$$(Av)_{r-}(0) = \alpha(Av)_{r-}(T), \ (Av)_{r+}(0) = \alpha(Av)_{r+}(T). \tag{12}$$

Let $p(t) = (Av)_{r-}(t) - v_{r-}(t), q(t) = (Av)_{r+}(t) - v_{r+}(t)$. (10), (11) and (12) provide that

$$p'(t) \geq 0, p(0) \geq \alpha p(T), \tag{13}$$

$$q'(t) \geq 0, q(0) \geq \alpha q(T). \tag{14}$$

By Lemma 5(i), $\forall t \in [0, T]$, $p(t) \geq 0, q(t) \geq 0$; that is, $Av \geq v$. $Au \leq u$ can be proven with the analogous method.

Now we turn to consider sequences $\{A^n v\}$ and $\{A^n u\}$. By nondecreasing property of A and $v \leq Av$, $Au \leq u$, we have

$$v \leq Av \leq A^2 v \leq \cdots \leq A^n v \leq \cdots \leq A^n u \leq \cdots \leq A^2 u \leq u.$$

$\forall t \in [0, T]$ and $m > n > 1$, Lemma 1, Lemma 2(iii) and (7) imply that

$$\begin{aligned} D(A^n v(t), A^m v(t)) &\leq \tfrac{1}{1-\alpha}\int_0^T D(f(\theta, A^{n-1}v(\lambda(\theta))), f(\theta, A^{m-1}v(\lambda(\theta))))d\theta \\ &\leq \tfrac{1}{1-\alpha}\int_0^T \varphi(\theta)d\theta \cdot d(A^{n-1}v, A^{m-1}v) \\ &\leq \left[\tfrac{1}{1-\alpha}\int_0^T \varphi(\theta)d\theta\right]^n \cdot d(v, A^{m-n}v) \\ &\leq \left[\tfrac{1}{1-\alpha}\int_0^T \varphi(\theta)d\theta\right]^n \cdot d(v, u). \end{aligned}$$

That is, $\{A^n v\}$ is a Cauchy sequence. According to the completeness of $C([0,T], R_F)$, $\{A^n v\}$ is convergent; that is, there exists $v^* \in C([0,T], R_F)$ such that $\{A^n v\}$ converges uniformly to v^*. Easily we can check that v^* satisfies $Av^* = v^*$. As a result, v^* is a (i)-differentiable solution to (2). An analogous result can be obtained for $\{A^n u\}$.

Suppose that $A^n u \to u^*$ and $x \in [v, u]$ is also a (i)-differentiable solution to (2); that is, $Ax = x$. Applying nondecreasing property of A, we have $A^n v \leq x \leq A^n u$ for $n \geq 1$. Lemma 4(v) implies $v^* \leq x \leq u^*$. Consequently, v^*, u^* are minimal and maximal (i)-differentiable solutions for (2) in the interval $[v, u]$. □

Theorem 2. *Suppose that $\alpha \in (1, +\infty)$, $f \in C([0,T] \times R_F, R_F)$.*

(i) There exist (ii)-differentiable functions $u, v \in C([0,T], R_F)$ satisfying $v \leq u$,

$$u'(t) \leq f(t, u(\lambda(t))),\ u(0) \leq \alpha u(T); \tag{15}$$

$$v'(t) \geq f(t, v(\lambda(t))),\ v(0) \geq \alpha v(T). \tag{16}$$

(ii) $\forall t \in [0,T]$, $f(t, \cdot)$ is decreasing on $[m_v, M_u]$ and satisfies

$$D(f(t,x), f(t,y)) \leq \varphi(t) \cdot D(x,y),\ x, y \in [m_v, M_u], \tag{17}$$

where $\varphi \in C([0,T], [0,\infty)$ and $\alpha \int_0^T \varphi(\theta) d\theta < \alpha - 1$.

Then there exist maximal and minimal (ii)-differentiable solutions for (2) in $[v, u]$.

Proof. Referring to (4), we denote

$$\begin{aligned} Bx(t) &= \tfrac{\alpha}{1-\alpha} \int_0^T f(\theta, x(\lambda(\theta)))d\theta \ominus \int_0^t (-1) f(\theta, x(\lambda(\theta)))d\theta \\ &= \tfrac{1}{\alpha-1} \int_0^t (-1) f(\theta, x(\lambda(\theta)))d\theta + \tfrac{\alpha}{\alpha-1} \int_t^T (-1) f(\theta, x(\lambda(\theta)))d\theta. \end{aligned}$$

For every $x \in C([0,T], R_F)$, Bx is (ii)-differentiable and $Bx(0) = \alpha Bx(T)$. Lemma 5(ii) implies that any $x \in C([0,T], R_F)$ satisfying $Bx = x$ is a (ii)-differentiable solution to (2). Here we prove that B is nondecreasing in the interval $[v, u]$.

Let $x, y \in [v, u]$ and $x \leq y$. Lemma 4(ii) and the decreasing property of f provide that $(-1)f(t, x(\lambda(t))) \leq (-1)f(t, y(\lambda(t)))$ for every $t \in [0,T]$. By Lemma 4(iii) and (iv), we have $Bx(t) \leq By(t), t \in [0,T]$. That is, B is nondecreasing on $[v, u]$.

Now, we prove $v \leq Bv$, $Bu \leq u$. By (16), we have

$$v'(t) \geq f(t, v(\lambda(t))) = (Bv)'(t), \tag{18}$$

$$v(0) \geq \alpha v(T),\ Bv(0) = \alpha Bv(T). \tag{19}$$

Then Lemma 3(ii) and (18) imply that

$$v'_{r-}(t) \geq (Bv)'_{r-}(t),\ v'_{r+}(t) \geq (Bv)'_{r+}(t),\ r \in [0,1]. \tag{20}$$

Hence, (19), (20) and Lemma 7(ii) imply that $v \leq Bv$. Similarly, we can also prove that $Bu \leq u$.

Applying the same method in the proof of Theorem 1, we can prove that there exist maximal and minimal (ii)-differentiable solutions to (2) in the interval $[v, u]$. □

If $\alpha \in (-\infty, 0)$, the upper and lower solutions method can not be applied directly. In fact, referring to (3) and (4), the corresponding integral operators for $\alpha \in (-1, 0)$ and $\alpha \in (-\infty, -1)$ can be written as

$$\begin{aligned} Fx(t) = & \ \tfrac{\alpha^2}{1-\alpha^2} \int_0^T f(\theta, x(\lambda(\theta)))d\theta + \tfrac{\alpha}{1-\alpha^2} \int_0^T f(\theta, x(\lambda(\theta)))d\theta \\ & + \int_0^t f(\theta, x(\lambda(\theta)))d\theta, \\ Hx(t) = & \ \tfrac{\alpha^2}{1-\alpha^2} \int_0^T f(\theta, x(\lambda(\theta)))d\theta + \tfrac{\alpha}{1-\alpha^2} \int_0^T f(\theta, x(\lambda(\theta)))d\theta \\ & \ominus \int_0^t (-f(\theta, x(\lambda(\theta))))d\theta. \end{aligned}$$

The nondecreasing or nonincreasing properties of f are not enough to guarantee the monotonicity of F and H; we need more hypotheses to discuss the existence of solution to (2).

$\forall x, y \in C([0, T], R_F)$ and $t \in [0, T]$, we denote

$$\begin{aligned} \tilde{F}(x, y)(t) = & \ \tfrac{\alpha^2}{1-\alpha^2} \int_0^T f(\theta, x(\lambda(\theta)))d\theta + \tfrac{\alpha}{1-\alpha^2} \int_0^T f(\theta, y(\lambda(\theta)))d\theta \\ & + \int_0^t f(\theta, x(\lambda(\theta)))d\theta. \end{aligned}$$

Theorem 3. *Suppose that* $\alpha \in (-1, 0)$, $f \in C([0, T] \times R_F, R_F)$.

(i) *There exist* $v, u \in C([0, T], R_F)$ *satisfying* $v \le u$ *and* $\tilde{F}(v, u), \tilde{F}(u, v) \in [v, u]$.
(ii) $\forall t \in [0, T]$, $f(t, \cdot)$ *is nondecreasing or nonincreasing on* $[m_v, M_u]$ *and satisfies*

$$D(f(t, x), f(t, y)) \le \varphi(t) \cdot D(x, y), \ x, y \in [m_v, M_u], \tag{21}$$

where $\varphi \in C([0, T], [0, \infty)$ *and* $\int_0^T \varphi(\theta)d\theta < 1 + \alpha$.

Then there exists at least one (i)-differentiable solution for (2) in $[v, u]$.

Proof. $\forall x \in C([0, T], R_F)$, Fx is (i)-differentiable and $Fx(0) = \alpha Fx(T)$. Moreover, Theorem 2 implies that any $x \in C([0, T], R_F)$ satisfying $Fx = x$ is also a (i)-differentiable solution to (2). We will prove that F is contraction mapping on $[v, u]$ and $F[v, u] \subseteq [v, u]$.

For every $t \in [0, T]$ and $x, y \in [v, u]$, (21) provides that

$$\begin{aligned} D(Fx(t), Fy(t)) & \le \tfrac{1}{1+\alpha} \int_0^T D(f(\theta, x(\lambda(\theta))), f(\theta, y(\lambda(\theta))))d\theta \\ & \le \tfrac{1}{1+\alpha} \int_0^T \varphi(\theta)d\theta \cdot d(x, y). \end{aligned}$$

Condition (ii) implies that F is contraction mapping on $[v, u]$.

On the other hand, $F[v, u] \subseteq [\tilde{F}(v, u), \tilde{F}(u, v)]$ if $f(t, \cdot)$ is nondecreasing for all $t \in [0, T]$; $F[v, u] \subseteq [\tilde{F}(u, v), \tilde{F}(v, u)]$ if $f(t, \cdot)$ is nonincreasing for all $t \in [0, T]$. Condition (i) guarantees that $F[v, u] \subseteq [v, u]$.

By Banach contraction mapping principle, there exists at least one (i)-differentiable solution for (2) in the interval $[v, u]$. □

Let $x, y \in C([0, T], R_F)$ and $t \in [0, T]$; we denote

$$\begin{aligned} \tilde{H}(x, y)(t) = & \ \tfrac{\alpha^2}{1-\alpha^2} \int_0^T f(\theta, y(\lambda(\theta)))d\theta + \tfrac{\alpha}{1-\alpha^2} \int_0^T f(\theta, x(\lambda(\theta)))d\theta \\ & \ominus \int_0^t (-f(\theta, y(\lambda(\theta))))d\theta. \end{aligned}$$

Theorem 4. *Suppose that* $\alpha \in (-\infty, -1)$, $f \in C([0, T] \times R_F, R_F)$.

(i) *There exist* $v, u \in C([0, T], R_F)$ *satisfying* $v \le u$ *and* $\tilde{H}(v, u), \tilde{H}(u, v) \in [v, u]$.
(ii) $\forall t \in [0, T]$, $f(t, \cdot)$ *is nondecreasing or nonincreasing on* $[m_v, M_u]$ *and satisfies*

$$D(f(t, x), f(t, y)) \le \varphi(t) \cdot D(x, y), \ x, y \in [m_v, M_u],$$

where $\varphi \in C([0, T], [0, \infty)$ *and* $\int_0^T \varphi(\theta)d\theta < 1 + \frac{1}{\alpha}$.

Then there exists at least one (ii)-differentiable solution to (2) in $[v, u]$.

Proof. For every $x, y \in C([0,T], R_F)$ and $t \in [0,T]$, we can check that

$$\begin{aligned}\tilde{H}(x,y)(t) = \ & \frac{1}{1-\alpha^2}\int_0^t f(\theta, y(\lambda(\theta)))d\theta + \frac{\alpha^2}{1-\alpha^2}\int_t^T f(\theta, y(\lambda(\theta)))d\theta \\ & + \frac{\alpha}{1-\alpha^2}\int_0^T f(\theta, x(\lambda(\theta)))d\theta.\end{aligned}$$

Consequently, $H[v,u] \subseteq [\tilde{H}(v,u), \tilde{H}(u,v)]$ if $f(t,\cdot)$ is nondecreasing for all $t \in [0,T]$, $H[v,u] \subseteq [\tilde{H}(u,v), \tilde{H}(v,u)]$ if $f(t,\cdot)$ is nonincreasing for all $t \in [0,T]$. Thus, $H[v,u] \subseteq [v,u]$.

On the other hand, for every $x, y \in [v,u]$ and $t \in [0,T]$,

$$\begin{aligned}D(Hx(t), Hy(t)) & \leq \frac{\alpha}{1+\alpha}\int_0^T D(f(\theta, x(\lambda(\theta))), f(\theta, y(\lambda(\theta))))d\theta \\ & \leq \frac{\alpha}{1+\alpha}\int_0^T \varphi(\theta)d\theta \cdot d(x,y).\end{aligned}$$

Condition (ii) implies that H has at least one fixed point in $[v,u]$. □

4. Conclusions

First-order delay differential equations are frequently applied to study models in economics, biology and so on. For example, the exponential growth model can be extended to $x'(t) = rx(\lambda(t))$. Some researchers try to consider the models in fuzzy cases; there are also some literatures discussing numerical algorithms of boundary value problems of fuzzy differential equations. However, we believe that it is necessary to discuss the existence of solutions before calculating the numerical solutions. In this paper, we provide some sufficient conditions for the existence of solutions to fuzzy delay differential equations. The results can be applied to estimate the existence and position of the solutions to fuzzy delay differential equation models.

Now, we introduce an example to verify our theorems.

Example 1. *Consider*

$$\begin{cases} x'(t) = \frac{1}{3}\cdot x(1-t) + (1,2,3), \\ x(0) = \frac{1}{2}x(1). \end{cases} \tag{22}$$

Let $u(t) = (10t+10)\cdot(1,2,3)$, $v(t) = \tilde{0}$. We can check that u, v are all (i)-differentiable and

$$\begin{aligned}& u'(t) = 10\cdot(1,2,3) \geq \frac{23}{3}\cdot[1,2,3] \geq \frac{1}{3}\cdot u(1-t) + (1,2,3), \\ & u(0) = 10\cdot(1,2,3) = \frac{1}{2}\cdot u(1).\end{aligned}$$

In addition,

$$\begin{aligned}& v'(t) = \tilde{0} \leq (1,2,3) = \frac{1}{3}\cdot v(1-t) + (1,2,3), \\ & v(0) = \tilde{0} = \frac{1}{2}v(1).\end{aligned}$$

Thus, Theorem 1(i) and (ii) are all satisfied; there exist maximal and minimal (i)-differentiable solutions to (22). We can get the approximate maximal and minimal (i)-differentiable solutions by calculating $A^n u$ and $A^n v$.

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