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On $\mathscr{H}$-Simulation Functions and Fixed Point Results in the Setting of ωt-Distance Mappings with Application on Matrix Equations

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Abstract: The concepts of b-metric spaces and ωt-distance mappings play a key role in solving various kinds of equations through fixed point theory in mathematics and other science. In this article, we study some fixed point results through these concepts. We introduce a new kind of function namely, $\mathscr{H}$-simulation function which is used in this manuscript together with the notion of ωt-distance mappings to furnish for new contractions. Many fixed point results are proved based on these new contractions as well as some examples are introduced. Moreover, we introduce an application on matrix equations to focus on the importance of our work.

Keywords: b-metric spaces; ωt-distance mappings; simulation functions; $\mathscr{H}$-simulation functions; fixed point

1. Introduction

Let $\mathscr{U}$ be a non empty set and $f : \mathscr{U} \to \mathscr{U}$ be a self mapping. A point $u' \in \mathscr{U}$ is called a fixed point of f if $fu' = u'$. If d is a metric on $\mathscr{U}$, then f is said to be contraction if there is $\eta \in [0, 1)$ such that $d(fu_1, fu_2) \leq \eta d(u_1, u_2)$, for each $u_1, u_2 \in \mathscr{U}$.

The novelty of fixed point theory in distance spaces appeared in 1922 by Banach [1] and known later by Banach contraction principle which asserts that a contraction on a complete metric space has a unique fixed point. Subsequently, several generalizations for this result are investigated, either by modifying the contraction conditions or by changing the setting of the distance spaces, for example see [2–14].

One well known generalization of metric spaces is b-metric spaces which were introduced by Bakhtin [15] and improved and named by Czerwik [16]. Then, it is used to investigate many fixed point results in the literature. This generalization enriched the fixed point theory in various ways: theorems, applications and many results. On the other hand, some authors generalized the notion of b-metric spaces to some spaces such as extended b-metric spaces, extended quasi b-metric spaces and ωt-distance mappings which were introduced by Kamran et al. [17], Nurwahyu [18] and

Hussain et al. [19], respectively. For a deeper knowledge concerning distance spaces and fixed point theory and functional analysis, we refer the reader to [20–22]
Henceforth, we consider the following notations: $\mathbb{R}$ the set of reals, $\mathbb{N}$ the set of naturals, $\mathbb{C}$ the set of complex numbers, $M_n(\mathbb{C})$ the set of all $n \times n$ matrices with complex entries and for a self mapping $f : \mathcal{U} \to \mathcal{U}$, $\mathcal{F}_f$ the set of all fixed points of f in $\mathcal{U}$.

2. Preliminary

The definition of b-metric spaces is given as following:

Definition 1. *[15] A function $b : \mathcal{U} \times \mathcal{U} \to [0, +\infty)$ is said to be b-metric if there is $s \in [1, +\infty)$ such that b satisfies:*

(b_1) *$b(u_1, u_2) = 0$ iff $u_1 = u_2$,*
(b_2) *$b(u_1, u_2) = b(u_2, u_1)$, for all $u_1, u_2 \in \mathcal{U}$,*
(b_3) *$b(u_1, u_2) \leq s[b(u_1, u_2) + b(u_2, u_3)]$, for all $u_1, u_2, u_3 \in \mathcal{U}$.*

The pair $(\mathcal{U}, b, s)$ is called a b-metric space.

The notion of ωt-distance mapping was introduced by Hussain et al. [19] in 2014 and given as following:

Definition 2. *[19] A function $\omega : \mathcal{U} \times \mathcal{U} \to [0, +\infty)$ is said to be an ωt-distance over a b-metric space $(\mathcal{U}, b, s)$ if ω satisfies:*

(ω_1) *$\omega(u_1, u_2) \leq s[\omega(u_1, u_2) + \omega(u_2, u_3)]$, for all $u_1, u_2, u_3 \in \mathcal{U}$,*
(ω_2) *$\omega(., u') \to [0, +\infty)$ is s-lower semi-continuous for all $u' \in \mathcal{U}$,*
(ω_3) *for any $\epsilon > 0$, there is $\gamma > 0$ such that*

$$\omega(u_1, u_2) \leq \gamma \text{ and } \omega(u_1, u_3) \leq \gamma \text{ imply } b(u_2, u_3) \leq \epsilon.$$

From now on, $(\mathcal{U}, b, s)$ is referred to a b-metric space, and ω is referred to an ωt-distance mapping over $(\mathcal{U}, b, s)$.

It is obviously that, every b-metric is an ωt-distance mapping.

Lemma 1. *[19] On $(\mathcal{U}, b, s)$, suppose we have two sequences (u_n) and (v_n) in $\mathcal{U}$. Let (λ_n) and (β_n) be sequences in $[0, +\infty)$ such that $\lambda_n \to 0$ and $\beta_n \to 0$. Then:*

1. *If $\omega(u_n, u) \leq \lambda_n$ and $\omega(u_n, v) \leq \beta_n$ for all $n \in \mathbb{N}$, then $u = v$.*
2. *If $\omega(u_n, v_n) \leq \lambda_n$ and $\omega(u_n, v) \leq \beta_n$ for all $n \in \mathbb{N}$, then $b(v_n, v) \to 0$.*
3. *If $\omega(u_n, u_m) \leq \lambda_n$ for all $n, m \in \mathbb{N}$ with $m > n$, then u_n is a Cauchy sequence.*
4. *If $\omega(u, u_n) \leq \lambda_n$ for all $n \in \mathbb{N}$, then u_n is a Cauchy sequence.*

Definition 3. *[23] Let Φ denote the set of all functions $\phi : [1, +\infty) \to [1, +\infty)$ that satisfy:*
(Φ_1) *ϕ is non decreasing and continuous on $[1, +\infty)$,*
(Φ_2) *for all $u' > 1$, $\lim_{n \to +\infty} \phi^n(u') = 1$.*

Remark 1. *[23] If $\phi \in \Phi$, then $\phi(1) = 1$ and $\phi(u') < u'$ for all $u' > 1$.*

Definition 4. *[24] Let Θ^* denotes the set of all functions $\theta^* : (0, +\infty) \to (1, +\infty)$ that satisfies:*
(Θ_1^*) *θ is non decreasing and continuous on $(0, +\infty)$,*
(Θ_2^*) *for each sequence $\{u_n'\}$ in $(0, +\infty)$, $\lim_{n \to +\infty} \theta^*(u_n') = 1$ if and only if $\lim_{n \to +\infty} t_n = 0$,*
(Θ_3^*) *there exist $\alpha \in (0, 1)$ and $\gamma \in (0, +\infty)$ such that $\lim_{v \to 0^+} \dfrac{\theta^*(u) - 1}{u^\alpha} = \gamma$.*

In this manuscript, we consider the class Θ to be defined as following:

Definition 5. *Let Θ denote the set of all continuous functions $\theta : [0,+\infty) \to [1,+\infty)$ that satisfy:*
(Θ_1) *θ is non decreasing on $[0,+\infty)$,*
(Θ_2) *for each sequence $\{u_n'\}$ in $[0,+\infty)$, $\lim_{n\to+\infty} \theta(u_n') = 1$ if and only if $\lim_{n\to+\infty} t_n = 0$.*

Remark 2. *If $\theta \in \Theta$, then $\theta^{-1}(\{1\}) = 0$.*

In 2015, Khojasteh et al. [25] introduced the concept of simulation functions in which they used it to unify several fixed point results in the literature. Then, significant results in fixed point theory using simulation functions were obtained, for example see [26–32]

Definition 6. *[25] A function $\zeta^* : [0,+\infty) \times [0,+\infty) \to \mathbb{R}$ is called a simulation function if it satisfies the following:*
(ζ_1^*) *$\zeta^*(0,0) = 0$,*
(ζ_2^*) *$\zeta^*(u_1,u_2) < u_1 - u_2$ for all $u_1,u_2 > 0$,*
(ζ_3^*) *If (u_n) and (u_n') are sequences in $[0,+\infty)$ such that $\lim_{n\to+\infty} u_n = \lim_{n\to+\infty} u_n' > 0$, then $\limsup_{n\to+\infty} \zeta^*(u_n,u_n') < 0$.*

Seong-Hoon Cho [33] introduced the following class of functions, namely $\mathscr{L}$-simulation functions and some new type of contractions by using $\mathscr{L}$-simulation functions:

Definition 7. *A function $\zeta' : [1,+\infty) \times [1,+\infty) \to \mathbb{R}$ is called $\mathscr{L}$-simulation function if it satisfies the following:*
(ζ_1') $\zeta'(1,1) = 1$,
(ζ_2') $\zeta'(u,u') < \frac{u'}{u}$,
(ζ_3') For each sequences $(u_n),(u_n')$ in $(1,+\infty)$, with $u_n \le u_n'$ for all $n \in \mathbb{N}$ $\lim_{n\to+\infty} u_n = \lim_{n\to+\infty} u_n' > 1$ implies $\limsup_{n\to+\infty} \zeta'(u_n,u_n') < 1$.

3. Main Results

We begin our work with the definition of $\mathscr{H}$-simulation functions and some examples on this notion. Then, we introduce the notion of $(\omega t,\theta,\phi)$-contractions with respect to $\mu \in \mathscr{H}$ to derive some results.

Definition 8. *A function $\mu : [1,+\infty) \times [1,+\infty) \to \mathbb{R}$ is called $\mathscr{H}$-simulation if $\mu(u,u') \le \frac{u'}{u}$ for all $u,u' \in [1,+\infty)$.*

We denote by $\mathscr{H}$ the set of all $\mathscr{H}$-simulation functions.

Remark 3. *Let $\mu \in \mathscr{H}$. If $(u_n),(u_n')$ are sequences in $[1,+\infty)$ with $1 \le \lim_{n\to+\infty} u_n' < \lim_{n\to+\infty} u_n$, then $\limsup_{n\to+\infty} \mu(u_n,u_n') < 1$.*

Now, we provide some examples on $\mathscr{H}$-simulation functions.

Example 1. *The following functions belong to $\mathscr{H}$:*

1. $\mu_1(u_1,u_2) = \dfrac{ku_2^r}{u_1}, k,r \in (0,1]$,
2. $\mu_2(u_1,u_2) = \dfrac{\min\{u_1,u_2\}}{\max\{u_1,u_2\}}$,

3. $\mu_3(u_1,u_2) = \dfrac{u_2}{u_1 + |\ln(\frac{u_2}{u_1})|}$,
4. $\mu_4(u_1,u_2) = \dfrac{u_2}{u_1 + \sqrt{u_2}}$,
5. $\mu_5(u_1,u_2) = \dfrac{u_2^2}{1 + u_1u_2}$,
6. *Let $f_1, f_2 : (0,+\infty) \to (0,+\infty)$ be continuous functions such that $f_1(r) < r$, and $f_2(r) \geq r$, for each $r \in (0,+\infty)$. Define $\mu_6(u_1,u_2) = \dfrac{f_1(u_2)}{f_2(u_1)}$.*

Note: Every $\mathscr{L}$-simulation function is $\mathscr{H}$-simulation while the converse isn't true in general as we can see in the following example.

Example 2. *Consider the function $\mu : [1,+\infty) \times [1,+\infty) \to \mathbb{R}$ which is defined as*

$$\mu(u_1,u_2) = 1 + \ln\left(\frac{u_2}{u_1}\right).$$

Then $\mu \in \mathscr{H}$ and $\mu \notin \mathscr{L}$.

Clearly $\mu(u_1,u_2) \leq \frac{u_2}{u_1}$ for all $u_1,u_2 \in [1,+\infty)$ and so, $\mu \in \mathscr{H}$.

To show that $\mu \notin \mathscr{L}$, consider the sequences $(u_n),(u_n^{'})$ in $(1,+\infty)$ such that $u_n = \frac{2n+3}{n+1}$, and $u_n^{'} = \frac{2n+1}{n}$. Then $u_n \leq u_n^{'}$ for all $n \in \mathbb{N}$ and $\lim_{n\to+\infty} u_n = \lim_{n\to+\infty} u_n^{'} = 2$, while

$$\begin{aligned}\limsup_{n\to+\infty} \zeta(u_n,u_n^{'}) &= \limsup_{n\to+\infty}\left(1+\ln\left(\frac{\frac{2n+1}{n}}{\frac{2n+3}{n+1}}\right)\right)\\ &= 1+\ln\left(\limsup_{n\to+\infty}\left(\frac{2n^2+3n+1}{2n^2+3n}\right)\right)\\ &= 1.\end{aligned}$$

Note: $\mu_2, \mu_3, \mu_4, \mu_5$ and μ_6 described in Example 1 are not members of $\mathscr{L}$.

Definition 9. *Suppose there is ω over $(\mathscr{U}, b, s)$ with $s \in [1,+\infty)$. A self mapping $f : \mathscr{U} \to \mathscr{U}$ is said to be $(\omega t, \theta, \phi)$-contraction with respect to μ if there exist $\theta \in \Theta$ and $\phi \in \Phi$ such that*

$$\mu(\theta(s\omega(fu_1,fu_2)), \phi\theta\omega(u_1,u_2)) \geq 1 \ \text{ for all } \ u_1,u_2 \in \mathscr{U}. \tag{1}$$

Lemma 2. *If f is $(\omega t,\theta,\phi)$-contraction, then for all $u_1,u_2 \in \mathscr{U}$, we have the following:*

1. *$\omega(u_1,u_2) > 0$ implies that $\omega(fu_1,fu_2) < \frac{1}{s}\omega(u_1,u_2)$,*
2. *$\omega(u_1,u_2) = 0$ implies that $\omega(fu_1,fu_2) = 0$.*

Proof. (1) Suppose $\omega(u_1,u_2) > 0$. Then Condition 1 implies that

$$\begin{aligned}1 &\leq \mu(\theta(s\omega(fu_1,fu_2),\phi\theta\omega(u_1,u_2))\\ &\leq \frac{\phi\theta\omega(u_1,u_2)}{\theta(s\omega(fu_1,fu_2))}\\ &< \frac{\theta\omega(u_1,u_2)}{\theta(s\omega(fu_1,fu_2))}.\end{aligned}$$

So $\theta(s\omega(fu_1,fu_2)) < \theta\omega(u_1,u_2)$. Since θ is non-decreasing, we have $s\omega(fu_1,fu_2) < \omega(u_1,u_2)$ and so, we get the result.

(2) Suppose $\omega(u_1,u_2) = 0$. By Condition 1, we have

$$1 \leq \theta(s\omega(fu_1,fu_2)) \leq \phi\theta\omega(u_1,u_2) = 1.$$

Hence the result. □

Lemma 3. *Suppose there is ω over $(\mathscr{U}, b, s)$ with $s \in [1, +\infty)$. Let $f : \mathscr{U} \to \mathscr{U}$ be an (ω, θ, ϕ)-contraction with respect to $\mu \in \mathscr{H}$. Then $\mathscr{F}_f$ contains at most one element.*

Proof. Assume that there are $u, v \in \mathscr{F}_f$. First, we claim that $\omega(u, v) = 0$. If $\omega(u, v) > 0$, then Lemma 2 implies that

$$\omega(u, v) = \omega(fu, fv) < \frac{1}{s}\omega(u, v).$$

a contradiction and so $\omega(u, v) = 0$. Similarly, we can get that $\omega(u, u) = 0$. $b(u, v) = 0$ and hence $u = v$. □

On $\mathscr{U}$, let $u_0 \in \mathscr{U}$ and $f : \mathscr{U} \to \mathscr{U}$ be a self mapping. Then we call the sequence (u_n), where $u_n = fu_{n-1}$, $n \in \mathbb{N}$ the Picard sequence generated by f at u_0.

Lemma 4. *Suppose there is ω over $(\mathscr{U}, b, s)$ with $s \in [1, +\infty)$. Let $f : \mathscr{U} \to \mathscr{U}$ be an (ω, θ, ϕ)-contraction with respect to $\mu \in \mathscr{H}$. Then*

$$\lim_{n \to +\infty} \omega(u_n, u_{n+1}) = 0 \text{ and } \lim_{n \to +\infty} \omega(u_{n+1}, u_n) = 0 \tag{2}$$

for any initial point $u_0 \in \mathscr{U}$, where (u_n) is the Picard sequence generated by f at u_0.

Proof. Let (u_n) be the Picard sequence generated by f at u_0. If there is $M \in \mathbb{N}$ such that $\omega(u_M, u_{M+1}) = 0$, then by Lemma 2, we get that $\omega(u_n, u_{n+1}) = 0$ for all $n \geq M$.

Assume that $\omega(u_n, u_{n+1}) > 0$ for all $n \in \mathbb{N}$. By Lemma 2, we have

$$\omega(u_n, u_{n+1}) < \frac{1}{s}\omega(u_{n-1}, u_n).$$

Thus $(\omega(u_n, u_{n+1}) : n \in \mathbb{N})$ is a non increasing sequence in $[0, +\infty)$. There is $c_0 \geq 0$ such that $\lim_{n \to +\infty} \omega(u_n, u_{n+1}) = c_0$. Suppose to the contrary; that is, $c_0 > 0$. Let $a_n = \theta s \omega(u_n, u_{n+1})$ and $b_n = \phi\theta\omega(u_{n-1}, u_n)$. Then $1 \leq \lim_{n \to +\infty} b_n < \lim_{n \to +\infty} a_n$. By (1) and Remark 3, we have

$$1 \leq \limsup_{n \to +\infty} \zeta(\theta s \omega(u_n, u_{n+1}), \phi\theta\omega(u_{n-1}, u_n)) < 1,$$

a contradiction. $\lim_{n \to +\infty} \omega(u_n, u_{n+1}) = 0$. By the same way we can show that $\lim_{n \to +\infty} \omega(u_{n+1}, u_n) = 0$. □

Lemma 5. *Suppose there is ω over $(\mathscr{U}, b, s)$ with $s \in [1, +\infty)$. Let $f : \mathscr{U} \to \mathscr{U}$ be an (ω, θ, ϕ)-contraction with respect to $\mu \in \mathscr{H}$. If there is $n_0 \in \mathbb{N}$ with $\omega(u_{n_0}, u_{n_0+1}) = 0$, then $u_{n_0+1} \in \mathscr{F}_f$. In addition, if there is $n_0 \in \mathbb{N}$ with $\omega(u_{n_0+1}, u_{n_0}) = 0$, then $u_{n_0} \in \mathscr{F}_f$.*

Proof. The proof follows from part (a) and part (c) of the definition of ω. □

Theorem 1. *Suppose $(\mathscr{U}, b, s)$ is complete with base $s \in [1, +\infty)$. Suppose that there are $\theta \in \Theta$ and $\phi \in \Phi$ such that $f : \mathscr{U} \to \mathscr{U}$ is an (ω, θ, ϕ)-contraction with respect to $\mu \in \mathscr{H}$ such that:*

$$\text{If } \nu \in X \text{ with } f\nu \neq \nu, \text{ then } \inf\{\omega(u, \nu) : u \in \mathscr{U}\} > 0. \tag{3}$$

Then $\mathscr{F}_f$ consists of only one element. Moreover, the sequence (u_n), where $u_{n+1} = fu_n$, $n \geq 0$ converges for any $u_0 \in \mathscr{U}$ and $\lim_{n \to +\infty} u_n \in \mathscr{F}_f$.

Proof. Let $u_0 \in \mathscr{U}$ and consider the Picard sequence (u_n) in $\mathscr{U}$ generated by f at u_0. According to Lemma 5, if there exists $n_0 \in \mathbb{N}$ such that $\omega(u_{n_0}, u_{n_0+1}) = 0$ or $\omega(u_{n_0+1}, u_{n_0}) = 0$, then $u_{n_0+1} \in \mathscr{F}_f$ or $u_{n_0} \in \mathscr{F}_f$, respectively. Therefore, we may assume that for each $n \in \mathbb{N}$, $\omega(u_n, u_{n+1}) \neq 0$ and $\omega(u_{n+1}, u_n) \neq 0$. By Lemma 4, we have $\lim_{n\to+\infty} \omega(u_n, u_{n+1}) = 0$ and $\lim_{n\to+\infty} \omega(u_{n+1}, u_n) = 0$. Now, we want to show that $\lim_{n,m\to+\infty} \omega(u_n, u_m) = 0$, i.e. (u_n) is a Cauchy sequence.

Assume the contrary; that is, $\lim_{n,m\to+\infty} \omega(u_n, u_m) \neq 0$. Thus there are $\epsilon > 0$ and two sub-sequences (u_{n_k}) and (u_{m_k}) of (u_n) such that (m_k) is chosen as the smallest index for which

$$\omega(u_{n_k}, u_{m_k}) \geq \epsilon,\ m_k > n_k > k. \tag{4}$$

This implies that

$$\omega(u_{n_k}, u_{m_k-1}) < \epsilon. \tag{5}$$

Set $\delta_k = \omega(u_{n_K-1}, u_{m_k})$. By Lemma 2, Equations (4) and (5) and (ω_1) of the definition of ω, we get

$$\begin{aligned} \epsilon \leq \omega(u_{n_k}, u_{m_k}) &\leq \tfrac{1}{s}\omega(u_{n_k-1}, u_{m_k-1}) \\ &\leq [\omega(u_{n_k-1}, u_{m_k}) + \omega(u_{m_k}, u_{m_k-1})]. \end{aligned}$$

By taking the limit inferior as $k \to +\infty$ and taking into account Equation (2), we get

$$\epsilon \leq \liminf_{k\to+\infty} \delta_k. \tag{6}$$

In addition,

$$\begin{aligned} \omega(u_{n_k-1}, u_{m_k}) &\leq \tfrac{1}{s}\omega(u_{n_k-2}, u_{m_k-1}) \\ &\leq [\omega(u_{n_k-2}, u_{n_k}) + \omega(u_{n_k}, u_{m_k-1})] \\ &< s[\omega(u_{n_k-2}, u_{n_k-1}) + \omega(u_{n_k-1}, u_{n_k})] + \epsilon]. \end{aligned}$$

By taking the limit superior as $k \to +\infty$ and taking into account (2), we get

$$\limsup_{k\to+\infty} \delta_k \leq \epsilon. \tag{7}$$

By Equations (6) and (7), we get

$$\lim_{k\to+\infty} \delta_k = \epsilon. \tag{8}$$

Now, set $\gamma_k = \omega(u_{n_k}, u_{m_k+1})$. By Lemma 2, we get

$$\omega(u_{n_k}, u_{m_k+1}) \leq \frac{1}{s}\omega(u_{n_k-1}, u_{m_k}).$$

By taking the limit superior to both sides, we get

$$\limsup_{k\to+\infty} \gamma_k \leq \frac{\epsilon}{s}. \tag{9}$$

On the other hand, we have

$$\epsilon \leq \omega(u_{n_k}, u_{m_k}) \leq s[\omega(u_{n_k}, u_{m_k+1}) + \omega(u_{m_k+1}, u_{m_k})].$$

By taking the limit inferior to both sides, we get

$$\frac{\epsilon}{s} \leq \liminf_{k\to+\infty} \gamma_k. \tag{10}$$

By Equations (9) and (10), we get

$$\lim_{k\to+\infty} \gamma_k = \frac{\epsilon}{s}. \tag{11}$$

By the properties of θ and ϕ, we get

$$\phi(\theta(\epsilon)) < \theta(\epsilon) = \theta(s\frac{\epsilon}{s}).$$

Now, by letting $a_k = \theta(s\gamma_k)$ and $b_k = \phi(\theta(\delta_k))$, then $\lim_{k\to+\infty} a_k > \lim_{k\to+\infty} b_k \geq 1$. Remark 3 and Condition (1) yield that

$$1 \leq \limsup_{k\to+\infty} \mu(a_k, b_k) < 1,$$

which is a contradiction. Therefore $\lim_{n,m\to+\infty} \omega(u_n, u_m) = 0$. Thus Lemma 1 implies that (u_n) is Cauchy. There is $v \in \mathscr{U}$ such that $\lim_{n\to+\infty} u_n = v$.

Since $\lim_{n,m\to+\infty} \omega(u_n, u_m) = 0$, then for any $r > 0$ there is $k_0 \in \mathbb{N}$ such that

$$\omega(u_n, u_m) \leq r \text{ for all } m > n \geq k_0.$$

The lower semi-continuity of ω implies that

$$\omega(u_n, v) \leq \liminf_{p\to+\infty} \omega(u_n, u_p) \leq r \text{ for all } m > n \geq k_0.$$

Suppose that $fv \neq v$. Then we have

$$\begin{aligned} 0 &< \inf\{\omega(u,v) : u \in \mathscr{U}\} \\ &\leq \inf\{\omega(u_n,v) : n \geq k_0\} \\ &\leq r, \end{aligned}$$

for every $r > 0$ which is a contradiction. Therefore $fv = v$. The uniqueness of v follows from Lemma 3. □

Corollary 1. *Suppose $(\mathscr{U}, b, s)$ is complete with base $s \in [1, +\infty)$, and there is ω over $(\mathscr{U}, b, s)$. Suppose that there are real numbers $a > 1$ and $\lambda \in (0, 1)$ such that $f : \mathscr{U} \to \mathscr{U}$ satisfies the following condition:*
If $u_1, u_2 \in \mathscr{U}$, then

$$a^{s\omega(fu_1,fu_2)} \leq a^{\lambda\omega(u_1,u_2)} - a^{-\lambda\omega(u_1,u_2)}. \tag{12}$$

In addition, assume that if $v \in X$ and $fv \neq v$, then

$$\inf\{\omega(u, v) : u \in \mathscr{U}\} > 0. \tag{13}$$

Then $\mathscr{F}_f$ consists of only one element.

Proof. Define $\mu : [1, +\infty) \times [1, +\infty) \to \mathbb{R}$, $\theta : [0, +\infty) \to [1, +\infty)$ and $\phi : [1, +\infty) \to [1, +\infty)$ by $\mu(u_1, u_2) = \dfrac{u_2^2}{1 + u_1u_2}$, $\theta(u) = a^u$ and $\phi(v) = v^\lambda$, respectively. Then $\mu \in \mathscr{H}$, $\theta \in \Theta$ and $\phi \in \Phi$. We now show that f is an $(\omega t, \theta, \phi)$-contraction with respect to μ. From Condition (12), we have

$$a^{s\omega(fu_1,fu_2)} \leq a^{\lambda\omega(u_1,u_2)} - a^{-\lambda\omega(u_1,u_2)}$$

iff

$$a^{2\lambda\omega(u_1,u_2)} - a^{s\omega(fu_1,fu_2)+\lambda\omega(u_1,u_2)} \geq 1$$

iff

$$(a^{\lambda\omega(u_1,u_2)})^2 \geq 1 + a^{s\omega(fu_1,fu_2)}a^{\lambda\omega(u_1,u_2)}$$

iff

$$\frac{(a^{\lambda\omega(u_1,u_2)})^2}{1 + a^{s\omega(fu_1,fu_2)}a^{\lambda\omega(u_1,u_2)}} \geq 1$$

iff

$$\frac{(\phi\theta\omega(u_1,u_2))^2}{1+\theta s\omega(fu_1,fu_2)\phi\theta\omega(u_1,u_2)} \geq 1$$

iff

$$\mu(\theta s\omega(fu_1,fu_2),\phi\theta\omega(u_1,u_2)) \geq 1.$$

Hence the result follows from Theorem 1. □

Corollary 2. *Suppose $(\mathscr{U},b,s)$ is complete with base $s \in [1,+\infty)$, and there is ω over $(\mathscr{U},b,s)$. Suppose that there is a real number $\lambda \in (0,1)$ such that $f:\mathscr{U}\to\mathscr{U}$ satisfies the following condition:*
If $u_1,u_2 \in \mathscr{U}$, then

$$|\lambda\omega(u_1,u_2) - s\omega(fu_1,fu_2)| \leq e^{\lambda\omega(u_1,u_2)} - e^{s\omega(fu_1,fu_2)}. \tag{14}$$

In addition, suppose that if $v \in X$ if $fv \neq v$, then

$$\inf\{\omega(u,v) : u \in \mathscr{U}\} > 0. \tag{15}$$

Then $\mathscr{F}_f$ consists of only one element.

Proof. Define $\mu:[1,+\infty)\times[1,+\infty)\to\mathbb{R}$, $\theta:[0,+\infty)\to[1,+\infty)$ and $\phi:[1,+\infty)\to[1,+\infty)$ by $\mu(u_1,u_2) = \dfrac{u_2}{u_1+|\ln(\frac{u_2}{u_1})|}$, $\theta(u)=e^u$ and $\phi(v)=v^\lambda$, respectively. Then $\mu \in \mathscr{H}$, $\theta \in \Theta$ and $\phi \in \Phi$. We now show that f is an $(\omega t,\theta,\phi)$-contraction with respect to μ. From Condition (14), we have

$$|\lambda\omega(u_1,u_2) - s\omega(fu_1,fu_2)| \leq e^{\lambda\omega(u_1,u_2)} - e^{s\omega(fu_1,fu_2)}$$

iff

$$e^{\lambda\omega(u_1,u_2)} \geq e^{s\omega(fu_1,fu_2)} + |\lambda\omega(u_1,u_2) - s\omega(fu_1,fu_2)|$$

iff

$$e^{\lambda\omega(u_1,u_2)} \geq e^{s\omega(fu_1,fu_2)} + \left|\ln(e^{\lambda\omega(u_1,u_2)-s\omega(fu_1,fu_2)})\right|$$

iff

$$e^{\lambda\omega(u_1,u_2)} \geq e^{s\omega(fu_1,fu_2)} + \left|\ln\left(\frac{e^{\lambda\omega(u_1,u_2)}}{e^{s\omega(fu_1,fu_2)}}\right)\right|$$

iff

$$\phi\theta\omega(u_1,u_2) \geq \theta s\omega(fu_1,fu_2) + \left|\ln\left(\frac{\phi\theta\omega(u_1,u_2)}{\theta s\omega(fu_1,fu_2)}\right)\right|$$

iff

$$\frac{\phi\theta\omega(u_1,u_2)}{\theta s\omega(fu_1,fu_2) + \left|\ln\left(\frac{\phi\theta\omega(u_1,u_2)}{\theta s\omega(fu_1,fu_2)}\right)\right|} \geq 1$$

iff

$$\mu(\theta s\omega(fu_1,fu_2),\phi\theta\omega(u_1,u_2)) \geq 1.$$

Hence the result follows from Theorem 1. □

4. Examples

Next, we illustrate our result by some examples.

Example 3. *Suppose $\mathscr{U}=\mathbb{C}$. Let f be a self mapping on $\mathscr{U}$ via $f(z)=\alpha z$ with α is real number in $[-\sqrt{8},\sqrt{8}]$. To show that $\mathscr{F}_f$ consist of only one element. Define $b:\mathscr{U}\times\mathscr{U}:[0,+\infty)$ via $b(z_1,z_2)=|z_1-z_2|^2$*

and $\omega : \mathcal{U} \times \mathcal{U} : [0, +\infty)$ *via* $\omega(z_1, z_2) = |z_2|^2$. *In addition, define* $\mu : [1, +\infty) \times [1, +\infty) \to \mathbb{R}$ *by* $\mu(u, u') = \frac{(u')^{\frac{1}{2}}}{u}$.

Moreover, define $\theta : [0, +\infty) \to [1, +\infty)$ *via* $\theta(u') = e^{u'}$ *and* $\phi : [1, +\infty) \to [1, +\infty)$ *by* $\phi(u') = \sqrt{u'}$. *Then* $(\mathcal{U}, b, s)$ *is complete b-metric space with* $s = 2$ *and* ω *is* ωt*-distance mapping over*$(\mathcal{U}, b, s)$, $\mu \in \mathcal{H}$, $\phi \in \Phi$ *and* $\theta \in \Theta$.

Now, we show that f *is an* $(\omega t, \theta, \phi)$*-contraction with respect to* μ*; i.e.,*

$$1 \leq \frac{\phi(\theta\omega(z_1, z_2)^{\frac{1}{2}}}{\theta 2\omega(fz_1, fz_2)} \; for\ all \;\; z_1, z_2 \in \mathcal{U}. \tag{16}$$

Now, for all $z_1, z_2 \in \mathcal{U}$, *we have*

$$\begin{aligned} 2\omega(fz_1, fz_2) &= 2\omega(\alpha z_1, \alpha z_2) \\ &= 2\alpha^2|z_2|^2 \\ &\leq \frac{1}{4}|z_2|^2 = \frac{1}{4}\omega(z_1, z_2). \end{aligned}$$

Hence

$$\begin{aligned} \theta 2\omega(fz_1, fz_2) &\leq \frac{1}{4}\omega(z_1, z_2) \\ &= e^{\frac{1}{4}\omega(z_1,z_2)} \\ &= \sqrt{(e^{\omega(z_1,z_2)})^{\frac{1}{2}}} \\ &= \phi(\theta\omega(z_1, z_2)^{\frac{1}{2}}). \end{aligned}$$

Utilizing Theorem 1, we get $\mathcal{F}_f$ *consists of only one element.*

Example 4. *Let* $\mathcal{U} = [0, 1]$. *Define* $\mu : [1, +\infty) \times [1, +\infty) \to \mathbb{R}$ *by* $\mu(u, u') = \frac{u'}{u}$. *Additionally, define* $b, \omega : \mathcal{U} \times \mathcal{U} \to [0, +\infty)$ *by* $b(u_1, u_2) = (u_1 - u_2)^2$ *and* $\omega(u_1, u_2) = \frac{1}{4}(u_1 - u_2)^2$. *Moreover, define* $\theta : [0, +\infty) \to [1, +\infty)$ *via* $\theta(u') = e^{u'}$ *and* $\phi : [1, +\infty) \to [1, +\infty)$ *by* $\phi(u') = \left(u'\right)^{\frac{9}{32}}$. *Then the function* $g : \mathcal{U} \to \mathcal{U}$ *which is defined by* $g(u) = \frac{1 - u^3}{4 - 2u^3}$ *has a unique fixed point in* $\mathcal{U}$.

Proof. It is clearly that

1. $(\mathcal{U}, b, s)$ is a complete b-metric space with $s = 2$, and also, ω is ωt-distance mapping over$(\mathcal{U}, b, s)$.
2. $\mu \in \mathcal{H}$ (see Example 2).
3. $\phi \in \Phi$ and $\theta \in \Theta$.

To show that $\mathcal{F}_g$ consists of only one element, it suffices to show that

$$\mu(\theta 2\omega(fu_1, fu_2), \phi\theta\omega(u_1, u_2) \geq 1 \;\; for\ all \;\; u_1, u_2 \in \mathcal{U};$$

i.e., we want to show that

$$\theta 2\omega(gu_1, gu_2) \leq \phi\theta\omega(u_1, u_2) \;\; for\ all \;\; u_1, u_2 \in \mathcal{U}.$$

Now,

$$\begin{aligned} 2\omega(gu_1, gu_2) &= \frac{1}{2}\left(\frac{1-u_1^3}{4-2u_1^3} - \frac{1-u_2^3}{4-2u_2^3}\right)^2 \\ &= \frac{1}{8}\left(\frac{(u_1-u_2)(u_1^2+u_1u_2+u_2^2)}{(2-u_1^3)^2(2-u_2^3)^2}\right)^2 \\ &\leq \frac{9}{128}\left(u_1-u_2\right)^2 \\ &= \frac{9}{32}\omega(u_1,u_2). \end{aligned}$$

$$\theta 2\omega(gu_1, gu_2) = e^{2\omega(gu_1,gu_2)} \leq e^{\frac{9}{32}\omega(u_1,u_2)} = \left(e^{\omega(u_1,u_2)}\right)^{\frac{9}{32}} = \phi\theta\omega(u_1,u_2).$$

Hence, Theorem 1 ensures that $\mathcal{F}_g$ consists of only one element. Using MATLAB, we can find that the fixed point of g is $u \simeq 0.248076921333013$. □

5. Applications

In this section, we highlight the novelty of our work by introducing some applications by utilizing Theorem 1.

Next, we show that for any real number $n \geq 2$, the equation

$$\sqrt{2}u = \frac{1+u^n}{n+u^n} \tag{17}$$

has a unique solution in [0,1].

Theorem 2. *Let $\mathcal{U} = [0,1]$. Define $\mu : [1,+\infty) \times [1,+\infty) \to \mathbb{R}$ by $\mu(u,u') = 1 + \ln(\frac{u'}{u})$. Additionally, define $b, \omega : \mathcal{U} \times \mathcal{U} \to [0,+\infty)$ by $b(u_1,u_2) = (u_1-u_2)^2$ and $\omega(u_1,u_2) = \frac{1}{4}(u_1-u_2)^2$. Moreover, define $\theta : [0,+\infty) \to [1,+\infty)$ via $\theta(u') = e^{u'}$ and $\phi : [1,+\infty) \to [1,+\infty)$ by $\phi(u') = \left(u'\right)^{\frac{(n-1)^2}{n^2}}$. Then for the function $f : \mathcal{U} \to \mathcal{U}$ which defined by $f(u) = \dfrac{1+u^n}{n\sqrt{2}+\sqrt{2}u^n}$, the set $\mathcal{F}_f$ consists of only one element.*

Proof. It is obviously that:

1. $(\mathcal{U}, b, s)$ is a complete b-metric space with $s = 2$. In addition, ω is an ωt-distance mapping over$(\mathcal{U}, b, s)$.
2. $\mu \in \mathcal{H}$ (see Example 2).
3. $\phi \in \Phi$ and $\theta \in \Theta$.

To show that $\mathcal{F}_f$ consists of only one element, it suffices to prove that

$$\mu(\theta 2\omega(fu_1, fu_2), \phi\theta\omega(u_1,u_2) \geq 1 \;\; for\ all \;\; u_1, u_2 \in \mathcal{U}, \tag{18}$$

which is equivalent to prove that

$$\theta 2\omega(fu_1, fu_2) \leq \phi\theta\omega(u_1,u_2) \;\; for\ all \;\; u_1, u_2 \in \mathcal{U}. \tag{19}$$

Now,

$$\begin{aligned}
2\omega(fu_1,fu_2) &= \frac{1}{2}\left(\frac{1+u_1^n}{n\sqrt{2}+\sqrt{2}u_1^n}-\frac{1+u_2^n}{n\sqrt{2}+\sqrt{2}u_2^n}\right)^2\\
&= \frac{1}{4(n+u_1^n)^2(n+u_2^n)^2}\Big((n-1)(u_1^n-u_2^n)\Big)^2\\
&\le \frac{(n-1)^2}{4n^4}\Big(u_1^n-u_2^n\Big)^2\\
&\le \frac{(n-1)^2}{4n^4}\Big(n(u_1-u_2)\Big)^2\\
&= \frac{(n-1)^2}{4n^2}\Big(u_1-u_2\Big)^2\\
&= \frac{(n-1)^2}{n^2}\omega(u_1,u_2).
\end{aligned}$$

$$\theta 2\omega(fu_1,fu_2)=e^{2\omega(fu_1,fu_2)}\le e^{\frac{(n-1)^2}{n^2}\omega(u_1,u_2)}=\left(e^{\omega(u_1,u_2)}\right)^{\frac{(n-1)^2}{n^2}}=\phi\theta\omega(u_1,u_2).$$

Hence, Theorem 1 ensures that $\mathscr{F}_f$ consists of only one element. There is $u\in\mathscr{U}$ such that $fu=u$; i.e., $u=\dfrac{1+u^n}{n\sqrt{2}+\sqrt{2}u^n}$. Hence Equation (17) has a unique solution. □

Now, we use Theorem 1 to confirm that for all $Q,A_i,B_i\in M_n(\mathbb{C})$, for $i\in\{1,2,\cdots,k\}$, the matrix equation

$$X=Q+\sum_{i=1}^{k}(A_iXB_i), \tag{20}$$

where $\sum_{i=1}^{k}\|A_i\|\,\|B_i\|=\lambda<1$ has a unique solution.

Let $\mathscr{Y}=M_n(\mathbb{C})$ and consider the spectral norm $\|.\|:\mathscr{Y}\to[0,+\infty)$ which known as $||A||=s_1$, where $s_1\ge s_2\ge\cdots\ge s_n$ are the singular-values of A. Clearly $(\mathscr{Y},\|.\|)$ is a Banach space since $\mathscr{Y}$ is a finite dimensional norm space.

Theorem 3. *Let $Q,A_i,B_i\in\mathscr{Y}$ for $i\in\{1,2,\cdots,k\}$ be such that $\sum_{i=1}^{k}\|A_i\|\,\|B_i\|=\lambda<1$. Then the matrix in Equation (20) has a unique solution in $\mathscr{Y}$. Moreover, for any matrix $X_0\in\mathscr{Y}$, the sequence $X_{n+1}=Q+\sum_{i=1}^{k}(A_iX_nB_i)$ converges to the solution of Equation (20).*

Proof. Let $b,\omega:\mathscr{Y}\times\mathscr{Y}\to[0,+\infty)$ be defined as $b(X,Y)=\|X-Y\|$ and $\omega(X,Y)=\frac{1}{3}\|X-Y\|$. Then, clearly b is a b-metric on $\mathscr{Y}$ with base $s=1$ and ω is an ωt-distance mapping. Let $\mu:[1,+\infty)\times[1,+\infty)\to\mathbb{R}$, $\theta:[0,+\infty)\to[1,+\infty)$ and $\phi:[1,+\infty)\to[1,+\infty)$ be defined as following: $\mu(u_1,u_2)=\dfrac{u_2}{u_1}$, $\theta(t)=e^t$ and $\phi(t)=t^\lambda$. Define $f:\mathscr{Y}\to\mathscr{Y}$ by $fX=Q+\sum_{i=1}^{k}(A_iXB_i)$.

Now, we prove that f is $(\omega t,\theta,\phi)$-contraction with respect to μ. To see this, let $X,Y\in\mathscr{Y}$. Then,

$$\begin{aligned}
\omega(fX, fY) &= \frac{1}{3}\left\|\sum_{i=1}^{k}(A_iXB_i) - \sum_{i=1}^{k}(A_iYB_i)\right\| \\
&= \frac{1}{3}\left\|\sum_{i=1}^{k}((A_iXB_i) - (A_iYB_i))\right\| \\
&= \frac{1}{3}\left\|\sum_{i=1}^{k}(A_i(X-Y)B_i)\right\| \\
&\leq \frac{1}{3}\sum_{i=1}^{k}\|(A_i(X-Y)B_i)\| \\
&\leq \frac{1}{3}\sum_{i=1}^{k}\|A_i\|\ \|(X-Y)\|\ \|B_i\| \\
&= \frac{1}{3}\|(X-Y)\|\ \sum_{i=1}^{k}\|A_i\|\ \|B_i\| \\
&= \lambda\ \omega(X, Y).
\end{aligned}$$

$e^{\omega(fX,fY)} \leq e^{\lambda\omega(X,Y)} = \left(e^{\omega(X,Y)}\right)^{\lambda}$. Hence, $\theta s\omega(fX, fY) \leq \phi\theta\omega(X, Y)$. Consequently, $\mathscr{F}_f$ consists of only one element. The matrix in Equation (20) has a unique solution. □

To illumine our application, consider the following example

Example 5. *Let $A_1, A_2, B_1, B_2, Q \in M_4(\mathbb{C})$ be given as following:*

$$A_1 = \begin{pmatrix} 0 & 0 & 0.5 & -0.5 \\ 0 & 0 & 0.5 & 0.5 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},\ B_1 = \begin{pmatrix} 0.15 & 0.15 & 0 & 0 \\ 0.15 & 0.15 & 0 & 0 \\ 0 & 0 & 0.02 & 0 \\ 0 & 0 & 0.02 & 0.04 \end{pmatrix}$$

$$A_2 = \begin{pmatrix} 0.025 & 0.05 & 0 & 0 \\ 0.05 & 0.05 & 0 & 0 \\ 0 & 0 & 0.1 & 0.1 \\ 0 & 0 & 0.1 & 0.1 \end{pmatrix},\ B_2 = \begin{pmatrix} 0 & 0 & 0.5 & 0.5 \\ 0 & 0 & 0 & 0.5 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},\ and$$

$$Q = \begin{pmatrix} 1 & -1 & 1 & -3 \\ 1 & 5 & 3 & 2 \\ 9 & 1 & 3 & 1 \\ 1 & 6 & 5 & 2 \end{pmatrix}.$$

One can find that $\sum_{i=1}^{2}\|A_i\|\ \|B_i\| = 0.5 < 1$. *Theorem 3 implies that the matrix equation $X = Q + A_1XB_1 + A_2XB_2$ has a unique solution, and the sequence $X_{n+1} = Q + A_1X_nB_1 + A_2X_nB_2$, for $n \geq 0$ converges to the unique solution for any initial matrix X_0.*

For instance, if we start at initial matrix $X_0 = \begin{pmatrix} 1 & 2 & 1 & 3 \\ 1 & -2 & 3 & 4 \\ -2 & 5 & 4 & 1 \\ 3 & 8 & 10 & 1 \end{pmatrix}$, *we find the solution using MATLAB at the 10th iteration which is* $X \simeq \begin{pmatrix} 1.4681 & -0.3926 & 1.0598 & -2.7607 \\ 2.7171 & 6.9520 & 3.2511 & 2.3746 \\ 10.9803 & 3.3849 & 3.7461 & 2.2526 \\ 1.6913 & 7.0959 & 5.5996 & 3.0472 \end{pmatrix}$.

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