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Fekete-Szegö Type Problems and Their Applications for a Subclass of q-Starlike Functions with Respect to Symmetrical Points

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Abstract: In this article, by using the concept of the quantum (or q-) calculus and a general conic domain $\Omega_{k,q}$, we study a new subclass of normalized analytic functions with respect to symmetrical points in an open unit disk. We solve the Fekete-Szegö type problems for this newly-defined subclass of analytic functions. We also discuss some applications of the main results by using a q-Bernardi integral operator.

Keywords: analytic functions; quantum (or q-) calculus; conic domain; q-derivative operator; Hankel determinant; Toeplitz matrices; Fekete-Szegö problem; q-Bernardi integral operator

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1. Introduction and Definitions

Let $\mathcal{A}$ denote the class of all functions f which are analytic in the open unit disk

$$\mathbb{E} = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}$$

and has the normalized Taylor-Maclaurin series expansion of the following form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1}$$

Let $\mathcal{S}$ be the subclass of all functions in $\mathcal{A}$ that are univalent in $\mathbb{E}$ (see [1]): If f and $g \in \mathcal{A}$, the function f is said to be subordinate to the function g, written as $f \prec g$, if there exists an analytic function w in $\mathbb{E}$, with

$$w(0) = 0 \quad \text{and} \quad |w(z)| < 1 \quad (z \in \mathbb{E}),$$

such that $f(z) = g\big(w(z)\big)$. Furthermore, the following equivalence will hold true (see [2]), if g is univalent in $\mathbb{E}$.

$$f(z) \prec g(z) \Longleftrightarrow f(0) = g(0) \quad \text{and} \quad f(\mathbb{E}) \subset g(\mathbb{E}).$$

Let $\mathcal{P}$ denote the well-known Carathéodory class of functions p, which are analytic in the open unit disk $\mathbb{E}$ with

$$\Re\big(p(z)\big) > 0 \quad \text{and} \quad p(0) = 1.$$

If $p \in \mathcal{P}$, then it has the form given by

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n, \tag{2}$$

where $|c_n| \leqq 2 \;\; (n \in \mathbb{N})$.

If f is univalent in $\mathbb{E}$ and $f(\mathbb{E})$ is a star-shaped domain with respect to the origin, then f is called starlike in $\mathbb{E}$ with respect to the origin. The analytical condition of a starlike function in $\mathbb{E}$ is given as follows:

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > 0 \quad (z \in \mathbb{E}).$$

The class of all such functions is denoted by $\mathcal{S}^*$. A function $f \in \mathcal{A}$ is said to be starlike with respect to symmetrical points (see [3]) if it satisfies the inequality:

$$\Re\left(\frac{zf'(z)}{f(z) - f(-z)}\right) > 0 \quad (z \in \mathbb{E}).$$

The class of all functions in $\mathcal{S}$ which are starlike with respect to symmetrical points is denoted by $\mathcal{S}_s^*$. Furthermore, we denote two interesting subclasses of $\mathcal{S}$ by k-$\mathcal{UCV}$ and k-$\mathcal{ST}$ $(0 \leqq k < \infty)$ of functions which are, respectively, k-uniformly convex and k-starlike in $\mathbb{E}$ defined for $0 \leqq k < \infty$ by

$$k\text{–}\mathcal{UCV} = \left\{ f : f \in \mathcal{S} \quad \text{and} \quad \Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > k\left|\frac{zf''(z)}{f^{-}(z)}\right| \quad (z \in \mathbb{E}) \right\}$$

and

$$k\text{–}\mathcal{ST} = \left\{ f : f \in \mathcal{S} \quad \text{and} \quad \Re\left(\frac{zf'(z)}{f(z)}\right) > k\left|\frac{zf'(z)}{f(z)} - 1\right| \quad (z \in \mathbb{E} \right\}.$$

Kanas et al. (see [4,5]; see also [6]) defined and studied classes of k-starlike functions and k-uniformly convex functions subject to the conic domain Ω_k $(k \geqq 0)$, where

$$\Omega_k = \left\{ u + iv : u^2 > k^2\left((u-1)^2 + v^2\right) \right\}. \tag{3}$$

For this conic domain, the following functions play the role of extremal functions:

$$p_k(z) = \begin{cases} A_1(z) & (k=0) \\ A_1(z) & (k=1) \\ A_3(z) & (0<k<1) \\ A_4(z) & (k>1), \end{cases} \tag{4}$$

where

$$A_1(z) = \frac{1+z}{1-z},$$

$$A_2(z) = 1 + \frac{2}{\pi^2}\left[\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2\right],$$

$$A_3(z) = 1 + \frac{2}{1-k^2}\sinh^2\left\{\begin{array}{c}\left(\frac{2}{\pi}\arccos k\right) \\ \left(\arctan(h\sqrt{z})\right)\end{array}\right\},$$

$$A_4(z) = 1 + \frac{1}{k^2-1}\sin\left(\frac{\pi}{2K(i)}\int_0^{\frac{u(z)}{\sqrt{t}}}\frac{1}{\sqrt{1-x^2}\sqrt{1-(ix)^2}}\,dx\right) + \frac{1}{1-k^2},$$

$i \in (0,1)$, and

$$k = \cosh\left(\frac{\pi K'(i)}{K(i)}\right),$$

$K(i)$ is the first kind of Legendre's complete elliptic integral (see, for details [4,5]). Indeed, from (4), we have

$$p_k(z) = 1 + P_1 z + P_2 z^2 + P_3 z^3 + \ldots \tag{5}$$

The quantum (or q-) calculus is an important tool which is used to study various families of analytic functions and has inspired the researchers due to its applications in mathematics and some other related disciplines. Srivastava (see, for details [7]) was the first who used the basic (or q-) hypergeometric functions in Geometric Functions Theory. The extension of the class of starlike functions in the quantum (or q-) calculus was first introduced in [8] by means of the q-difference operator. After that, some remarkable research work was conducted by many mathematicians, which has played an important role in Geometric Function Theory. In particular, Srivastava et al. [9,10] studied the class of q-starlike functions related with the Janowski functions. Mahmood et al. [11] studied the class of q-starlike functions associated with conic regions. The upper bound of the third Hankel determinant for a class of q-starlike functions was investigated in [12] (see also [9]). Kanas and Raducanu [13] introduced the q-analogue of the Ruscheweyh operator by using the concept of convolution and studied some of its properties (see also [11,14–20]). Many other q-derivative and q-integral operators can be written by using the idea of convolution (we refer, for details, to [21–24]). For a comprehensive review of the quantum (or q-)-calculus literature, we refer to a recently-published survey-cum-expository review article by Srivastava [25]. In this article, we will use the conic domain $\Omega_{k,q}$ and the quantum (or q-) calculus to define and investigate new subclasses of starlike functions with respect to symmetrical points in the open unit disk $\mathbb{E}$. We will investigate the Hankel determinant, the Toeplitz matrices and the Fekete-Szegö inequalities, and discuss some applications of the main results by using the q-Bernardi integral operator.

We first give some basic definitions of the quantum (or q-) calculus that will help us in the upcoming sections. We also provide some notations and concepts used in this investigation.

Definition 1. *Let $q \in (0,1)$ and the q-factorial $[n]_q!$ be defined as follows:*

$$[n]_q! = \begin{cases} 1 & (n = 0) \\ \prod_{k=1}^{n-1} [k]_q & (n \in \mathbb{N}). \end{cases} \tag{6}$$

Definition 2. *The generalized q-Pochhammer symbol $[t]_{n,q}$ $(t \in \mathbb{C})$ is defined as follows:*

$$[t]_{n,q} = \frac{(q^t, q)_n}{(1-q)n} = \begin{cases} 1 & (n = 0) \\ [t]_q [t+1]_q [t+2]_q \ldots [t+n-1]_q & (n \in \mathbb{N}). \end{cases}$$

Definition 3. *The q-Gamma function is defined as follows:*

$$\Gamma_q(t+1) = [t]_q \Gamma_q(t) \quad \text{and} \quad \Gamma_q(1) = 1 \quad (t > 0)\,.$$

Definition 4. *(see [26]) For $f \in \mathcal{A}$, the q-derivative operator or q-difference operator are defined as follows:*

$$D_q f(z) = \frac{f(z) - f(qz)}{(1-q)z} \quad (z \in \mathbb{E}). \tag{7}$$

From (1) and (7), we have

$$D_q f(z) = 1 + \sum_{n=2}^{\infty} [n]_q a_n z^{n-1}. \tag{8}$$

Moreover, for $n \in \mathbb{N}$ and $z \in \mathbb{E}$, we get

$$D_q z^n = [n]_q z^{n-1}, \quad D_q \left\{ \sum_{n=1}^{\infty} a_n z^n \right\} = \sum_{n=1}^{\infty} [n]_q a_n z^{n-1}.$$

When $q \to 1-$, the q-difference operator D_q approaches the ordinary differential operator:

$$\lim_{q \to 1-} \left(D_q f\right)(z) = f'(z).$$

Definition 5. *(see [8]) We say that a function $f \in \mathcal{A}$ belongs to the class $\mathcal{S}_q^*$ if*

$$f(0) = 1 = f'(0) \tag{9}$$

and

$$\left| \frac{z\left(D_q f\right)(z)}{f(z)} - \frac{1}{1-q} \right| \leqq \frac{1}{1-q}. \tag{10}$$

By applying the principle of subordination, the conditions (9) and (10) can be written as follows (see [27]):

$$\frac{z\left(D_q f\right)(z)}{f(z)} \prec \frac{1+z}{1-qz}.$$

Now, making use of the quantum (or q-) calculus and the principle of subordination, we define q-starlike and q-convex functions with respect to symmetrical points as follows.

Definition 6. *An analytic function f is said to be in the class $\mathcal{S}_s^*(q)$ if*

$$\left| \frac{2z\left(D_q f\right)(z)}{f(z)-f(-z)} - \frac{1}{1-q} \right| \leqq \frac{1}{1-q}. \tag{11}$$

By applying the principle of subordination, the condition (11) can be written as follows:

$$\frac{2z\left(D_q f\right)(z)}{f(z)-f(-z)} \prec \frac{1+z}{1-qz}.$$

Definition 7. *(see [9]) Let $k \in [0, \infty)$ and $q \in (0,1)$. A function p is said to be in the class k-$\mathcal{P}_q$ if and only if*

$$p(z) \prec p_{k,q}(z), \tag{12}$$

where

$$p_{k,q}(z) = 2p_k(z)\left\{(1+q) + (1-q)\, p_k(z)\right\}^{-1} \tag{13}$$

and $p_k(z)$ is given by (5).

Geometrically, a function $p \in k$-$\mathcal{P}_q$ takes on all values from the domain $\Omega_{k,q}$, which is defined as follows:

$$\Omega_{k,q} = \left\{ w : \Re\left(\frac{(1+q)\,w}{(q-1)\,w+2} \right) > k \left| \frac{(1+q)\,w}{(q-1)\,w+2} - 1 \right| \right\}.$$

Remark 1. *If $q \to 1-$, then $\Omega_{k,q} = \Omega_k$ is given by (3).*

Remark 2. *For $q \to 1-$, then k-$\mathcal{P}_q = \mathcal{P}(p_k)$, where $\mathcal{P}(p_k)$ is defined in [4].*

In the present investigation, by using the quantum (or q-) calculus and the general conic domain $\Omega_{k,q}$, we focus on the Hankel determinant, the Toeplitz matrices and the Fekete-Szegö problems for the function class $\mathcal{S}_s^*(q)$.

Definition 8. *An analytic function f is said to be in the class k-$\mathcal{S}_s^*(q)$ if*

$$\frac{2z\left(D_q f\right)(z)}{f(z)-f(-z)} \in k-\mathcal{P}_q$$

or, equivalently,

$$\Re\left(\frac{(1+q)\,\frac{2z\left(D_q f\right)(z)}{f(z)-f(-z)}}{(q-1)\,\frac{2z\left(D_q f\right)(z)}{f(z)-f(-z)}+2} \right) > k \left| \frac{(1+q)\,\frac{2z\left(D_q f\right)(z)}{f(z)-f(-z)}}{(q-1)\,\frac{2z\left(D_q f\right)(z)}{f(z)-f(-z)}+2} - 1 \right|. \tag{14}$$

Special Case:
For $k = 0$ and $q \to 1-$, then the class k-$\mathcal{S}_s^*(q)$ reduces to $\mathcal{S}_s^*$ (see [3]).

Let $n \in \mathbb{N}_0$ and $j \in \mathbb{N}$. The jth Hankel determinant was introduced and studied in [28]:

$$H_j(n) = \begin{vmatrix} a_n & a_{n+1} \cdots a_{n+j-1} \\ a_{n+1} & a_{n+2} \cdots a_{n+j-2} \\ \cdots & \cdots \quad \cdots \\ \cdots & \cdots \quad \cdots \\ a_{n+j-1} & a_{n+j-2} \cdots a_{n+2j-2} \end{vmatrix}$$

where $a_1 = 1$. Several authors have studied $H_j(n)$. In particular, sharp upper bounds on $H_2(2)$ were obtained in [29–31] for several classes. The Hankel determinant $H_2(1)$ represents a Fekete-Szegö functional $|a_3 - a_2^2|$. This functional has been further generalized as $|a_3 - \mu a_2^2|$ for some real or complex μ and also the functional $|a_2 a_4 - a_3^2|$ is equivalent to $H_2(2)$ [30]. Babalola [32] studied the Hankel determinant $H_3(1)$.

The symmetric Toeplitz determinant $T_j(n)$ is defined as follows:

$$T_j(n) = \begin{vmatrix} a_n & a_{n+1} \cdots a_{n+j-1} & \\ a_{n+1} & \cdots & \\ \vdots & \vdots & \vdots \\ a_{n+j-1} & \cdots & a_n \end{vmatrix},$$

so that

$$T_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_2 \end{vmatrix}, \quad T_2(3) = \begin{vmatrix} a_3 & a_4 \\ a_4 & a_3 \end{vmatrix}, \quad T_3(2) = \begin{vmatrix} a_2 & a_3 & a_4 \\ a_3 & a_2 & a_3 \\ a_4 & a_3 & a_2 \end{vmatrix},$$

and so on. The problem of finding the best possible bounds for $||a_{n+1}| - |a_n||$ has a long history (see [33]). It is known from [33] that

$$||a_{n+1}| - |a_n|| < c,$$

for a constant c.

Lemma 1. (see [31]) *If p is analytic in $\mathbb{E}$ and of the form (2), then*

$$2c_2 = c_1^2 + x(4 - c_1^2)$$

and

$$4c_3 = c_1^3 + 2(4 - c_1^2)c_1 x - (4 - c_1^2)c_1 x^2 + 2(4 - c_1^2)(1 - \left|x^2\right|)z$$

and, for some $x, z \in \mathbb{C}$, with $|z| \leqq 1$, and $|x| \leqq 1$.

Lemma 2. *(see also [34]) If p is analytic in $\mathbb{E}$ and of the form (2), and if $\mu \in \mathbb{C}$ $(1 \leqq k \leqq n-1)$, then*

$$|c_n - \mu c_k c_{n-k}| \leqq 2 \max\left(1, |2\mu - 1|\right).$$

Lemma 3. *(see [35]; see also [33]) If the function p given by (2) is analytic in $\mathbb{E}$, then*

$$|c_n| \leqq 2 \quad (n \in \mathbb{N}).$$

The above inequality is sharp for the function f given by

$$f(z) = \frac{1+z}{1-z}.$$

Lemma 4. *(see [35]) If p is analytic in $\mathbb{E}$ and of the form (2), then*

$$\left|c_2 - v c_1^2\right| \leqq \begin{cases} -4v + 2 & (v < 0) \\ 2 & (0 \leqq v \leqq 1) \\ 4v - 2 & (v > 1) \end{cases}$$

The equality holds true for the function p given by

$$p(z) = \frac{1+z}{1-z}$$

or by one of its rotations, when $v < 0$ or $v > 1$. In addition, the equality holds true for the function p given by

$$p(z) = \frac{1+z^2}{1-z^2}$$

or by one of its rotations, when $0 < v < 1$. if $v = 0$, the equality holds true if and only if

$$p(z) = \left(\frac{1}{2} + \frac{1}{2}\lambda\right)\frac{1+z}{1-z} + \left(\frac{1}{2} - \frac{1}{2}\lambda\right)\frac{1-z}{1+z}, \qquad (0 \leqq \lambda \leqq 1)$$

or one of its rotations. If $v = 1$, the equality holds true if and only if $p(z)$ is the reciprocal of one of the functions such that the equality holds true in the case when $v = 0$. In addition, the above upper bound is sharp and it can be improved as follows when:

$$\left|c_2 - vc_1^2\right| + v\left|c_1\right|^2 \leqq 2 \quad (0 < v \leqq \frac{1}{2})$$

and

$$\left|c_2 - vc_1^2\right| + (1-v)\left|c_1\right|^2 \leqq 2 \quad (\frac{1}{2} < v \leqq 1).$$

2. Main Results

Theorem 1. *Let the function f given by (1) belong to the class k-$\mathcal{S}_s^*(q)$. Then*

$$|a_2| \leqq \frac{1}{2}P_1,$$

$$|a_3| \leqq \frac{1}{2q}\left\{P_1 + \left|P_2 - P_1 + \left(\frac{(q-1)P_1^2}{2}\right)\right|\right\}$$

and

$$|a_4| \leqq \frac{1}{2(1+q^2)}\left\{P_1 + \left|2\left(P_2 - P_1\right) + \frac{(2q^2-2q+1)}{2q}P_1^2\right| + \left|P_3 + P_1 - 2P_2 - \frac{(2q^2-2q+1)}{2q}\left(P_1^2 + P_1P_2\right) + \frac{(q-1)\left(q^2-q+1\right)}{4q}\,P_1^3\right|\right\}.$$

Proof. For $f \in k$-$\mathcal{S}_s^*(q)$, we have

$$\frac{2z\left(D_q f\right)(z)}{f(z) - f(-z)} = h(z) \prec \mathcal{H}_k(z), \tag{15}$$

where

$$\mathcal{H}_k(z) = 2p_k(z)\left[(1+q) + (1-q)\,p_k(z)\right]^{-1}$$

and $p_k(z)$ is given by (5).

The function $p(z)$ with $p(0) = 1$ is given as follows:

$$p(z) = \frac{1 + \mathcal{H}_k^{-1}(h(z))}{1 - \mathcal{H}_k^{-1}(h(z))} = 1 + c_1 z + c_2 z^2 + \cdots. \tag{16}$$

After some computation involving (16), we have

$$h(z) = \mathcal{H}_k\left(\frac{p(z)+1}{p(z)-1}\right).$$

Therefore, we find that

$$\mathcal{H}_k\left(\frac{p(z)+1}{p(z)-1}\right) = 1+\left(\frac{q+1}{2}\right)\left[\frac{P_1c_1}{2}z+\left\{\frac{P_1c_2}{2}+\frac{1}{4}\left(P_2-P_1+\frac{(q-1)P_1^2}{2}\right)c_1^2\right\}z^2\right.$$
$$+\left\{\frac{P_1c_3}{2}+\left(\frac{P_2}{2}-\frac{P_1}{2}+\frac{(q-1)P_1^2}{4}\right)c_1c_2+\left(\frac{P_1}{8}-\frac{P_2}{4}\right.\right.$$
$$\left.\left.\left.-\frac{(q-1)P_1^2}{8}+\frac{P_3}{8}-\frac{1}{8}(q-1)P_1P_2-\frac{1}{32}(q-1)^2P_1^3\right)c_1^3\right\}z^3\right]. \tag{17}$$

We also have

$$\frac{2z\left(D_qf\right)(z)}{f(z)-f(-z)} = 1+[2]_q\,a_2z+q\,[2]_q\,a_3z^2+\left\{[4]_q\,a_4-[2]_q\,a_2a_3\right\}z^3+\cdots. \tag{18}$$

Comparing the corresponding coefficients in (17) and (18) along with Lemma 3, we obtain the required result. □

Theorem 2. *Let the analytic function $f\in\mathcal{A}$ be in the class k-$\mathcal{S}_s^*(q)$. Then*

$$T_3(2) \leqq \left[\frac{P_1}{2}+\frac{1}{2\left(1+q^2\right)}\left(\Omega_1+\Omega_2\right)\right]\left[\frac{P_1^2}{4}+16\left|\Omega_3\right|+\frac{P_1^2}{2q^2}+2P_1^2\Omega_5\left|2-\frac{\Omega_4}{\Omega_5P_1^2}\right|\right],$$

where

$$\Omega_1 = P_1+\left|2P_2-2P_1+\Omega_6P_1^2\right|,$$

and

$$\Omega_2 = \left|P_3+P_1-2P_2-\Omega_6P_1^2+\Omega_7P_1P_2+(q-1)\,\Omega_8P_1^3\right|.$$

Furthermore, we have

$$\Omega_3 = 2P_1\Omega_5\left(\frac{P_3}{8}+\frac{P_1}{8}-\frac{P_2}{4}-\frac{1}{8}\Omega_6P_1^2+\frac{1}{8}\Omega_7P_1P_2+\frac{1}{8}(q-1)\,\Omega_8P_1^3\right)$$
$$-\frac{1}{2q^2}\left(\frac{P_2}{4}-\frac{P_1}{4}+(q-1)\,\frac{P_1^2}{8}\right)^2,$$

$$\Omega_4 = \frac{P_1}{2q^2}\left(\frac{P_2}{4}-\frac{P_1}{4}+(q-1)\,\frac{P_1^2}{8}\right)-2P_1\Omega_5\left(\frac{P_2}{2}-\frac{P_1}{2}-\frac{1}{4}\,\Omega_6P_1^2\right),$$

$$\Omega_5 = \frac{1}{16\left(1+q^2\right)},$$

$$\Omega_6 = \left(\frac{2q^2-2q+1}{2q}\right),$$

$$\Omega_7 = \left(\frac{1-2q^2+2q}{2q}\right)$$

and

$$\Omega_8 = \left(\frac{q^2 - q + 1}{4q}\right).$$

Here P_1 and P_2 are given in (5).

Proof. By comparison of coefficients in (17) and (18), we can obtain

$$a_2 = \frac{1}{4} P_1 c_1 \tag{19}$$

$$a_3 = \frac{1}{2q}\left\{\frac{1}{2} P_1 c_2 + \left(\frac{P_2}{4} - \frac{P_1}{4} + \left(\frac{(q-1)P_1^2}{8}\right) c_1^2\right)\right\} \tag{20}$$

$$\begin{aligned} a_4 = \frac{1}{2(1+q^2)} & \left[\frac{1}{2} P_1 c_3 + \left(\frac{P_2 - P_1}{2} + \frac{(2q^2 - 2q + 1)}{8q} P_1^2\right) c_1 c_2\right. \\ & + \left\{\frac{1}{8}\left(P_3 + P_1 - 2P_2 - \frac{1}{2q}\left(2q^2 - 2q + 1\right) P_1^2\right)\right. \\ & \left. + \frac{(1 - 2q^2 + 2q)}{16q} P_1 P_2 + \frac{(q-1)\left(q^2 - q + 1\right)}{32q} P_1^3 \right\} c_1^3. \end{aligned} \tag{21}$$

A detailed calculation for $T_3(2)$ yields

$$T_3(2) = (a_2 - a_4)\left(a_2^2 - 2a_3^2 + a_2 a_4\right).$$

Now, if $f \in k\text{-}\mathcal{S}_s^*(q)$, then we have

$$|a_2 - a_4| \leqq |a_2| + |a_4|,$$

$$|a_2 - a_4| \leqq \frac{1}{2} P_1 + \frac{1}{2\left(1+q^2\right)}\left(\Omega_1 + \Omega_2\right). \tag{22}$$

We need to maximize $\left|a_2^2 - 2a_3^2 + a_2 a_4\right|$ for $f \in k\text{-}\mathcal{S}_s^*(q)$. Thus, by writing a_2, a_3, a_4 in terms of c_1, c_2, c_3, with the help of (19) and (21), we get

$$\left|a_2^2 - 2a_3^2 + a_2 a_4\right| \leqq \left|\frac{P_1^2 c_1^2}{4} + \Omega_3 c_1^4 - \Omega_4 c_1^2 c_2 - \frac{P_1^2 c_2^2}{8q^2} + \Omega_5 P_1^2 c_1 c_3\right|. \tag{23}$$

Finally, applying the triangle inequality, Lemma 2 and Lemma 3 along with (22) and (23), we obtained the required result. □

Theorem 3. *If an analytic function $f \in \mathcal{A}$ is in the class $k\text{-}\mathcal{S}_s^*(q)$, then*

$$\left|a_2 a_4 - a_3^2\right| \leqq \frac{p_1^2}{4q^2}.$$

Proof. Making use of (19), (20) and (21), we have

$$a_2 a_4 - a_3^2 = \lambda_1 c_1 c_3 + \lambda_2 c_1^2 c_2 - \lambda_3 c_2^2 + \lambda_4 c_1^4, \tag{24}$$

where

$$\lambda_1 = \frac{p_1^2}{16\left(1+q^2\right)},$$

$$\lambda_2 = \left(\frac{1}{16q^2\left(1+q^2\right)}\right) p_1^2 - \left(\frac{1}{16q^2\left(1+q^2\right)}\right) p_1 p_2 + \left(\frac{2-q}{64q^2\left(1+q^2\right)}\right) p_1^3,$$

$$\lambda_3 = \frac{p_1^2}{16q^2}$$

and

$$\lambda_4 = \left(\frac{1}{16\left(1+q^2\right)}\right) p_1 p_3 - \left(\frac{1}{64q^2\left(1+q^2\right)}\right) p_1^2 + \left(\frac{1}{32q^2\left(1+q^2\right)}\right) p_1 p_2 + \left(\frac{q-2}{128q^2\left(1+q^2\right)}\right) p_1^3$$

$$+ \left(\frac{4q^2-4q^3-q+2}{128q^2\left(1+q^2\right)}\right) p_1^2 p_2 + \left(\frac{q-1}{256q^2\left(1+q^2\right)}\right) p_1^4 - \left(\frac{1}{64q^2}\right) p_2^2.$$

By using Lemma 1, we take

$$\mathrm{Y} = 4 - c_1^2 \qquad \text{and} \qquad \mathrm{Z} = \left(1-|x|^2\right) z.$$

Without loss of generality, we assume that $c = c_1$ $(0 \leqq c \leqq 2)$, so that

$$a_2 a_4 - a_3^2 = \frac{1}{4}\left(\lambda_1 + 2\lambda_2 - \lambda_3 + 4\lambda_4\right) c^4 + \frac{1}{2}\left(\lambda_1 + \lambda_2 - \lambda_3\right) \mathrm{Y} c^2 x$$

$$- \frac{\lambda_1}{4} \mathrm{Y} c^2 x^2 - \frac{\lambda_3}{4} \mathrm{Y}^2 x^2 + \frac{\lambda_1}{2} c \mathrm{YZ}. \tag{25}$$

Taking the moduli on both sides of (25) and using the triangle inequality, we find that

$$\left|a_2 a_4 - a_3^2\right| \leqq \left|\frac{1}{4}\left(\lambda_1 + 2\lambda_2 - \lambda_3 + 4\lambda_4\right)\right| c^4 + \left|\frac{1}{2}\left(\lambda_1 + \lambda_2 - \lambda_3\right)\right| \mathrm{Y} c^2 |x|$$

$$+ \left|\frac{\lambda_1}{4}\right| \mathrm{Y} c^2 |x|^2 + \left|\frac{\lambda_3}{4}\right| \mathrm{Y}^2 |x|^2 + \left|\frac{\lambda_1}{2}\right| \left(1-|x|^2\right) c\mathrm{Y}.$$

This can be written as follows:

$$\left|a_2 a_4 - a_3^2\right| \leqq |A_\lambda| c^4 + |B_\lambda| |x| \mathrm{Y} c^2 + \left|\frac{\lambda_1}{4}\right| |x|^2 \mathrm{Y} c^2 + \left|\frac{\lambda_3}{4}\right| |x|^2 \mathrm{Y}^2 + \left|\frac{\lambda_1}{2}\right| \left(1-|x|^2\right) \mathrm{Y} c$$

$$= \mathcal{G}(|x|), \tag{26}$$

where

$$A_\lambda = \frac{1}{4}\left(\lambda_1 + 2\lambda_2 - \lambda_3 + 4\lambda_4\right)$$

$$B_\lambda = \frac{1}{2}\left(\lambda_1 + \lambda_2 - \lambda_3\right).$$

Now, trivially, we have

$$\mathcal{G}'(c, |x|) > 0$$

on the closed interval $[0, 1]$, which shows that $\mathcal{G}(c, |x|)$ is an increasing function in the interval $[0, 1]$. Therefore, the maximum value occurs at $x = 1$ and we have

$$\max\{\mathcal{G}(c, |1|)\} = \mathcal{G}(c),$$

$$\mathcal{G}(c, |1|) = |A_\lambda| c^4 + |B_\lambda| \Upsilon c^2 + \left|\frac{\lambda_1}{4}\right| \Upsilon c^2 + \left|\frac{\lambda_3}{4}\right| \Upsilon^2$$

and

$$\mathcal{G}(c) = |A_\lambda| c^4 + |B_\lambda| \Upsilon c^2 + \left|\frac{\lambda_1}{4}\right| \Upsilon c^2 + \left|\frac{\lambda_3}{4}\right| \Upsilon^2.$$

Hence, by putting $\Upsilon = 4 - c_1^2$ and after some simplification, we have

$$\mathcal{G}(c) = \left(|A_\lambda| - |B_\lambda| - \left|\frac{\lambda_1}{4}\right| + \left|\frac{\lambda_3}{4}\right|\right) c^4 + 4\left(|B_\lambda| + \left|\frac{\lambda_1}{4}\right| - \left|\frac{\lambda_3}{2}\right|\right) c^2 + 4\,|\lambda_3|.$$

We consider $\mathcal{G}'(c) = 0$, for the optimum value of $\mathcal{G}(c)$, which implies that $c = 0$. Thus, $\mathcal{G}(c)$ has a maximum value at $c = 0$. Hence, the maximum value of $\mathcal{G}(c)$ is given by

$$\max\{\mathcal{G}(c)\} = 4\,|\lambda_3|, \tag{27}$$

which occurs at $c = 0$ or

$$c^2 = \frac{4\left(|B_\lambda| + \left|\frac{\lambda_1}{4}\right| - \left|\frac{\lambda_3}{2}\right|\right)}{|A_\lambda| - |B_\lambda| - \left|\frac{\lambda_1}{4}\right| + \left|\frac{\lambda_3}{4}\right|}.$$

Hence, by putting

$$\lambda_3 = \frac{p_1^2}{16q^2}$$

in (27) and after some simplification, we obtain the desired result. □

For $q \to 1-$, $k = 0$, and $p_1 = 2$ in Theorem 3, we have the following known result for the class $\mathcal{S}_s^*$.

Corollary 1. *(see [36]) If an analytic function $f \in \mathcal{A}$ that belongs to the class $\mathcal{S}^*$, then*

$$\left|a_2 a_4 - a_3^2\right| \leqq 1.$$

2.1. The Fekete-Szegö Problem

Theorem 4. *Let the function $f \in \mathcal{A}$ given by (1) belong to the class k-$\mathcal{S}_s^*(q)$. Then*

$$\left|a_3 - \mu a_2^2\right| \leqq \begin{cases} \frac{1}{4q}\left(2P_2 + (q(1-\mu) - 1)P_1^2\right) & (\mu \leqq \delta_1) \\ \frac{P_1}{2q} & (\delta_1 \leqq \mu \leqq \delta_2) \\ -\frac{1}{4q}\left(2P_2 + (q(1-\mu) - 1)P_1^2\right) & (\mu \leqq \delta_1), \end{cases}$$

where

$$\delta_1 = \frac{2P_2 + P_1\left[(q-1)P_1 - 2\right]}{qP_1^2},$$

$$\delta_2 = \frac{2P_2 + P_1\left[(q-1)P_1 + 2\right]}{qP_1^2}.$$

Proof. From (19) and (20), we have

$$a_3 - \mu a_2^2 = \frac{P_1}{4q}\left(c_2 - vc_1^2\right),$$

where

$$v = \frac{1}{2}\left(1 - \frac{P_2}{P_1} - \frac{(q-1)P_1}{2} + \frac{\mu q P_1}{2}\right). \tag{28}$$

By applying the triangle inequality and Lemma 4, we obtain Theorem 4. □

If we set $k = 0$ and $q \to 1-$ in Theorem 4, we thus obtain the following known result.

Corollary 2. *(see [37]) If an analytic function $f \in \mathcal{S}_s^*(\phi)$, then*

$$\left|a_3 - \mu a_2^2\right| \leqq \begin{cases} \frac{1}{2}(P_2 - \frac{\mu}{2}P_1^2) & (\mu \leqq \delta_1) \\ \frac{P_1}{2} & (\delta_1 \leqq \mu \leqq \delta_2) \\ -\frac{1}{2}(P_2 - \frac{\mu}{2}P_1^2) & (\mu \leqq \delta_1), \end{cases}$$

where

$$\delta_1 = \frac{2(P_2 - P_1)}{P_1^2},$$

$$\delta_2 = \frac{2(P_2 + P_1)}{P_1^2}.$$

Let $\delta_1 \leqq \mu \leqq \delta_2$. Then, in view of Lemma 4, Theorem 4 can be improved as follows.

Theorem 5. *If the function f given by (1) belongs to the class $\mathcal{S}_s^*(q)$ and if*

$$\delta_1 \leqq \mu \leqq \delta_3 = \frac{2P_2}{qP_1},$$

then

$$\left|a_3 - \mu a_2^2\right| + \frac{1}{qP_1^2}\left(2(P_2 - P_1) - (q-1)P_1^2 + \mu q P_1^2\right)|a_2|^2 \leqq \frac{P_1}{2q}.$$

Furthermore, if $\delta_3 \leqq \mu \leqq \delta_2$*, then*

$$\left|a_3 - \mu a_2^2\right| + \frac{1}{qP_1^2}\left(2\left(P_2 + P_1\right) - (q-1)P_1^2 - \mu q P_1^2\right)|a_2|^2 \leqq \frac{P_1}{2q}.$$

If we set $k = 0$ and $q \to 1-$, we obtain the following known result.

Corollary 3. *(see [37]) If an analytic function* $f \in \mathcal{S}_s^*(\phi)$ *and if*

$$\delta_1 \leqq \mu \leqq \delta_3 = \frac{2P_2}{P_1^2},$$

then

$$\left|a_3 - \mu a_2^2\right| + \frac{1}{qP_1^2}\left(2\left(P_2 - P_1\right) + \mu P_1^2\right)|a_2|^2 \leqq \frac{P_1}{2}.$$

Moreover, if

$$\delta_3 \leqq \mu \leqq \delta_2,$$

then

$$\left|a_3 - \mu a_2^2\right| + \frac{1}{P_1^2}\left(2\left(P_2 + P_1\right) - \mu P_1^2\right)|a_2|^2 \leqq \frac{P_1}{2}.$$

2.2. Applications of the Main Results

In this section, firstly we recall that the Bernardi integral operator F_β given in [38] as follows:

$$F_\beta(f(z)) = \frac{1+\beta}{z^\beta}\int_0^z t^{\beta-1}f(t)dt \qquad (f \in \mathcal{A}, \beta > -1).$$

The q-integral of the function f on $[0, z]$ is defined as follows (see, for example [39]):

$$\int_0^z f(t)d_qt = (1-q)\,z\sum_{k=0}^{\infty} q^k f(q^k z),$$

and q-integral of the function z^n is given by

$$\int_0^z z^n d_qt = \frac{z^{n+1}}{[n+1]_q}, \tag{29}$$

where $n \neq -1$ and for $q \to 1-$, Equation (29) becomes

$$\int_0^z h_1(t)dt = \frac{z^{n+1}}{n+1}.$$

Noor [39] introduced the q-Bernardi integral operator $\mathcal{B}_q(z)$ as follows:

$$\mathcal{B}_q(z) = F_\beta(f(z)) = \frac{[1+\beta]_q}{z^\beta}\int_0^z t^{\beta-1}f(t)d_qt \quad (\beta > -1). \tag{30}$$

Let $f \in \mathcal{A}$. Then, by using Equations (29) and (8), we obtain the following power series for the function $\mathcal{B}_q(z)$ in the open unit disk $\mathbb{E}$ as follows:

$$\mathcal{B}_q(z) = z + \sum_{n=2}^{\infty}\frac{[1+\beta]_q}{[n+\beta]_q}a_n z^n. \tag{31}$$

Clearly, $\mathcal{B}_q(z)$ is analytic in the open unit disk $\mathbb{E}$.

Let

$$\mathcal{B}_n = \frac{[1+\beta]_q}{[n+\beta]_q} \qquad (n \geqq 1). \tag{32}$$

Applying Theorem 1 on Equation (31), we obtain the following result.

Theorem 6. *If the function $\mathcal{B}_q(z)$ given by (31) belongs to the class k-$\mathcal{S}_s^*(q)$, where $k \in [0,1]$, then*

$$|a_2| \leqq \frac{1}{2\mathcal{B}_2} P_1,$$

$$|a_3| \leqq \frac{(1+q)}{2\left([3]_q \mathcal{B}_3 - 1\right)} \left\{ P_1 + \left| P_2 - P_1 + \left(\frac{(q-1)P_1^2}{2} \right) \right| \right\}$$

and

$$\begin{aligned} |a_4| \leqq \frac{(1+q)}{2[4]_q \mathcal{B}_4} \Bigg[P_1 &+ \left| 2(P_2 - P_1) + \frac{2\left([3]_q \mathcal{B}_3 - 1\right)(q-1) + (1+q)}{2\left([3]_q \mathcal{B}_3 - 1\right)} P_1^2 \right| \\ &+ \left| P_3 + P_1 - 2P_2 - \left(\frac{2\left([3]_q \mathcal{B}_3 - 1\right)(q-1) + (1+q)}{2\left([3]_q \mathcal{B}_3 - 1\right)} \right) \left(P_1^2 + P_1 P_2\right) \right. \\ &\left. + \frac{(q-1)}{4} \left(\frac{(q-1)\left([3]_q \mathcal{B}_3 - 1\right) + (1+q)}{\left([3]_q \mathcal{B}_3 - 1\right)} \right) P_1^3 \right| \Bigg], \end{aligned}$$

where $\mathcal{B}_2$, $\mathcal{B}_3$ and $\mathcal{B}_4$ are given in (32).

Applying Theorem 2 to Equation (31), we obtain the following result.

Theorem 7. *If the function $\mathcal{B}_q(z)$ given by (31) belongs to the class k-$\mathcal{S}_s^*(q)$, then*

$$\begin{aligned} T_3(2) \leqq \left\{ \frac{P_1}{2\mathcal{B}_2} + \frac{1+q}{2[4]_q \mathcal{B}_4} (\Omega_{10} + \Omega_{11}) \right\} \Bigg\{ & \frac{P_1^2}{4\mathcal{B}_2^2} + 16|\Omega_{12}| + \frac{(1+q)^2 P_1^2}{2\left([3]_q \mathcal{B}_3 - 1\right)^2} \\ & + 2P_1^2 \Omega_{14} \left| 2 - \frac{\Omega_{13}}{P_1^2 \Omega_{14}} \right| \Bigg\}, \end{aligned}$$

where

$$\Omega_{10} = P_1 + \left|2P_2 - 2P_1 + \Omega_{15} P_1^2\right|,$$

$$\Omega_{11} = \left|P_3 + P_1 - 2P_2 + \Omega_{15} P_1^2 + \Omega_{16} P_1 P_2 + (q-1)\,\Omega_{17} P_1^3\right|,$$

$$\Omega_{12} = 2P_1 \Omega_{14} \left(\frac{P_3}{8} + \frac{P_1}{8} - \frac{P_2}{4} - \frac{1}{8}\Omega_{15} P_1^2 + \frac{1}{8}\Omega_{16} P_1 P_2 + \frac{1}{8}(q-1)\,\Omega_{17} P_1^3\right) - \frac{(1+q)^2}{2\left([3]_q \mathcal{B}_3 - 1\right)^2} \left(\frac{P_2}{4} - \frac{P_1}{4} + (q-1)\frac{P_1^2}{8}\right)^2,$$

$$\Omega_{13} = \frac{(1+q)^2 P_1}{2\left([3]_q \mathcal{B}_3 - 1\right)^2} \left(\frac{P_2}{4} - \frac{P_1}{4} + (q-1)\frac{P_1^2}{8}\right) + 2P_1 \Omega_{14} \left(\frac{P_2}{2} - \frac{P_1}{2} + \frac{1}{4}\Omega_{15} P_1^2\right),$$

$$\Omega_{14} = \frac{(1+q)}{16 \mathcal{B}_2 \mathcal{B}_4 [4]_q},$$

$$\Omega_{15} = \left(\frac{2\left([3]_q \mathcal{B}_3 - 1\right)(q-1) + (1+q)}{2\left([3]_q \mathcal{B}_3 - 1\right)}\right),$$

$$\Omega_{16} = \left(\frac{(1+q) - 2\left([3]_q \mathcal{B}_3 - 1\right)(q-1)}{2\left([3]_q \mathcal{B}_3 - 1\right)}\right),$$

$$\Omega_{17} = \left(\frac{\left([3]_q \mathcal{B}_3 - 1\right)(q-1) + (1+q)}{4\left([3]_q \mathcal{B}_3 - 1\right)}\right),$$

where P_1 and P_2 are given in (5).

Applying Theorem 3 to Equation (31), we obtain the following result.

Theorem 8. *If the function $\mathcal{B}_q(z)$ given by (31) belongs to the class k-$\mathcal{S}_s^*(q)$, then*

$$\left|a_2 a_4 - a_3^2\right| \leqq \frac{(1+q)^2 p_1^2}{4\left([3]_q \mathcal{B}_3 - 1\right)^2}.$$

For $q \to 1-$, $k = 0$, $\beta = 0$ and $p_1 = 2$ in Theorem (8), we have the following known result for the class $\mathcal{S}_s^*$.

Corollary 4. *(see [36]) Let $f \in \mathcal{S}_s^*$ be of the form (1). Then*

$$\left|a_2 a_4 - a_3^2\right| \leqq 1.$$

Theorem 9. *If the function $\mathcal{B}_q(z)$ given by (31) belongs to the class k-$\mathcal{S}_s^*(q)$, then*

$$\left|a_3-\mu a_2^2\right| \leqq \begin{cases} \frac{(1+q)}{4\left([3]_q\mathcal{B}_3-1\right)}\left(2P_2+\left((q-1)-\mu\frac{\left([3]_q\mathcal{B}_3-1\right)}{(1+q)\mathcal{B}_2^2}\right)P_1^2\right) & (\mu \leqq \delta_1) \\ \frac{(1+q)P_1}{4\left([3]_q\mathcal{B}_3-1\right)}, & (\delta_1 \leqq \mu \leqq \delta_2) \\ -\frac{(1+q)P_1}{4\left([3]_q\mathcal{B}_3-1\right)}\left(2P_2+\left((q-1)-\mu\frac{\left([3]_q\mathcal{B}_3-1\right)}{(1+q)\mathcal{B}_2^2}\right)P_1^2\right) & (\mu \leqq \delta_1), \end{cases}$$

where

$$\delta_1 = \frac{(1+q)\,\mathcal{B}_2^2}{\left([3]_q\,\mathcal{B}_3-1\right)}\left(\frac{2P_2+P_1\left((q-1)\,P_1-2\right)}{P_1^2}\right)$$

and

$$\delta_2 = \frac{(1+q)\,\mathcal{B}_2^2}{\left([3]_q\,\mathcal{B}_3-1\right)}\left(\frac{2P_2+P_1\left((q-1)\,P_1+2\right)}{P_1^2}\right).$$

If we set $k=0$, $\beta=0$ and $q\to 1-$ in Theorem 9, we obtain the following known result.

Corollary 5. *(see [37]) If an analytic function $f\in\mathcal{A}$ belongs to the class $\mathcal{S}_s^*(\phi)$, then*

$$\left|a_3-\mu a_2^2\right| \leqq \begin{cases} \frac{1}{2}(P_2-\frac{\mu}{2}P_1^2) & (\mu \leqq \delta_1) \\ \frac{P_1}{2} & (\delta_1 \leqq \mu \leqq \delta_2) \\ -\frac{1}{2}\left(P_2-\frac{\mu}{2}P_1^2\right) & (\mu \geqq \delta_2), \end{cases}$$

where

$$\delta_1 = \frac{2\,(P_2-P_1)}{P_1^2}$$

and

$$\delta_2 = \frac{2\,(P_2+P_1)}{P_1^2}.$$

3. Conclusions

We have made use of the general conic domain $\Omega_{k,q}$ and the quantum (or q-) calculus to introduce and investigate several new subclasses of q-starlike functions with respect to symmetrical points in open unit disk $\mathbb{E}$. We have studied some interesting results such as the Hankel determinant, the Toeplitz matrices, and the Fekete-Szegö inequalities. We have also discussed some applications of our main results by using a q-Bernardi integral operator.

For further investigation, we can easily follow a known relationship between the q-analysis and (p,q)-analysis (see [25] (p. 340, Equations (9.1), (9.2) and (9.3))) and the results for the q-analogues, which we have included in this paper for $0<q<1$, can then be easily transformed into the related results for the (p,q)-analogues with $0<q<p\leqq 1$ by adding a rather redundant (or superfluous) parameter p (see, for details [25] (p. 340)).

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