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Existence and Stability Analysis for Fractional Impulsive Caputo Difference-Sum Equations with Periodic Boundary Condition

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Abstract: In this paper, by using the Banach contraction principle and the Schauder's fixed point theorem, we investigate existence results for a fractional impulsive sum-difference equations with periodic boundary conditions. Moreover, we also establish different kinds of Ulam stability for this problem. An example is also constructed to demonstrate the importance of these results.

Keywords: existence; impulse; Ulam–Hyers stability; fractional Caputo difference equations; boundary value problem

JEL Classification: 39A05; 39A12

1. Introduction

In this paper, we study the following periodic boundary value problem for fractional impulsive difference-sum equations:

$$\begin{aligned} &\Delta^{\alpha}_{C}u(t) = F\Big[t+\alpha-1, u(t+\alpha-1), \Psi^{\gamma}u(t+\alpha-1)\Big], \;\; t\in\mathbb{N}_{0,T},\; t+\alpha-1\neq t_k \\ &Au(\alpha-1)+B\Delta^{-\beta}u(\alpha+\beta-1) = Cu(T+\alpha)+D\Delta^{-\beta}u(T+\alpha+\beta) \end{aligned} \tag{1}$$

where the impluse conditions subjected to $I_k, J_k : \mathbb{R}\to\mathbb{R}$ are given by

$$\begin{aligned} &\Delta u(t_k) = I_k\left(u(t_k-1)\right), \;\; k=1,2,\ldots,p \\ &\Delta\left(\Delta^{-\beta}u(t_k+\beta)\right) = J_k\left(\Delta^{-\beta}u(t_k+\beta-1)\right), \;\; k=1,2,\ldots,p \end{aligned} \tag{2}$$

and $\alpha,\beta\in(0,1)$; $\mathbb{N}_{0,T} := \{0,1,\ldots,T\}$; $t_k\in\mathbb{N}_{\alpha,T+\alpha-1}$ and $t_0=\alpha-1<t_1<t_2<\ldots<t_p<T+\alpha$, $t_{k+1}-t_k\geq 2$; $\Delta u(t_k) = u(t_k+1)-u(t_k)$; $\Delta\left(\Delta^{-\beta}u(t_k+\beta)\right) = \Delta^{-\beta}u(t_k+\beta+1)-\Delta^{-\beta}u(t_k+\beta)$; $A,B,C,D\in\mathbb{R}$; $F:\mathbb{N}_{\alpha-1,T+\alpha}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is a continuous function ; and for $\varphi:\mathbb{N}_{\alpha-1,T+\alpha}\times\mathbb{N}_{\alpha-1,T+\alpha}\to[0,\infty)$,

$$(\Psi^{\gamma}u)(t) := [\Delta^{-\gamma}\varphi\, u](t+\gamma) = \frac{1}{\Gamma(\gamma)}\sum_{s=\alpha-\gamma-1}^{t-\gamma}(t-\sigma(s))^{\underline{\gamma-1}}\varphi(t,s+\gamma)\,u(s+\gamma).$$

Fractional calculus has been gaining more attention over the past decade as this calculus has been addressed to various problems used in science and engineering; see [1–8]. For fractional difference calculus theory, which is the discrete case of fractional calculus, there is still not much interest among mathematical researchers. Basic knowledge of fractional difference calculus can be found in [9]. Some interesting results on fractional difference calculus have been studied, which has helped to develop the basic theory of this calculus; see [10–40] and references cited therein. The extension of applications of fractional difference calculus; see [41–43] and references cited therein.

Currently, the studies of boundary value problems for fractional difference equations are shown both extensive and more complex of conditions. There are some recent papers to study the existence and stability results of fractional difference equation [44–51]. However, a few paper has been made to develop the theory of the existence and stability results of fractional difference equations with impulse [52,53].

These results are the motivation for this research. By using the Banach contraction principle and the Schauder's fixed point theorem, we aim to prove the existence results for the problem (1) and (2). Moreover, we also provide a condition for the different kinds of Ulam stability for the problem (1) and (2). Then, we present an illustrative example.

2. Preliminaries

In this section, we recall some notations, definitions and lemmas which will be used in the main results.

Definition 1. *The generalized falling function is defined as*

$$t^{\underline{\alpha}} := \frac{\Gamma(t+1)}{\Gamma(t+1-\alpha)},$$

when $t+1-\alpha$ is not a pole of the Gamma function. If $t+1$ is not a pole and when $t+1-\alpha$ is a pole, then $t^{\underline{\alpha}} = 0$.

Definition 2. *For $\alpha > 0$ and f defined on $\mathbb{N}_a := \{a, a+1, \ldots\}$, the fractional sum of order α of f is defined as*

$$\Delta^{-\alpha} f(t) := \frac{1}{\Gamma(\alpha)} \sum_{s=a}^{t-\alpha} (t-\sigma(s))^{\underline{\alpha-1}} f(s),$$

where $t \in \mathbb{N}_{a+\alpha}$ and $\sigma(s) = s+1$.

Definition 3. *For $\alpha > 0$, $N \in \mathbb{N}$ is satisfied with $\alpha \in (N-1, N]$ and f defined on $\mathbb{N}_a$, the Riemann-Liouville fractional delta difference of order α of f is defined as*

$$\Delta^{\alpha} f(t) := \Delta^N \Delta^{-(N-\alpha)} f(t) = \frac{1}{\Gamma(-\alpha)} \sum_{s=a}^{t+\alpha} (t-\sigma(s))^{\underline{-\alpha-1}} f(s),$$

where $t \in \mathbb{N}_{a+N-\alpha}$. The Caputo fractional difference of order α of f is defined as

$$\Delta_C^{\alpha} f(t) := \Delta^{-(N-\alpha)} \Delta^N f(t) = \frac{1}{\Gamma(N-\alpha)} \sum_{s=a}^{t-(N-\alpha)} (t-\sigma(s))^{\underline{N-\alpha-1}} \Delta^N f(s),$$

where $t \in \mathbb{N}_{a+N-\alpha}$. If $\alpha = N$, then $\Delta^{\alpha} f(t) = \Delta_C^{\alpha} f(t) = \Delta^N f(t)$.

Lemma 1 ([10]). *Let $\alpha > 0$ and $\alpha \in (N-1, N]$. Then*

$$\Delta^{-\alpha} \Delta_C^{\alpha} y(t) = y(t) + C_0 + C_1 t^{\underline{1}} + C_2 t^{\underline{2}} + \ldots + C_{N-1} t^{\underline{N-1}},$$

for some $C_i \in \mathbb{R}$, $0 \le i \le N-1$.

Now, we aim to study the following linear variant of the boundary value problem (1) and (2).

Lemma 2. *Let* $\Lambda \neq 0$; $\alpha, \beta \in (0,1)$; $A, B, C, D \in \mathbb{R}$; $t_0 = \alpha - 1 < t_1 < t_2 < \ldots < t_p < T+\alpha$; *for* $k = 1, 2, \ldots, p$, $t_{k+1} - t_k \ge 2$ *and* $t_k \in \mathbb{N}_{\alpha, T+\alpha-1}$; $I_k, J_k : \mathbb{R} \to \mathbb{R}$ *and* $h : \mathbb{N}_{\alpha-1, T+\alpha} \to \mathbb{R}$ *be continuous. Then the following problem*

$$\begin{aligned} &\Delta_C^\alpha u(t) = h(t+\alpha-1), \ \ t \in \mathbb{N}_{0,T}, \ t+\alpha-1 \neq t_k \\ &Au(\alpha-1) + B\Delta^{-\beta} u(\alpha+\beta-1) = Cu(T+\alpha) + D\Delta^{-\beta} u(T+\alpha+\beta) \end{aligned} \tag{3}$$

where the impluse conditions subjected to I_k, J_k *are given by*

$$\begin{aligned} &\Delta u(t_k) = I_k\left(u(t_k-1)\right), \ \ k = 1, 2, \ldots, p \\ &\Delta\left(\Delta^{-\beta} u(t_k+\beta)\right) = J_k\left(\Delta^{-\beta} u(t_k+\beta-1)\right), \ \ k = 1, 2, \ldots, p \end{aligned} \tag{4}$$

has the unique solution which is in a form

$$u(t) = \begin{cases} \dfrac{1}{\Lambda}\Big[C\displaystyle\sum_{i=1}^{p} I_i\left(u(t_i-1)\right) + D\sum_{i=1}^{p} J_i\left(\Delta^{-\beta} u(t_i+\beta-1)\right) \\ \quad + C\Phi_1[h] + D\Phi_2[h]\Big] + \dfrac{1}{\Gamma(\alpha)}\displaystyle\sum_{s=t_0}^{t-1}(t-s+\alpha-2)^{\underline{\alpha-1}} h(s), \quad t \in \mathbb{N}_{t_0, t_1} \\ \\ \dfrac{1}{\Lambda}\Big[C\displaystyle\sum_{i=1}^{p} I_i\left(u(t_i-1)\right) + D\sum_{i=1}^{p} J_i\left(\Delta^{-\beta} u(t_i+\beta-1)\right) \\ \quad + C\Phi_1[h] + D\Phi_2[h]\Big] + \displaystyle\sum_{i=1}^{k} I_i\left(u_{t_i-1}\right) \\ \quad + \dfrac{1}{\Gamma(\alpha)}\displaystyle\sum_{i=1}^{k}\sum_{s=t_{i-1}}^{t_i-1}(t_i-s+\alpha-2)^{\underline{\alpha-1}} h(s) \\ \quad + \dfrac{1}{\Gamma(\alpha)}\displaystyle\sum_{s=t_k}^{t-1}(t-s+\alpha-2)^{\underline{\alpha-1}} h(s), \qquad t \in \mathbb{N}_{t_k+1, t_{k+1}} \end{cases} \tag{5}$$

where $\Delta u(t_k) = u(t_k+1) - u(t_k)$; $\Delta\left(\Delta^{-\beta} u(t_k+\beta)\right) = \Delta^{-\beta} u(t_k+\beta+1) - \Delta^{-\beta} u(t_k+\beta)$; *the functionals* $\Phi_1[h], \Phi_2[h]$ *and the constant* Λ *are defined by*

$$\Phi_1[h] := \frac{1}{\Gamma(\alpha)}\left[\sum_{i=1}^{p}\sum_{s=t_{i-1}}^{t_i-1}(t_i-s+\alpha-2)^{\underline{\alpha-1}} h(s) + \sum_{s=t_p}^{T+\alpha-1}(T+2\alpha-s-2)^{\underline{\alpha-1}} h(s)\right] \tag{6}$$

$$\begin{aligned} \Phi_2[h] := \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\Bigg[&\sum_{i=1}^{p}\sum_{r=t_{i-1}+1}^{t_i}\sum_{s=t_{i-1}}^{r-1}(t_i+\beta-s-1)^{\underline{\beta-1}}(r-s+\alpha-2)^{\underline{\alpha-1}} h(s) \\ &+ \sum_{s=t_p+1}^{T+\alpha}\sum_{s=t_k}^{r-1}(T+\alpha+\beta-s-1)^{\underline{\beta-1}}(r-s+\alpha-2)^{\underline{\alpha-1}} h(s)\Bigg] \end{aligned} \tag{7}$$

$$\Lambda := A + B - C - \frac{D}{\Gamma(\beta)}\left[\sum_{i=1}^{p}\sum_{s=t_{i-1}}^{t_i}(t_i-s+\beta-1)^{\underline{\beta-1}} + \sum_{s=t_p}^{T+\alpha}(T+\alpha+\beta-s-1)^{\underline{\beta-1}}\right]. \tag{8}$$

Proof. For $t \in \mathbb{N}_{t_0,t_1}$, by Lemma 1 and taking the fractional sum of order α for (3) and (4), a general solution can be written as

$$\begin{aligned} u(t) &= C_0 + \frac{1}{\Gamma(\alpha)} \sum_{s=0}^{t-\alpha} (t-\sigma(s))^{\underline{\alpha-1}} h(s+\alpha-1) \\ &= C_0 + \frac{1}{\Gamma(\alpha)} \sum_{s=t_0}^{t-1} (t-s+\alpha-2)^{\underline{\alpha-1}} h(s) \end{aligned} \tag{9}$$

By substituting $t = t_1$ into (9), we have

$$u(t_1) = C_0 + \frac{1}{\Gamma(\alpha)} \sum_{s=t_0}^{t_1-1} (t_1-s+\alpha-2)^{\underline{\alpha-1}} h(s). \tag{10}$$

If $t \in \mathbb{N}_{t_1+1,t_2}$, then we have

$$\begin{aligned} u(t) &= u(t_1+1) + \frac{1}{\Gamma(\alpha)} \sum_{s=t_1}^{t-1} (t-s+\alpha-2)^{\underline{\alpha-1}} h(s) \\ &= \Delta u(t_1) + u(t_1) + \frac{1}{\Gamma(\alpha)} \sum_{s=t_1}^{t-1} (t-s+\alpha-2)^{\underline{\alpha-1}} h(s) \\ &= I_1\left(u_{t_1}-1\right) + \left[C_0 + \frac{1}{\Gamma(\alpha)} \sum_{s=t_0}^{t_1-1} (t_1-s+\alpha-2)^{\underline{\alpha-1}} h(s) \right] \\ &\quad + \frac{1}{\Gamma(\alpha)} \sum_{s=t_1}^{t-1} (t-s+\alpha-2)^{\underline{\alpha-1}} h(s). \end{aligned} \tag{11}$$

If $t \in \mathbb{N}_{t_2+1,t_3}$, then we have

$$\begin{aligned} u(t) &= u(t_2+1) + \frac{1}{\Gamma(\alpha)} \sum_{s=t_2}^{t-1} (t-s+\alpha-2)^{\underline{\alpha-1}} h(s) \\ &= \Delta u(t_2) + u(t_2) + \frac{1}{\Gamma(\alpha)} \sum_{s=t_2}^{t-1} (t-s+\alpha-2)^{\underline{\alpha-1}} h(s) \\ &= I_2\left(u_{t_2}-1\right) + \Bigg[I_1\left(u_{t_1}-1\right) + C_0 + \frac{1}{\Gamma(\alpha)} \sum_{s=t_0}^{t_1-1} (t_1-s+\alpha-2)^{\underline{\alpha-1}} h(s) \\ &\quad + \frac{1}{\Gamma(\alpha)} \sum_{s=t_1}^{t_2-1} (t_2-s+\alpha-2)^{\underline{\alpha-1}} h(s) \Bigg] + \frac{1}{\Gamma(\alpha)} \sum_{s=t_2}^{t-1} (t-s+\alpha-2)^{\underline{\alpha-1}} h(s). \end{aligned} \tag{12}$$

Repeating the process, the solution $u(t)$ for $t \in \mathbb{N}_{t_k+1,t_{k+1}}$ $(k = 1, 2, \ldots, p)$ can be written as

$$\begin{aligned} u(t) &= C_0 + \sum_{i=1}^{k} I_i\left(u_{t_i}-1\right) + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k} \sum_{s=t_{i-1}}^{t_i-1} (t_i-s+\alpha-2)^{\underline{\alpha-1}} h(s) \\ &\quad + \frac{1}{\Gamma(\alpha)} \sum_{s=t_k}^{t-1} (t-s+\alpha-2)^{\underline{\alpha-1}} h(s). \end{aligned} \tag{13}$$

Next, for $t \in \mathbb{N}_{t_0+\beta, t_1+\beta}$, taking the fractional sum of order β for (9), we have

$$\begin{aligned}\Delta^{-\beta}u(t) &= \frac{C_0}{\Gamma(\beta)}\sum_{s=\alpha-1}^{t-\beta}(t-\sigma(s))^{\underline{\beta-1}} \\ &\quad+\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\sum_{s=\alpha-1}^{t-\beta}\sum_{s=0}^{r-\alpha}(t-\sigma(r))^{\underline{\beta-1}}(r-\sigma(s))^{\underline{\alpha-1}}h(s).\end{aligned} \tag{14}$$

Thus, for $t \in \mathbb{N}_{t_0,t_1}$, we have

$$\begin{aligned}\Delta^{-\beta}u(t+\beta) &= \frac{C_0}{\Gamma(\beta)}\sum_{s=t_0}^{t}(t+\beta-s-1)^{\underline{\beta-1}} \\ &\quad+\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\sum_{s=t_0+1}^{t}\sum_{s=t_0}^{r-1}(t+\beta-r-1)^{\underline{\beta-1}}(r-s+\alpha-2)^{\underline{\alpha-1}}h(s).\end{aligned} \tag{15}$$

By substituting $t = t_1$ into (15), we have

$$\begin{aligned}\Delta^{-\beta}u(t_1+\beta) &= \frac{C_0}{\Gamma(\beta)}\sum_{s=t_0}^{t_1}(t_1+\beta-s-1)^{\underline{\beta-1}} \\ &\quad+\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\sum_{s=t_0+1}^{t_1}\sum_{s=t_0}^{r-1}(t_1+\beta-r-1)^{\underline{\beta-1}}(r-s+\alpha-2)^{\underline{\alpha-1}}h(s).\end{aligned} \tag{16}$$

If $t \in \mathbb{N}_{t_1+1,t_2}$, then we have

$$\begin{aligned}\Delta^{-\beta}u(t+\beta) &= \Delta^{-\beta}u(t_1+\beta+1)+\frac{C_0}{\Gamma(\beta)}\sum_{s=t_1}^{t}(t+\beta-s-1)^{\underline{\beta-1}} \\ &\quad+\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\sum_{s=t_1+1}^{t}\sum_{s=t_1}^{r-1}(t+\beta-r-1)^{\underline{\beta-1}}(r-s+\alpha-2)^{\underline{\alpha-1}}h(s) \\ &= \Delta\left(\Delta^{-\beta}u(t_1+\beta)\right)+\Delta^{-\beta}u(t_1+\beta)+\frac{C_0}{\Gamma(\beta)}\sum_{s=t_1}^{t}(t+\beta-s-1)^{\underline{\beta-1}} \\ &\quad+\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\sum_{s=t_1+1}^{t}\sum_{s=t_1}^{r-1}(t+\beta-r-1)^{\underline{\beta-1}}(r-s+\alpha-2)^{\underline{\alpha-1}}h(s) \\ &= J_1\left(\Delta^{-\beta}u(t_1+\beta-1)\right)+\left[\frac{C_0}{\Gamma(\beta)}\sum_{s=t_0}^{t_1}(t_1+\beta-s-1)^{\underline{\beta-1}}\right. \\ &\quad\left.+\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\sum_{s=t_0+1}^{t_1}\sum_{s=t_0}^{r-1}(t_1+\beta-r-1)^{\underline{\beta-1}}(r-s+\alpha-2)^{\underline{\alpha-1}}h(s)\right] \\ &\quad+\frac{C_0}{\Gamma(\beta)}\sum_{s=t_1}^{t}(t+\beta-s-1)^{\underline{\beta-1}} \\ &\quad+\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\sum_{s=t_1+1}^{t}\sum_{s=t_1}^{r-1}(t+\beta-r-1)^{\underline{\beta-1}}(r-s+\alpha-2)^{\underline{\alpha-1}}h(s).\end{aligned} \tag{17}$$

If $t \in \mathbb{N}_{t_2+1,t_3}$, then we have

$$\begin{aligned}
\Delta^{-\beta}u(t+\beta) &= \Delta^{-\beta}u(t_2+\beta+1) + \frac{C_0}{\Gamma(\beta)}\sum_{s=t_2}^{t}(t+\beta-s-1)^{\underline{\beta-1}} \\
&\quad + \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\sum_{s=t_2+1}^{t}\sum_{s=t_2}^{r-1}(t+\beta-r-1)^{\underline{\beta-1}}(r-s+\alpha-2)^{\underline{\alpha-1}}h(s) \\
&= \Delta\left(\Delta^{-\beta}u(t_2+\beta)\right) + \Delta^{-\beta}u(t_2+\beta) + \frac{C_0}{\Gamma(\beta)}\sum_{s=t_2}^{t}(t+\beta-s-1)^{\underline{\beta-1}} \\
&\quad + \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\sum_{s=t_2+1}^{t}\sum_{s=t_2}^{r-1}(t+\beta-r-1)^{\underline{\beta-1}}(r-s+\alpha-2)^{\underline{\alpha-1}}h(s) \\
&= J_1\left(\Delta^{-\beta}u(t_1+\beta-1)\right) + J_2\left(\Delta^{-\beta}u(t_2+\beta-1)\right) \\
&\quad + \frac{C_0}{\Gamma(\beta)}\left[\sum_{s=t_0}^{t_1}(t_1+\beta-s-1)^{\underline{\beta-1}} + \sum_{s=t_1}^{t_2}(t_2+\beta-s-1)^{\underline{\beta-1}}\right] \\
&\quad + \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\left[\sum_{s=t_0+1}^{t_1}\sum_{s=t_0}^{r-1}(t_1+\beta-r-1)^{\underline{\beta-1}}(r-s+\alpha-2)^{\underline{\alpha-1}}h(s)\right. \\
&\quad \left. + \sum_{s=t_1+1}^{t_2}\sum_{s=t_1}^{r-1}(t_2+\beta-r-1)^{\underline{\beta-1}}(r-s+\alpha-2)^{\underline{\alpha-1}}h(s)\right] \\
&\quad + \frac{C_0}{\Gamma(\beta)}\sum_{s=t_2}^{t}(t+\beta-s-1)^{\underline{\beta-1}} \\
&\quad + \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\sum_{s=t_2+1}^{t}\sum_{s=t_2}^{r-1}(t+\beta-r-1)^{\underline{\beta-1}}(r-s+\alpha-2)^{\underline{\alpha-1}}h(s).
\end{aligned} \tag{18}$$

Repeating the process, the solution $u(t)$ for $t \in \mathbb{N}_{t_k+1,t_{k+1}}$ $(k = 1, 2, \ldots, p)$ can be written as

$$\begin{aligned}
\Delta^{-\beta}u(t) &= \sum_{i=1}^{k}J_i\left(\Delta^{-\beta}u_{t_i}+\beta-1\right) \\
&\quad + \frac{C_0}{\Gamma(\beta)}\left[\sum_{i=1}^{k}\sum_{s=t_{i-1}}^{t_i-1}(t_i-s+\alpha-2)^{\underline{\beta-1}}h(s) + \sum_{s=t_k}^{t}(t-s+\beta-1)^{\underline{\beta-1}}\right] \\
&\quad + \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\left[\sum_{i=1}^{k}\sum_{s=t_{i-1}+1}^{t_i}\sum_{s=t_{i-1}}^{r-1}(t_i+\beta-r-1)^{\underline{\beta-1}}(r-s+\alpha-2)^{\underline{\alpha-1}}h(s)\right. \\
&\quad \left. + \sum_{s=t_k+1}^{t}\sum_{s=t_k}^{r-1}(t+\beta-r-1)^{\underline{\beta-1}}(r-s+\alpha-2)^{\underline{\alpha-1}}h(s)\right].
\end{aligned} \tag{19}$$

By substituting $t = \alpha - 1$ into (9), (16); $t = T + \alpha$ into (13), (19); and using the condition $Au(\alpha - 1) + B\Delta^{-\beta}u(\alpha + \beta - 1) = Cu(T + \alpha) + D\Delta^{-\beta}u(T + \alpha + \beta)$, we have

$$C_0 = \frac{1}{\Lambda}\left[C\sum_{i=1}^{p}I_i\left(u(t_i-1)\right) + D\sum_{i=1}^{p}J_i\left(\Delta^{-\beta}u(t_i+\beta-1)\right) + C\Phi_1[h] + D\Phi_2[h]\right] \tag{20}$$

where $\Phi_1[h], \Phi_2[h]$ and Λ are defined in (6)–(8), respectively. Substituting C_0 into (9) and (13), we have (5). $\square$

3. Main Results

3.1. Existence and Uniqueness Solution

Let $\mathcal{C} = C(\mathbb{N}_{\alpha-1,T+\alpha}, \mathbb{R})$ be the Banach space equipped with the norm $\|u\| = \max_{t\in\mathbb{N}_{\alpha-1,T+\alpha}} |u(t)|$. Defined the operator $\mathcal{T} : \mathcal{C} \to \mathcal{C}$ by

$$(\mathcal{T}u)(t) := \begin{cases} \dfrac{1}{\Lambda}\Bigg[C\displaystyle\sum_{i=1}^{p} I_i\left(u(t_i-1)\right) + D\sum_{i=1}^{p} J_i\left(\Delta^{-\beta}u(t_i+\beta-1)\right) + C\Phi_1^*[Fu] + D\Phi_2^*[Fu]\Bigg] \\ + \dfrac{1}{\Gamma(\alpha)}\displaystyle\sum_{s=t_0}^{t-1}(t-s+\alpha-2)^{\underline{\alpha-1}}F\left[s,u(s),\Psi^\gamma u(s)\right], \qquad t\in\mathbb{N}_{t_0,t_1} \\ \\ \dfrac{1}{\Lambda}\Bigg[C\displaystyle\sum_{i=1}^{p} I_i\left(u(t_i-1)\right) + D\sum_{i=1}^{p} J_i\left(\Delta^{-\beta}u(t_i+\beta-1)\right) \\ + C\Phi_1^*[Fu] + D\Phi_2^*[Fu]\Bigg] + \displaystyle\sum_{i=1}^{k} I_i\left(u_{t_i}-1\right) \\ + \dfrac{1}{\Gamma(\alpha)}\displaystyle\sum_{i=1}^{k}\sum_{s=t_{i-1}}^{t_i-1}(t_i-s+\alpha-2)^{\underline{\alpha-1}}F\left[s,u(s),\Psi^\gamma u(s)\right] \\ + \dfrac{1}{\Gamma(\alpha)}\displaystyle\sum_{s=t_k}^{t-1}(t-s+\alpha-2)^{\underline{\alpha-1}}F\left[s,u(s),\Psi^\gamma u(s)\right], \qquad t\in\mathbb{N}_{t_k+1,t_{k+1}} \end{cases} \tag{21}$$

where the functionals $\Phi_1^*[Fu]$ and $\Phi_2^*[Fu]$ are defined as

$$\Phi_1^*[Fu] = \frac{1}{\Gamma(\alpha)}\Bigg[\sum_{i=1}^{p}\sum_{s=t_{i-1}}^{t_i-1}(t_i-s+\alpha-2)^{\underline{\alpha-1}}F\left[s,u(s),\Psi^\gamma u(s)\right] + \sum_{s=t_p}^{T+\alpha-1}(T+2\alpha-s-2)^{\underline{\alpha-1}}F\left[s,u(s),\Psi^\gamma u(s)\right]\Bigg] \tag{22}$$

$$\Phi_2^*[Fu] = \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\Bigg[\sum_{i=1}^{p}\sum_{r=t_{i-1}+1}^{t_i}\sum_{s=t_{i-1}}^{r-1}(t_i+\beta-s-1)^{\underline{\beta-1}}(r-s+\alpha-2)^{\underline{\alpha-1}}\times F\left[s,u(s),\Psi^\gamma u(s)\right] + \sum_{s=t_p+1}^{T+\alpha}\sum_{s=t_k}^{r-1}(T+\alpha+\beta-s-1)^{\underline{\beta-1}}(r-s+\alpha-2)^{\underline{\alpha-1}}\times F\left[s,u(s),\Psi^\gamma u(s)\right]\Bigg] \tag{23}$$

and constant Λ is given in (8). Observe that the operator in (21) has the fixed points which are the solutions of the problem (1) and (2).

Theorem 1. *Let $F : \mathbb{N}_{\alpha-1,T+\alpha}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$, $I_k, J_k : \mathbb{R}\to\mathbb{R}$, $k = 1,2,\ldots,p$ be continuous; $\varphi : \mathbb{N}_{\alpha-1,T+\alpha}\times\mathbb{N}_{\alpha-1,T+\alpha}\to[0,\infty)$ be continuous with $\varphi_0 = \max\{\varphi(t-1,s) : (t,s)\in\mathbb{N}_{\alpha-1,T+\alpha}\times\mathbb{N}_{\alpha-1,T+\alpha}\}$. In addition, suppose that*

(H_1) *There exist constants $\ell_1,\ell_2 > 0$ such that*

$$\left|F[t,u(t),(\Psi^\gamma u)(t)] - F[t,v(t),(\Psi^\gamma v)(t)]\right| \le \ell_1|u-v| + \ell_2|(\Psi^\gamma u)-(\Psi^\gamma v)|$$

for each $t\in\mathbb{N}_{\alpha-1,\alpha+T}$ and $u,v\in\mathbb{R}$.

(H_2) *There exist constants* $\lambda_1, \lambda_2 > 0$ *such that*

$$\begin{aligned} \left|I_k(u) - I_k(v)\right| &\leq \lambda_1 |u - v| \\ \left|J_k\left(\Delta^{-\beta}u\right) - J_k\left(\Delta^{-\beta}v\right)\right| &\leq \lambda_2 |\Delta^{-\beta}u - \Delta^{-\beta}v| \end{aligned}$$

for each $u, v \in \mathcal{C}$ *and* $k = 1, 2, \ldots, p$.

(H_3) $\mathcal{L}\mathcal{P} + \lambda_1 \mathcal{Q} + \lambda_2 \mathcal{R} \ < \ 1,$

where

$$\mathcal{L} = \ell_1 + \ell_2 \varphi_0 \frac{(T + \alpha + \gamma)^{\underline{\gamma}}}{\Gamma(\gamma + 1)} \tag{24}$$

$$\mathcal{P} = \left(\left|\frac{C}{\Lambda}\right| + 1\right) \Omega_1 + \left|\frac{D}{\Lambda}\right| \Omega_2 \tag{25}$$

$$\mathcal{Q} = \frac{p(p+1)}{2} \left(\left|\frac{C}{\Lambda}\right| + 1\right) \tag{26}$$

$$\mathcal{R} = \left|\frac{D}{\Lambda}\right| \frac{(T + \alpha + \beta)^{\underline{\beta}}}{\Gamma(\beta + 1)} \tag{27}$$

$$\Omega_1 = \left[\frac{p(p+1)}{2} + 1\right] \frac{(T + \alpha)^{\underline{\alpha}}}{\Gamma(\alpha + 1)} \tag{28}$$

$$\Omega_2 = \left[\frac{p(p+1)}{2} + 1\right] \frac{(T + \alpha)^{\underline{\alpha}} (T + \alpha + \beta)^{\underline{\beta}}}{\Gamma(\alpha + 1)\Gamma(\beta + 1)}. \tag{29}$$

Therefore, the problem (1) and (2) has a unique solution on $\mathbb{N}_{\alpha-1,\alpha+T}$.

Proof. We will show that $\mathcal{T}$ is a contraction. Let

$$\mathcal{H}|u - v|(t) := \left|F[t, u(t), (\Psi^\gamma u)(t)] - F[t, v(t), (\Psi^\gamma v)(t)]\right|.$$

For any $u, v \in \mathcal{C}$, we have

$$\begin{aligned} &\left|\Phi_1^*[Fu] - \Phi_1^*[Fv]\right| \\ &= \frac{1}{\Gamma(\alpha)} \left| \sum_{i=1}^{p} \sum_{s=t_{i-1}}^{t_i - 1} (t_i - s + \alpha - 2)^{\underline{\alpha - 1}} \, \mathcal{H}|u - v|(s) + \sum_{s=t_p}^{T+\alpha-1} (T + 2\alpha - s - 2)^{\underline{\alpha - 1}} \, \mathcal{H}|u - v|(s) \right| \\ &\leq \frac{[\ell_1 |u - v| + \ell_2 |(\Psi^\gamma u) - (\Psi^\gamma v)|]}{\Gamma(\alpha)} \left| \sum_{i=1}^{p} \sum_{s=t_{i-1}}^{t_i - 1} (t_i - s + \alpha - 2)^{\underline{\alpha - 1}} + \sum_{s=t_p}^{T+\alpha-1} (T + 2\alpha - s - 2)^{\underline{\alpha - 1}} \right| \\ &\leq \left[\ell_1 + \ell_2 \varphi_0 \frac{(T + \alpha + \gamma)^{\underline{\gamma}}}{\Gamma(\gamma + 1)}\right] |u - v| \left[\frac{p(p+1)}{2} + 1\right] \frac{(T + \alpha)^{\underline{\alpha}}}{\Gamma(\alpha + 1)} \\ &= \mathcal{L}\Omega_1 |u - v|. \end{aligned} \tag{30}$$

Similary, we get

$$|\Phi_2^*[Fu] - \Phi_2^*[Fv]| \leq \mathcal{L}\Omega_2 |u - v|. \tag{31}$$

Next, for each $t \in \mathbb{N}_{t_k+1,t_{k+1}}$, $k = 1, 2, \ldots, p$, we obtain

$$\begin{aligned}
&|(\mathcal{T}u)(t) - (\mathcal{T}v)(t)| \\
\leq\ & \frac{1}{\Lambda}\Bigg| C \sum_{i=1}^{p} \left| I_i(u_{t_i} - 1) - I_k(v_{t_i} - 1) \right| \\
&+ D \sum_{i=1}^{p} \left| J_i\left(\Delta^{-\beta} u(t_i + \beta - 1)\right) - J_i\left(\Delta^{-\beta} v(t_i + \beta - 1)\right) \right| \\
&+ C\left|\Phi_1^*[Fu] - \Phi_1^*[Fv]\right| + D\left|\Phi_2^*[Fu] - \Phi_2^*[Fv]\right| \Bigg| + \sum_{i=1}^{k} \left| I_i(u_{t_i} - 1) - I_k(v_{t_i} - 1) \right| \\
&+ \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k} \sum_{s=t_{i-1}}^{t_i - 1} (t_i - s + \alpha - 2)^{\underline{\alpha - 1}}\, \mathcal{H}|u - v|(s) \\
&+ \frac{1}{\Gamma(\alpha)} \sum_{s=t_k}^{t-1} (t - s + \alpha - 2)^{\underline{\alpha - 1}}\, \mathcal{H}|u - v|(s) \\
\leq\ & \frac{1}{|\Lambda|}\Bigg[|C| \frac{p(p+1)}{2} \lambda_1 |u - v| + |D| \frac{p(p+1)}{2} \lambda_2 \left| \Delta^{-\beta} u - \Delta^{-\beta} v \right| + |C| \mathcal{L}\Omega_1 |u - v| \\
&+ |D|\mathcal{L}\Omega_2 |u - v| \Bigg] + \frac{p(p+1)}{2} \lambda_1 |u - v| + \left[\frac{p(p+1)}{2} + 1 \right] \frac{(T + \alpha)^{\underline{\alpha}}}{\Gamma(\alpha + 1)} \mathcal{L}|u - v| \\
\leq\ & \Bigg\{ \frac{1}{|\Lambda|}\left[|C| \frac{p(p+1)}{2} \lambda_1 + |C|\mathcal{L}\Omega_1 + |D|\mathcal{L}\Omega_2 \right] + \frac{p(p+1)}{2} \lambda_1 + \left[\frac{p(p+1)}{2} + 1 \right] \\
& \frac{(T + \alpha)^{\underline{\alpha}}}{\Gamma(\alpha + 1)} \mathcal{L} \Bigg\} |u - v| + \left| \frac{D}{\Lambda} \right| \frac{p(p+1)}{2} \lambda_2 \frac{(T + \alpha + \beta)^{\underline{\beta}}}{\Gamma(\beta + 1)} |u - v| \\
\leq\ & \left[\mathcal{L}\mathcal{P} + \lambda_1 \mathcal{Q} + \lambda_2 \mathcal{R} \right] |u - v|. \qquad (32)
\end{aligned}$$

Obviously, for each $t \in \mathbb{N}_{t_0,t_1}$, we obtain $\left|(\mathcal{T}u)(t) - (\mathcal{T}v)(t)\right| < |u - v|$.

Thus, for each $t \in \mathbb{N}_{\alpha-1,T+\alpha}$, we have $\left\|(\mathcal{T}u)(t) - (\mathcal{T}v)(t)\right\| \leq \|u - v\|$.

So, $\mathcal{T}$ is a contraction. By the Banach contraction principle, we can conclude that $\mathcal{T}$ has a fixed point. Hence, the problem (1) and (2) has a unique solution on $t \in \mathbb{N}_{\alpha-1,\alpha+T}$. $\square$

3.2. Existence of at Least One Solution

In the next result, we use of the Schauder's fixed point theorem to discuss the existence of at least one solution of (1) and (2).

Theorem 2. *Suppose that (H_1) and (H_3) hold. Then, there exists at least one solution for the problem (1) and (2).*

Proof. We organize the proof as follows:

Step I. Examine that $\mathcal{T}$ map bounded sets into bounded sets in $B_R = \{u \in \mathcal{C} : \|u\| \leq R\}$. Let $\max|F(t,0,0,0)| = K$, $\max|I_k(u)| = M$, $\max\left|J_k\left(\Delta^{-\beta}u\right)\right| = M$, and choose a constant

$$R \geq \frac{\mathcal{P}K}{1 - \mathcal{L}\mathcal{P} + M\mathcal{Q} + N\mathcal{R}}. \qquad (33)$$

Let

$$\left|\mathcal{S}(t,u,0)\right| = \left|F\left[t,u,\Delta^{-\beta}u\right] - F[t,0,0]\right| + \left|F[t,0,0]\right|.$$

For any $u, v \in \mathcal{C}$, we obtain

$$\begin{aligned}
&\left|\Phi_1^*[Fu]\right| \\
&= \frac{1}{\Gamma(\alpha)}\left|\sum_{i=1}^{p}\sum_{s=t_{i-1}}^{t_i-1}(t_i - s + \alpha - 2)^{\underline{\alpha-1}}\left|\mathcal{S}(s,u,0)\right| + \sum_{s=t_p}^{T+\alpha-1}(T + 2\alpha - s - 2)^{\underline{\alpha-1}}\left|\mathcal{S}(s,u,0)\right|\right| \\
&\leq \left(\left[\ell_1 + \ell_2\varphi_0\frac{(T+\alpha+\gamma)^{\underline{\gamma}}}{\Gamma(\gamma+1)}\right]|u| + K\right)\Omega_1 \\
&= \mathcal{L}\Omega_1 R + K\Omega_1.
\end{aligned} \tag{34}$$

Similarly, we obtain

$$|\Phi_2^*[Fu]| \leq \mathcal{L}\Omega_2 R + K\Omega_2. \tag{35}$$

So, for each $t \in \mathbb{N}_{t_k+1,t_{k+1}}$, $k = 1, 2, \ldots, p$, we get

$$|(\mathcal{T}u)(t)| \leq \left[\mathcal{L}\mathcal{P} + M\mathcal{Q} + N\mathcal{R}\right]R + K\mathcal{P} \leq R. \tag{36}$$

Obviously, for each $t \in \mathbb{N}_{t_0,t_1}$, we have $|\mathcal{T}u(t)| < R$.

Therefore, $\|\mathcal{T}u\| \leq R$ for each $t \in \mathbb{N}_{\alpha-1,T+\alpha}$, which implies that $\mathcal{F}$ is uniformly bounded.

Step II. It is obvious that the operator $\mathcal{T}$ is continuous on B_R since the continuity of F.

Step III. Examine that $\mathcal{T}$ is equicontinuous on B_R. For any $\epsilon > 0$, there exists $\delta > 0$ such that for $\tau_1, \tau_2 \in \mathbb{N}_{\alpha-1,T+\alpha}$ with $\tau_1 < \tau_2$

$$|\tau_2^{\underline{\alpha}} - \tau_1^{\underline{\alpha}}| < \tfrac{\epsilon\Gamma(\alpha+1)}{\|F\|} \quad \text{whenever} \quad |\tau_2 - \tau_1| < \delta.$$

Then, we obtain

$$\begin{aligned}
\left|(\mathcal{T}u)(\tau_2) - (\mathcal{T}u)(\tau_1)\right| &\leq \frac{\|F\|(T-\omega_0)^{\alpha}}{\Gamma_q(\alpha+1)\Gamma_q(\beta+1)}\left|\frac{1}{\Gamma(\alpha)}\sum_{s=t_k}^{\tau_2-1}(\tau_2 - s + \alpha - 2)^{\underline{\alpha-1}}\right. \\
&\quad \left. - \frac{1}{\Gamma(\alpha)}\sum_{s=t_k}^{\tau_1-1}(\tau_1 - s + \alpha - 2)^{\underline{\alpha-1}}\right| \\
&= \frac{\|F\|}{\Gamma(\alpha+1)}|\tau_2^{\underline{\alpha}} - \tau_1^{\underline{\alpha}}| < \epsilon.
\end{aligned} \tag{37}$$

This implies that the set $\mathcal{T}(B_R)$ is an equicontinuous set.

Hence, by the Arzelá–Ascoli theorem, we deduce that $\mathcal{T} : \mathcal{C} \to \mathcal{C}$ is completely continuous. Therefore, it follows from the Schauder's fixed point theorem that problem (1)-(2) has at least one solution. □

4. Ulam Stability Analysis Results

Based on the concept in Wang et al. [54], we provide Ulam's type stability concepts for the problem (1) and (2). Consider the inequalities:

$$\begin{cases}
\left|\Delta_C^{\alpha} z(t) - F\left[t+\alpha-1, z(t+\alpha-1), \Psi^{\gamma} z(t+\alpha-1)\right]\right| \leq \epsilon \\
|\Delta u(t_k) - I_k\left(u(t_k - 1)\right)| \leq \epsilon \\
\left|\Delta\left(\Delta^{-\beta} u(t_k+\beta)\right) - J_k\left(\Delta^{-\beta} u(t_k+\beta-1)\right)\right| \leq \epsilon,
\end{cases} \tag{38}$$

$$\begin{cases} \left|\Delta_C^\alpha z(t) - F\Big[t+\alpha-1, z(t+\alpha-1), \Psi^\gamma z(t+\alpha-1)\Big]\right| \le \epsilon\rho(t) \\ |\Delta u(t_k) - I_k\left(u(t_k-1)\right)| \le \epsilon\psi_1 \\ \left|\Delta\left(\Delta^{-\beta}u(t_k+\beta)\right) - J_k\left(\Delta^{-\beta}u(t_k+\beta-1)\right)\right| \le \epsilon\psi_2, \end{cases} \tag{39}$$

for $t \in \mathbb{N}_{0,T}$, $t+\alpha-1 \neq t_k, k = 1, 2, \ldots, p$.

Definition 4. *The problem (1) and (2) is the Ulam–Hyers stable if there exists a real number $c_{F,p} > 0$ such that for each $\epsilon > 0$ and for each solution $z \in \mathcal{C}$ of the inequality (38), there exists a solution $u \in \mathcal{C}$ of problem (1) and (2) with*

$$|z(t) - u(t)| \le c_{F,p}\epsilon. \tag{40}$$

The problem (1) and (2) is the generalized Ulam–Hyers stable, if we substitute the function $\theta_{F,p}(\epsilon) \in C(\mathbb{R}^+, \mathbb{R}^+)$, $\theta_{F,p}(0) = 0$ for the constant $c_{F,p}\epsilon$ on inequality (40).

Definition 5. *The problem (1) and (2) is the Ulam–Hyers–Rassias stable with respect to (ρ, ψ_1, ψ_2) if there exists a real number $c_{F,p,\rho} > 0$ such that for each $\epsilon, \psi_1, \psi_2 > 0$ and for each solution $z \in \mathcal{C}$ of the inequality (40), there exists a solution $u \in \mathcal{C}$ of problem (1) and (2) with*

$$|z(t) - u(t)| \le c_{F,p,\rho}\epsilon\left(\rho(t) + \psi_1 + \psi_2\right). \tag{41}$$

The problem (1) and (2) is the generalized Ulam–Hyers–Rassias stable, if we substitute the function $\rho^*(t)$ for the function $\epsilon\rho(t)$ and the constants ψ_1^*, ψ_2^* for on the constants $\epsilon\psi_1, \epsilon\psi_2$ on inequalities (39) and (41).

Remark 1. *A function $z \in \mathcal{C}$ is a solution of the inequality (38) if and only if there exist $\phi \in \mathcal{C}$ (depend on z) and sequence ω_k, κ_k, $k = 1, 2, \ldots, p$ with $\omega = \max\{\omega_k\}, \kappa = \max\{\kappa_k\}$, such that*

- (i) $|\phi(t)| \le \epsilon$ *for* $t \in \mathbb{N}_{t_k+1, t_{k+1}}$, *and* $|\omega|, |\kappa| \le \epsilon$;
- (ii) $\Delta_C^\alpha z(t) = F\big[t+\alpha-1, z(t+\alpha-1), \Psi^\gamma z(t+\alpha-1)\big] + \phi(t+\alpha-1)$ *for* $t \in \mathbb{N}_{0,T}$, $t+\alpha-1 \neq t_k$;
- (iii) $\Delta u(t_k) = I_k\left(u(t_k-1)\right) + \omega_k$;
- (iv) $\Delta\left(\Delta^{-\beta}u(t_k+\beta)\right) = J_k\left(\Delta^{-\beta}u(t_k+\beta-1)\right) + \kappa_k$.

Remark 2. *A function $z \in \mathcal{C}$ is a solution of the inequality (39) if and only if there exist $\phi \in \mathcal{C}$ (depend on z) and sequence ω_k, κ_k, $k = 1, 2, \ldots, p$ with $\omega = \max\{\omega_k\}, \kappa = \max\{\kappa_k\}$,*

- (i) $|\phi(t)| \le \epsilon\rho(t)$ *for* $t \in \mathbb{N}_{t_k+1, t_{k+1}}$, $|\omega| \le \epsilon\psi_1$ *and* $|\kappa| \le \epsilon\psi_2$;
- (ii) $\Delta_C^\alpha z(t) = F\big[t+\alpha-1, z(t+\alpha-1), \Psi^\gamma z(t+\alpha-1)\big] + \phi(t+\alpha-1)$ *for* $t \in \mathbb{N}_{0,T}$, $t+\alpha-1 \neq t_k$;
- (iii) $\Delta u(t_k) = I_k\left(u(t_k-1)\right) + \omega_k$;
- (iv) $\Delta\left(\Delta^{-\beta}u(t_k+\beta)\right) = J_k\left(\Delta^{-\beta}u(t_k+\beta-1)\right) + \kappa_k$.

Lemma 3. *If $z \in \mathcal{C}$ is a solution of the inequality (38), then for $\epsilon > 0$, z is solution of the inequality*

$$|z(t) - (\mathcal{T}z)(t)| \le \epsilon\left(\mathcal{P} + \mathcal{Q} + \mathcal{R}\right), \tag{42}$$

where $\mathcal{P}$, $\mathcal{Q}$ and $\mathcal{R}$ are defined in (25)–(27).

Proof. From Remark 1 and Lemma 2, we have

$$\begin{aligned} z(t) &= \frac{1}{\Lambda}\left[C\sum_{i=1}^{p}\left[I_i\left(u(t_i-1)\right)+\omega_i\right]+D\sum_{i=1}^{p}\left[J_i\left(\Delta^{-\beta}u(t_i+\beta-1)\right)+\kappa_i\right]\right. \\ &\quad \left.+C\Phi_1^*[Fu+\phi]+D\Phi_2^*[Fu+\phi]\right]+\sum_{i=1}^{k}\left[I_i\left(u_{t_i}-1\right)+\omega_i\right] \\ &\quad +\frac{1}{\Gamma(\alpha)}\sum_{i=1}^{k}\sum_{s=t_{i-1}}^{t_i-1}(t_i-s+\alpha-2)^{\underline{\alpha-1}}\left[F\left[s,u(s),\Psi^{\gamma}u(s)\right]+\phi(s)\right] \\ &\quad +\frac{1}{\Gamma(\alpha)}\sum_{s=t_k}^{t-1}(t-s+\alpha-2)^{\underline{\alpha-1}}\left[F\left[s,u(s),\Psi^{\gamma}u(s)\right]+\phi(s)\right]. \end{aligned} \tag{43}$$

Hence, we obtain

$$\begin{aligned} |z(t)-(\mathcal{T}z)(t)| &\leq \frac{1}{|\Lambda|}\left[|C|\sum_{i=1}^{p}|\omega_i|+|D|\sum_{i=1}^{p}|\kappa_i|+|C|\Phi_1^*[|\phi|]+D\Phi_2^*[|\phi|]\right] \\ &\quad +\sum_{i=1}^{k}|\omega_i|+\frac{1}{\Gamma(\alpha)}\sum_{i=1}^{k}\sum_{s=t_{i-1}}^{t_i-1}(t_i-s+\alpha-2)^{\underline{\alpha-1}}\,|\phi(s)| \\ &\quad +\frac{1}{\Gamma(\alpha)}\sum_{s=t_k}^{t-1}(t-s+\alpha-2)^{\underline{\alpha-1}}\,|\phi(s)| \\ &\leq \frac{p(p+1)}{2}\left[\left(\left|\frac{C}{\Lambda}\right|+1\right)|\omega|+\left|\frac{D}{\Lambda}\right||\kappa|\right]+\frac{\|\phi\|}{|\Lambda|}\left[|C|\Omega_1+|D|\Omega_2\right] \\ &\quad +\frac{\|\phi\|}{\Gamma(\alpha+1)}\left[\frac{p(p+1)}{2}(T+\alpha)^{\underline{\alpha}}+t^{\underline{\alpha}}\right] \\ &\leq \|\phi\|\mathcal{P}+|\omega|\mathcal{Q}+|\kappa|\mathcal{R} \\ &\leq \epsilon\left(\mathcal{P}+\mathcal{Q}+\mathcal{R}\right) \end{aligned} \tag{44}$$

where $\mathcal{P},\mathcal{Q},\mathcal{R},\Omega_1,\Omega_2$ are defined in (25)–(29). This completes the proof. □

Theorem 3. *Suppose that* (H_1)*–*(H_3) *hold with* $1-(\mathcal{LP}+\lambda_1\mathcal{Q}+\lambda_2\mathcal{R})\neq 0$*. Then, problem* (1) *and* (2) *is Ulam–Hyers stable.*

Proof. Suppose $z\in\mathcal{C}$ is a solution of (38) and assume that $x(t)$ is the unique solution of problem (1) and (2). Consider

$$\begin{aligned} |z(t)-x(t)| &= \left|z(t)-(\mathcal{T}z)(t)+(\mathcal{T}z)(t)-x(t)\right| \\ &\leq \left|z(t)-(\mathcal{T}z)(t)\right|+\left|(\mathcal{T}z)(t)-x(t)\right| \\ &\leq \epsilon\left(\mathcal{P}+\mathcal{Q}+\mathcal{R}\right)+\left(\mathcal{LP}+\lambda_1\mathcal{Q}+\lambda_2\mathcal{R}\right)|z(t)-x(t)|, \end{aligned} \tag{45}$$

from which we obtain

$$|z(t)-x(t)| \leq \frac{\epsilon\left(\mathcal{P}+\mathcal{Q}+\mathcal{R}\right)}{\left|1-\left(\mathcal{LP}+\lambda_1\mathcal{Q}+\lambda_2\mathcal{R}\right)\right|} \tag{46}$$

By setting $c_{F,p}=\frac{(\mathcal{P}+\mathcal{Q}+\mathcal{R})}{\left|1-(\mathcal{LP}+\lambda_1\mathcal{Q}+\lambda_2\mathcal{R})\right|}$, we have

$$|z(t)-x(t)|\leq c_{F,p}\epsilon.$$

Hence, the problem (1) and (2) is Ulam–Hyers stable. Next, by setting $\theta_{F,p}(\epsilon) = c_{F,p}\epsilon$ with $\theta_{F,p}(0) = 0$, the problem (1) and (2) is generalized Ulam–Hyers stable. □

Lemma 4. *If $z \in \mathcal{C}$ is a solution of the inequality (39) and assume that*

$$(H_4) \quad \frac{p(p+1)}{2}\left[\left(\left|\frac{C}{\Lambda}\right|+1\right)|\omega| + \left|\frac{D}{\Lambda}\right||\kappa|\right] + \frac{\|\phi\|}{|\Lambda|}\left[|C|\Omega_1 + |D|\Omega_2\right] + \frac{\|\phi\|}{\Gamma(\alpha+1)}\left[\frac{p(p+1)}{2}(T+\alpha)^{\frac{\alpha}{\alpha}} + t^{\frac{\alpha}{\alpha}}\right] \leq \epsilon\mathcal{O}\left[\rho(t)+\psi_1+\psi_2\right],$$

then for $\epsilon > 0$, z is solution of the following inequality

$$|z(t) - (\mathcal{T}z)(t)| \leq \epsilon\,\mathcal{O}\left[\rho(t)+\psi_1+\psi_2\right], \tag{47}$$

where $\mathcal{O} = \max\{\mathcal{P}, \mathcal{Q}, \mathcal{R}\}$ and $\mathcal{P}, \mathcal{Q}, \mathcal{R}, \Omega_1, \Omega_2$ are defined in (25)–(29).

Proof. By the same argument of the proof Lemma 3, we have

$$\begin{aligned}
|z(t) - (\mathcal{T}z)(t)| &\leq \frac{p(p+1)}{2}\left[\left(\left|\frac{C}{\Lambda}\right|+1\right)|\omega| + \left|\frac{D}{\Lambda}\right||\kappa|\right] + \frac{\|\phi\|}{|\Lambda|}\left[|C|\Omega_1 + |D|\Omega_2\right] \\
&\quad + \frac{\|\phi\|}{\Gamma(\alpha+1)}\left[\frac{p(p+1)}{2}(T+\alpha)^{\frac{\alpha}{\alpha}} + t^{\frac{\alpha}{\alpha}}\right] \qquad (48) \\
&\leq \epsilon\left[\rho(t)\mathcal{P} + \psi_1\mathcal{Q} + \psi_2\mathcal{R}\right] \\
&\leq \epsilon\mathcal{O}\left[\rho(t)+\psi_1+\psi_2\right]. \qquad (49)
\end{aligned}$$

This completes the proof. □

Theorem 4. *Suppose that (H_1)–(H_3) hold with $1 - (\mathcal{L}\mathcal{P} + \lambda_1\mathcal{Q} + \lambda_2\mathcal{R}) \neq 0$. Then, problem (1) and (2) is Ulam–Hyers–Rassias stable.*

Proof. Suppose $z \in \mathcal{C}$ is a solution of (39) and assume that $x(t)$ is the unique solution of problem (1) and (2). Consider

$$\begin{aligned}
|z(t) - x(t)| &= \big|z(t) - (\mathcal{T}z)(t) + (\mathcal{T}z)(t) - x(t)\big| \\
&\leq \big|z(t) - (\mathcal{T}z)(t)\big| + \big|(\mathcal{T}z)(t) - x(t)\big| \\
&\leq \epsilon\mathcal{O}\big(\rho(t)+\psi_1+\psi_2\big) + \big(\mathcal{L}\mathcal{P} + \lambda_1\mathcal{Q} + \lambda_2\mathcal{R}\big)|z(t) - x(t)|, \qquad (50)
\end{aligned}$$

from which we obtain

$$|z(t) - x(t)| \leq \frac{\epsilon\mathcal{O}\left[\rho(t)+\psi_1+\psi_2\right]}{\left|1 - (\mathcal{L}\mathcal{P} + \lambda_1\mathcal{Q} + \lambda_2\mathcal{R})\right|} \tag{51}$$

By setting $c_{F,p} = \frac{\mathcal{O}}{\left|1-(\mathcal{L}\mathcal{P}+\lambda_1\mathcal{Q}+\lambda_2\mathcal{R})\right|}$, we have

$$|z(t) - x(t)| \leq c_{F,p}\,\epsilon\left[\rho(t)+\psi_1+\psi_2\right].$$

Hence, the problem (1) and (2) is Ulam–Hyers–Rassias stable. Next, by setting $\rho^*(t) + \psi_1^* + \psi_2^* = \epsilon\left[\rho(t)+\psi_1+\psi_2\right]$, the problem (1) and (2) is generalized Ulam–Hyers–Rassias stable. □

5. An Example

To show the application of our theorems, we provide the fractional impulsive difference-sum equations with periodic boundary conditions of the form

$$\begin{aligned}
&\Delta_C^{\frac{1}{2}}u(t) = \frac{\cos^2\left(t-\frac{1}{2}\right)\pi}{2\left(t-\frac{199}{2}\right)^2}\left[\frac{u^2+|u|}{|u|+1}+e^{-\sin^2\left(t-\frac{1}{2}\right)\pi}\Psi^{\frac{1}{3}}u\left(t-\frac{1}{2}\right)\right],\ t\in\mathbb{N}_{0,T},\ t-\frac{1}{2}\neq t_k \\
&10u\left(\frac{1}{2}\right)+20\Delta^{-\frac{2}{3}}u\left(\frac{1}{6}\right) = 5u\left(\frac{21}{2}\right)+15\Delta^{-\frac{2}{3}}u\left(\frac{67}{6}\right)
\end{aligned} \tag{52}$$

where the impluse conditions subjected to $I_k, J_k : \mathbb{R}\to\mathbb{R}$ are given by

$$\begin{aligned}
&\Delta u(t_k) = \frac{1}{k+100}\cos|u(t_k-1)|,\ \ k=1,2,3 \\
&\Delta\left(\Delta^{-\frac{2}{3}}u\left(t_k+\frac{2}{3}\right)\right) = \frac{1}{(k+50)^2}\tan^{-1}\left|\Delta^{-\frac{2}{3}}u\left(t_k-\frac{1}{3}\right)\right|,\ \ k=1,2,3
\end{aligned} \tag{53}$$

and $\varphi(t,s)=\frac{e^{-|s-t|}}{(t+10)^3}$ and $t_k=\frac{1}{2}+2k\ \left(t_1=\frac{5}{2}, t_2=\frac{9}{2}, t_3=\frac{13}{2}\right)$.

Here $\alpha=\frac{1}{2}$, $\beta=\frac{2}{3}$, $\gamma=\frac{1}{3}$, $T=10$, $p=3$, $A=10$, $B=20$, $C=5$, $D=15$, and $F\left[t,u(t),\Psi^\gamma u(t)\right]=\frac{\cos^2\pi t}{2(t+100)^2}\left[\frac{u^2+|u|}{|u|+1}+e^{-\sin^2\pi t}\Psi^{\frac{1}{3}}u\left(t\right)\right]$.

We can find that

$$\begin{gathered}
\Theta = 6.74071,\ \ |\Lambda| = 75.57065,\ \ \Omega_1 = 25.90098,\ \ \Omega_2 = 144.78000, \\
\mathcal{P} = 46.7733,\ \ \mathcal{Q} = 7.46312\ \text{ and }\ \mathcal{R} = 0.73969.
\end{gathered}$$

Let $t\in\mathbb{N}_{\frac{1}{2},\frac{21}{2}}$ and $u,v\in\mathbb{R}$, we have

$$\left|F\left[t,u,\Psi^\gamma u\right]-F\left[t,v,\Psi^\gamma v\right]\right| \leq \frac{1}{2\left(\frac{1}{2}+100\right)^2}|u-v|+\frac{1}{2e^\pi\left(\frac{1}{2}+100\right)^2}\left|\Psi^\gamma u-\Psi^\gamma v\right|.$$

So, (H_1) holds with $\ell_1=0.000495$, $\ell_2=5.6095\times10^{-6}$, and we have $\varphi_0=0.00907$.

For all $u,v\in\mathcal{C}$ and $k=1,2,3$

$$I_k(u)-I_k(v)\leq\frac{1}{101}|u-v|\ \text{ and }\ J_k\left(\Delta^{-\beta}u\right)-I_k\left(\Delta^{-\beta}v\right)\leq\frac{1}{51^2}\left|\Delta^{-\beta}u-\Delta^{-\beta}v\right|$$

So, (H_2) holds with $\lambda_1=\frac{1}{101}$, $\lambda_2=\frac{1}{51^2}$.

Finally, (H_3) holds with

$$\mathcal{LP}+\lambda_1\mathcal{Q}+\lambda_2\mathcal{R}\ \approx\ 0.07418\ <\ 1.$$

Hence, by Theorem 1, the boundary value problem (52) and (53) has a unique solution.

In view of Theorem 3, we have

$$c_{F,p}=\frac{\mathcal{P}+\mathcal{Q}+\mathcal{R}}{|1-(\mathcal{LP}+\lambda_1\mathcal{Q}+\lambda_2\mathcal{R})|}\ \approx\ 2128.38328.$$

Therefore, problem (52) and (53) is Ulam–Hyers stable and hence generalized Ulam–Hyers stable.

By setting

$$c_{F,p,\rho} = \frac{\mathcal{O}}{|1 - (\mathcal{L}\mathcal{P} + \lambda_1\mathcal{Q} + \lambda_2\mathcal{R})|} \approx 44.03976, \text{ and}$$

$$\rho(t) = 0.02412\, t^{\underline{\alpha}} + 1.12568,\ \psi_1 = 0.13677,\ \psi_2 = 0.01698.$$

Thus, by Theorem 4, problem (52) and (53) is Ulam–Hyers–Rassias stable and also generalized Ulam–Hyers–Rassias stable.

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