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Singular Special Curves in 3-Space Forms

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Abstract: We study the singular Bertrand curves and Mannheim curves in the 3-dimensional space forms. We introduce the geometrical properties of such special curves. Moreover, we get the relationships between singularities of original curves and torsions of another mate curves.

Keywords: Bertrand curves; Mannheim curves; space forms; singularities

1. Introduction

In classical differential geometry, Bertrand curves as special curves have been deeply explored in Euclidean space; see [1]. For instance, in [2] Pears proved that a Bertrand curve in $\mathbb{R}^n$ must belong to a 3-dimensional subspace $\mathbb{R}^3 \subset \mathbb{R}^n$. In [3], Izumiya and Takeuchi constructed Bertrand curves from spherical curves and verified a fact that two disjoint minimal asymptotic curves on a ruled surface both of which are transversal to rulings are Bertrand curves in [4]. Moreover, mathematicians also studied Bertrand curves in other spaces, such as in the 3-dimensional sphere space $\mathbb{S}^3$ [5], in the 3-dimensional Riemannian space forms [6] and in non-flat 3-dimensional space forms [7,8].

Mannheim curves as another kind of special curves are broadly concerned. In [9], Liu and Wang focused on the Mannheim mate and showed the necessary and sufficient conditions for the existence of curves. Meanwhile, Mannheim curves also have been studied in the 3-dimensional Riemannian space forms [10] and in non-flat 3-dimensional space forms [11].

Though we cannot construct a Frenet–Serret frame of a smooth curve at a singularity, Takahashi and Honda defined a framed curve in $\mathbb{R}^n$, see [12,13]. If a smooth curve has a moving frame at every point, then we call it a framed curve. Notice that a framed curve may be having singularities. Framed curves are a generalization of Legendre curves and regular curves. For the regular Bertrand and Mannheim curves, Takahashi and Honda found that the existence condition is not sufficient. It turns out that the non-degenerate condition, that the curvature does not vanish, is needed. In [14], the authors added the non-degenerate condition when proving a regular curve is a Bertrand or Mannheim curve. They discussed a framed curve in $\mathbb{R}^3$, under what conditions, can be either a Bertrand or Mannheim curve. They found an interesting fact. If a framed curve is a Bertrand curve, then it is also a Mannheim curve. This result is not true for the regular case. In this paper, we concentrate on singular Bertrand and Mannheim curves in 3-space forms and we find out the relationship between singular points and the torsion τ.

We assume here that all maps and manifolds are C^∞ unless otherwise stated.

2. Preliminaries

We now review some basic notions and present the local differential geometry of Frenet type framed base curves in 3-space forms.

Let $\mathbb{R}^4_v$ be a 4-dimensional semi-Euclidean space with index v, where the standard metric of $\mathbb{R}^4_v$ is

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = -\sum_{i=1}^{v} x_i y_i + \sum_{j=v+1}^{4} x_j y_j,$$

where $\boldsymbol{x} = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4$, $\boldsymbol{y} = (y_1, y_2, y_3, y_4) \in \mathbb{R}^4$, and v =0 or 1. For a non-zero vector $\boldsymbol{x} \in \mathbb{R}^4_v$, if $\langle \boldsymbol{x}, \boldsymbol{x} \rangle > 0$, $\langle \boldsymbol{x}, \boldsymbol{x} \rangle = 0$ or $\langle \boldsymbol{x}, \boldsymbol{x} \rangle < 0$, then we call $\boldsymbol{x}$ spacelike, lightlike or timelike, respectively. We call $\|\boldsymbol{x}\| = \sqrt{|\langle \boldsymbol{x}, \boldsymbol{x} \rangle|}$ the norm of a given vector $\boldsymbol{x}$.

For any $\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3 \in \mathbb{R}^4_v$, the wedge product of them is

$$\boldsymbol{x}_1 \wedge \boldsymbol{x}_2 \wedge \boldsymbol{x}_3 = \begin{vmatrix} \delta \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 & \mathbf{e}_4 \\ x_1^1 & x_2^1 & x_3^1 & x_4^1 \\ x_1^2 & x_2^2 & x_3^2 & x_4^2 \\ x_1^3 & x_2^3 & x_3^3 & x_4^3 \end{vmatrix},$$

where $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, \boldsymbol{e}_4\}$ stands for a canonical basis of $\mathbb{R}^4_v$, $\boldsymbol{x}_i = (x_1^i, x_2^i, x_3^i, x_4^i)$ and $\delta = (-1)^v$, $i = 1, 2, 3$, $v = 0, 1$. Therefore, we have

$$\langle \boldsymbol{x}, \boldsymbol{x}_1 \wedge \boldsymbol{x}_2 \wedge \boldsymbol{x}_3 \rangle = \det(\boldsymbol{x}, \boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3),$$

we also say that $\boldsymbol{x}_1 \wedge \boldsymbol{x}_2 \wedge \boldsymbol{x}_3$ is pseudo-orthogonal to any $\boldsymbol{x}_i$, $i = 1, 2, 3$.

We define $\mathbb{M}^3(c) \subset \mathbb{R}^4_v$ the 3-dimensional space forms with constant curvature c. Therefore, we know that $\mathbb{M}^3(c)$ is the 3-dimensional Euclidean space $\mathbb{R}^3$ if $c = 0$, $\mathbb{M}^3(c)$ is the 3-dimensional sphere space $\mathbb{S}^3$ if $c = 1$, and $\mathbb{M}^3(c)$ is the 3-dimensional hyperbolic space $\mathbb{H}^3$ if $c = -1$. We assume that $\mathbb{S}^{2+|c|}$ is a $(2 + |c|)$-dimensional sphere space.

In [12], Honda and Takahashi introduced the notion of framed curves that is a smooth curve with a moving frame in $\mathbb{R}^n$. Although the framed curve has a moving frame at the singular point, we still want to construct a Frenet–Serret frame of the framed curve. In [13], Honda introduced a special framed curve, namely the Frenet type framed base curve, having a Frenet–Serret type frame at the singular point.

Now we will extend this idea to 3-space forms.

Definition 1. *We say that $\boldsymbol{\gamma} = \boldsymbol{\gamma}(t) : I \to \mathbb{M}^3(c)$ is a Frenet type framed base curve if there exists a smooth function $\alpha : I \to \mathbb{R}$ and a regular curve $\boldsymbol{T} : I \to \mathbb{S}^{2+|c|}$ satisfying $c\langle \boldsymbol{\gamma}(t), \boldsymbol{T}(t) \rangle = 0$ and $\dot{\boldsymbol{\gamma}}(t) = \alpha(t)\boldsymbol{T}(t)$ for all $t \in I$. Then $\alpha(t)$ and $\boldsymbol{T}(t)$ are called a speed function and a unit tangent vector of $\boldsymbol{\gamma}(t)$, respectively.*

By the above definition, t_0 is a singular point of $\boldsymbol{\gamma}$ if and only if $\alpha(t_0) = 0$. Now we construct a moving frame of $\boldsymbol{\gamma}$ in 3-dimensional space forms. We give the normal vector of $\boldsymbol{\gamma}(t)$ by $\boldsymbol{n}(t) = \dot{\boldsymbol{T}}(t) + c\alpha(t)\boldsymbol{\gamma}(t)$. If $\langle \dot{\boldsymbol{T}}(t), \dot{\boldsymbol{T}}(t) \rangle \neq c\alpha^2(t)$, then we can define the unit principal normal vector $\boldsymbol{N}(t)$ by

$$\boldsymbol{N}(t) = \frac{\dot{\boldsymbol{T}}(t) + c\alpha(t)\boldsymbol{\gamma}(t)}{\|\dot{\boldsymbol{T}}(t) + c\alpha(t)\boldsymbol{\gamma}(t)\|}.$$

The binormal vector $\boldsymbol{B}$ can be expressed as

$$\begin{cases} \boldsymbol{B}(t) = \boldsymbol{T}(t) \wedge \boldsymbol{N}(t), & \text{c=0;} \\ \boldsymbol{B}(t) = \boldsymbol{\gamma}(t) \wedge \boldsymbol{T}(t) \wedge \boldsymbol{N}(t), & \text{c=}\pm 1. \end{cases}$$

Therefore, we get an orthonormal frame $\{\boldsymbol{T}(t), \boldsymbol{N}(t), \boldsymbol{B}(t)\}$ along $\boldsymbol{\gamma}(t)$ in $\mathbb{M}^3(c)$.

The Frenet–Serret type formula holds:

$$\begin{cases} \dot{\boldsymbol{T}}(t) = -c\alpha(t)\boldsymbol{\gamma}(t) + \kappa(t)\boldsymbol{N}(t), \\ \dot{\boldsymbol{N}}(t) = -\kappa(t)\boldsymbol{T}(t) + \tau(t)\boldsymbol{B}(t), \\ \dot{\boldsymbol{B}}(t) = -\tau(t)\boldsymbol{N}(t), \end{cases}$$

where $\kappa(t)$ and $\tau(t)$ are the curvature and torsion of $\boldsymbol{\gamma}(t)$, respectively. Note that $\kappa(t)$ and $\tau(t)$ are dependent on a choice of parametrization. The condition $\langle \dot{\boldsymbol{T}}(t), \dot{\boldsymbol{T}}(t) \rangle \neq c\alpha^2(t)$ means that the curvature does not vanish. We call such $\boldsymbol{\gamma}$ a non-degenerate curve (cf. [14]).

For any point $\boldsymbol{\gamma}(t)$ in the curve $\boldsymbol{\gamma}$, the geodesics in $\mathbb{M}^3(c)$ starting at $\boldsymbol{\gamma}(t)$ with the velocity $\boldsymbol{\delta}(t)$ is defined as

$$\Gamma_t^{\gamma}(v) = \exp_{\boldsymbol{\gamma}(t)}(v\boldsymbol{\delta}(t)) = f(v)\boldsymbol{\gamma}(t) + g(v)\boldsymbol{\delta}(t),\ v \in \mathbb{R},$$

where the functions f and g are expressed as

$$\begin{cases} f(v) = 1, & g(v) = v, & \mathbb{M}^3(c) = \mathbb{R}^3, \\ f(v) = \cos v, & g(v) = \sin v, & \mathbb{M}^3(c) = \mathbb{S}^3, \\ f(v) = \cosh v, & g(v) = \sinh v, & \mathbb{M}^3(c) = \mathbb{H}^3, \end{cases}$$

and $c = 0,\ 1$ or -1 if $\mathbb{M}^3(c)$ is $\mathbb{R}^3$, $\mathbb{S}^3$ or $\mathbb{H}^3$, respectively.

3. Bertrand Curves of Frenet Type Framed Base Curves in 3-Space Forms

In [14], Honda and Takahashi added the existence condition of Bertrand curves in $\mathbb{R}^3$. They stressed that the Bertrand mate must be a non-degenerate curve. Through the above definition of Frenet type framed base curve, we know that the Frenet type framed base curve is a non-degenerate curve. Now we give the definition of the Bertrand curve of a Frenet type framed base curve in $\mathbb{M}^3(c)$.

Definition 2. *A Frenet type framed base curve $\boldsymbol{\gamma} = \boldsymbol{\gamma}(t) : I \to \mathbb{M}^3(c)$ is called a Bertrand curve of a Frenet type framed base curve if there is another Frenet type framed base curve $\boldsymbol{\beta} = \boldsymbol{\beta}(t) : I \to \mathbb{M}^3(c)$ $(\boldsymbol{\gamma} \neq \boldsymbol{\beta})$ such that the principal normal geodesics of $\boldsymbol{\gamma}$ and $\boldsymbol{\beta}$ are parallel at corresponding points. We call $\boldsymbol{\beta}$ the Bertrand mate of $\boldsymbol{\gamma}$.*

Assume that $\boldsymbol{\gamma}(t)$ is a Bertrand curve of a Frenet type framed base curve and $\boldsymbol{\beta}(t)$ is the Bertrand mate of $\boldsymbol{\gamma}$, then there exists a constant a satisfying

$$\boldsymbol{\beta}(t) = f(a)\boldsymbol{\gamma}(t) + g(a)\boldsymbol{N}(t).$$

Then we have some conclusions similar to the regular curve case in $\mathbb{M}^3(c)$, for more detail [6,7].

Proposition 1. *Let $\boldsymbol{\gamma}$ be a Bertrand curve of a Frenet type framed base curve in $\mathbb{M}^3(c)$ and $\boldsymbol{\beta}$ be the Bertrand mate of $\boldsymbol{\gamma}$, then the following properties hold.*

(1) The tangent vectors of $\boldsymbol{\gamma}$ make a constant angle with the tangent vectors of $\boldsymbol{\beta}$ at corresponding points.
(2) The binormal vectors of $\boldsymbol{\gamma}$ make a constant angle with the binormal vectors of $\boldsymbol{\beta}$ at corresponding points.

In the paper, we assume that $f(a)g(a) \neq 0$. Otherwise, we have the fact that $\boldsymbol{\gamma} = \pm\boldsymbol{\beta}$ or $\boldsymbol{\gamma}$ is a regular Bertrand curve in $\mathbb{M}^3(c)$.

Proposition 2. *Let $\boldsymbol{\gamma}$ be a Bertrand curve of a Frenet type framed base curve in $\mathbb{M}^3(c)$ and $\boldsymbol{\beta}$ be the Bertrand mate of $\boldsymbol{\gamma}$. Then there exist two constants a and θ satisfying the following formulas*

(1) $(\alpha(t)f(a) - \kappa(t)g(a))\sin\theta = \tau(t)g(a)\cos\theta,$
(2) $(\alpha_\beta(t)f(a) + \epsilon\kappa_\beta(t)g(a))\sin\theta = \tau_\beta(t)g(a)\cos\theta,$
(3) $\alpha(t)\alpha_\beta(t)\cos^2\theta = (\alpha(t)f(a) - \kappa(t)g(a))(\alpha_\beta(t)f(a) + \epsilon\kappa_\beta(t)g(a)),$
(4) $\alpha(t)\alpha_\beta(t)\sin^2\theta = \tau(t)\tau_\beta(t)g^2(a),$

where $\epsilon = \pm 1$, θ is the constant angle between the tangent vectors of $\boldsymbol{\gamma}$ and $\boldsymbol{\beta}$, $\kappa(t)$, $\tau(t)$, $\alpha(t)$, $\kappa_\beta(t)$, $\tau_\beta(t)$ and $\alpha_\beta(t)$ denote the curvature, torsion and speed function of $\boldsymbol{\gamma}$ and $\boldsymbol{\beta}$, respectively.

Proposition 3. *If $\boldsymbol{\gamma}$ is a plane Frenet type framed base curve in $\mathbb{M}^3(c)$, then $\boldsymbol{\gamma}$ is a Bertrand curve. If $\boldsymbol{\gamma}$ is a Frenet type framed base curve in $\mathbb{M}^3(c)$ and plane curve $\boldsymbol{\beta}$ is the Bertrand mate of $\boldsymbol{\gamma}$, then $\boldsymbol{\gamma}$ is a plane curve.*

Theorem 1. *A Frenet type framed base curve $\boldsymbol{\gamma}$ in $\mathbb{M}^3(c)$ is a Bertrand curve if and only if (1) $\boldsymbol{\gamma}$ is a plane curve or (2) $\lambda_1\kappa(t) + \lambda_2\tau(t) = \alpha(t)$ and $\lambda_2\kappa(t) - \lambda_1\tau(t) + c\lambda_1\lambda_2\alpha(t) \neq 0$, where $\lambda_1(\neq 0)$ and λ_2 are constants.*

Proof. Firstly, we suppose that $\boldsymbol{\gamma}$ is a space Bertrand curve. By Proposition 2 (1), we obtain

$$\lambda_1\kappa(t) + \lambda_2\tau(t) = \alpha(t),$$

for constants $\lambda_1 = g(a)/f(a)$ and $\lambda_2 = g(a)\cos\theta/f(a)\sin\theta$. Let

$$\boldsymbol{\beta}(t) = f(a)\boldsymbol{\gamma}(t) + g(a)\boldsymbol{N}(t)$$

be the Bertrand mate of $\boldsymbol{\gamma}$. We assume $\boldsymbol{T}_\beta(t) = \cos\theta\boldsymbol{T}(t) + \sin\theta\boldsymbol{B}(t)$. Taking the derivative of $\boldsymbol{T}_\beta$, we obtain the following formula

$$\kappa_\beta(t)\boldsymbol{N}_\beta(t) = (c\alpha_\beta(t)f(a) - c\alpha(t)\cos\theta)\boldsymbol{\gamma}(t) + (c\alpha_\beta(t)g(a) + \kappa(t)\cos\theta - \tau(t)\sin\theta)\boldsymbol{N}(t).$$

Because of the definition of Bertrand curves, we have

$$\frac{d}{dv}\,|_{v=a}\,\Gamma_t^{\gamma}(v) = \epsilon\boldsymbol{N}_\beta(t).$$

By $f^2(a) + cg^2(a) = 1$ and $\lambda_1\cos\theta = \lambda_2\sin\theta$, then we have

$$\epsilon\kappa_\beta(t) = \frac{f(a)\sin\theta(\lambda_2\kappa(t) - \lambda_1\tau(t) + c\lambda_1\lambda_2\alpha(t))}{\lambda_1}.$$

Since the Bertrand mate $\boldsymbol{\beta}$ is a non-degenerate curve, that means $\kappa_\beta(t) \neq 0$, for all $t \in I$. Therefore that concludes the proof.

Conversely, let us assume that $\lambda_1\kappa(t) + \lambda_2\tau(t) = \alpha(t)$ for certain constants $\lambda_1(\neq 0)$ and λ_2. We define another curve $\boldsymbol{\beta}$ in $\mathbb{M}^3(c)$ given by $\boldsymbol{\beta}(t) = f(a)\boldsymbol{\gamma}(t) + g(a)\boldsymbol{N}(t)$, where a is a constant number such that $g(a) = \lambda_1 f(a)$. By taking the derivative of $\boldsymbol{\beta}$, we see that

$$\alpha_\beta(t)\boldsymbol{T}_\beta(t) = f(a)\tau(t)\sqrt{\lambda_1^2+\lambda_2^2}\frac{\lambda_2\boldsymbol{T}(t) + \lambda_1\boldsymbol{B}(t)}{\sqrt{\lambda_1^2+\lambda_2^2}}.$$

Therefore, we assume that

$$\boldsymbol{T}_\beta(t) = \frac{\lambda_2\boldsymbol{T}(t) + \lambda_1\boldsymbol{B}(t)}{\sqrt{\lambda_1^2+\lambda_2^2}}.$$

Continuing to take the derivative, we get

$$\begin{aligned}\kappa_\beta(t)\boldsymbol{N}_\beta(t) &= (c\alpha_\beta(t)f(a) - \frac{\lambda_2 c\alpha(t)}{\sqrt{\lambda_1^2+\lambda_2^2}})\boldsymbol{\gamma}(t) + (c\alpha_\beta(t)g(a) + \frac{\lambda_2\kappa(t) - \lambda_1\tau(t)}{\sqrt{\lambda_1^2+\lambda_2^2}})\boldsymbol{N}(t)\\ &= \frac{f(a)(\lambda_2\kappa(t) - \lambda_1\tau(t) + c\lambda_1\lambda_2\alpha(t))}{\sqrt{\lambda_1^2+\lambda_2^2}}(-cg(a)\boldsymbol{\gamma}(t) + f(a)\boldsymbol{N}(t)).\end{aligned}$$

Therefore, the principal normal vector of $\boldsymbol{\beta}(t)$ is

$$\boldsymbol{N}_\beta(t) = -\epsilon c g(a)\boldsymbol{\gamma}(t) + \epsilon f(a)\boldsymbol{N}(t).$$

Then we obtain the principal normal geodesic starting at a point $\boldsymbol{\beta}(t_0)$

$$\Gamma^{\beta}_{t_0}(v) = f(v)\boldsymbol{\beta}(t_0) + g(v)\boldsymbol{N}_\beta(t_0) = f(v+\epsilon a)\boldsymbol{\gamma}(t_0) + g(v+\epsilon a)\boldsymbol{N}(t_0).$$

□

For a regular Bertrand curve $\boldsymbol{\gamma}$, if the torsion of $\boldsymbol{\gamma}$ vanishes at some point, then $\boldsymbol{\gamma}$ is a plane curve. However, for a Bertand curve of a Frenet type framed base curve, if there exists t_0 satisfying $\tau(t_0) = 0$, then it is either a plane curve or a space curve which has singular points. So we can see that these points at which torsion vanishes have relationships with the singular points of Bertrand curves. Under the above assumption, we come to the following conclusion.

Corollary 1. *Let $\boldsymbol{\gamma}$ be a Bertrand curve of a Frenet type framed base curve in $\mathbb{M}^3(c)$ and $\boldsymbol{\beta}$ be the Bertrand mate of $\boldsymbol{\gamma}$. Then $\alpha_\beta(t) = \pm f(a)\tau(t)\sqrt{\lambda_1^2 + \lambda_2^2}$ and $\alpha(t) = f(a)\tau_\beta(t)\sqrt{\lambda_1^2 + \lambda_2^2}$.*

Proof. By the above discussion, we know $\alpha_\beta(t) = \pm f(a)\tau(t)\sqrt{\lambda_1^2 + \lambda_2^2}$. Using the Frenet frame of $\boldsymbol{\gamma}$, the Frenet frame of $\boldsymbol{\beta}$ can be expressed as

$$\begin{aligned} \beta(t) &= f(a)\boldsymbol{\gamma}(t) + g(t)\boldsymbol{N}(t), \\ \boldsymbol{T}_\beta(t) &= \cos\theta \boldsymbol{T}(t) + \sin\theta \boldsymbol{B}(t), \\ \epsilon \boldsymbol{N}_\beta(t) &= -cg(a)\boldsymbol{\gamma}(t) + f(a)\boldsymbol{N}(t), \\ \epsilon \boldsymbol{B}_\beta(t) &= -\sin\theta \boldsymbol{T}(t) + \cos\theta \boldsymbol{B}(t). \end{aligned}$$

Since $\dot{\boldsymbol{B}}_\beta(t) = -\tau_\beta(t)\boldsymbol{N}_\beta(t)$, then $\tau_\beta(t) = \alpha(t)\sin\theta/\lambda_1 f(a)$. Meanwhile, we have $\lambda_1 = \sin\theta\sqrt{\lambda_1^2 + \lambda_2^2}$. □

4. Mannheim Curves of Frenet Type Framed Base Curves in 3-Space Forms

Definition 3. *A Frenet type framed base curve $\boldsymbol{\gamma} = \boldsymbol{\gamma}(t) : I \to \mathbb{M}^3(c)$ is called a Mannheim curve of a Frenet type framed base curve if there is another Frenet type framed base curve $\boldsymbol{\beta} = \boldsymbol{\beta}(t) : I \to \mathbb{M}^3(c)$ $(\boldsymbol{\gamma} \neq \boldsymbol{\beta})$ such that the principal normal geodesic of $\boldsymbol{\gamma}$ and the binormal geodesic of $\boldsymbol{\beta}$ are parallel at corresponding points. We call $\boldsymbol{\beta}$ the Mannheim mate of $\boldsymbol{\gamma}$.*

Suppose that $\boldsymbol{\gamma}(t)$ and $\boldsymbol{\beta}(t)$ are a pair of Mannheim curves of Frenet type framed base curves, then there is a constant a satisfying

$$\boldsymbol{\beta}(t) = f(a)\boldsymbol{\gamma}(t) + g(a)\boldsymbol{N}(t).$$

More details are available from [9,10].

Theorem 2. *A Frenet type framed base curve $\boldsymbol{\gamma}$ in $\mathbb{M}^3(c)$ is a Mannheim curve if and only if it simultaneously satisfies the following equations,*

(1) $f(a)\kappa(t) + cg(a)\alpha(t) \neq 0$,
(2) $f(a)(\dot{\kappa}(t)\tau(t) - \kappa(t)\dot{\tau}(t)) + cg(a)(\dot{\alpha}(t)\tau(t) - \alpha(t)\dot{\tau}(t)) \neq 0$,
(3) $\alpha(t)\kappa(t)(f^2(a) - cg^2(a)) + c\alpha^2(t)f(a)g(a) = f(a)g(a)(\kappa^2(t) + \tau^2(t))$.

Proof. Let $\gamma(t)$ be a Mannheim curve of a Frenet type framed base curve in $\mathbb{M}^3(c)$ and $\boldsymbol{\beta}(t)$ be a Mannheim mate of γ. Suppose that $\boldsymbol{\beta}(t) = f(a)\gamma(t) + g(a)\boldsymbol{N}(t)$ and a is a constant such that $f(a)g(a) \neq 0$. Differentiating $\boldsymbol{\beta}(t)$ with respect to t,

$$\alpha_\beta(t)\boldsymbol{T}_\beta(t) = (f(a)\alpha(t) - g(a)\kappa(t))\boldsymbol{T}(t) + g(a)\tau(t)\boldsymbol{B}(t). \tag{1}$$

Assume that there is a function $\theta(t) : I \to \mathbb{R}$ such that

$$\boldsymbol{T}_\beta(t) = \cos\theta(t)\boldsymbol{T}(t) + \sin\theta(t)\boldsymbol{B}(t). \tag{2}$$

Due to (1) and (2), we have

$$(f(a)\alpha(t) - g(a)\kappa(t))\sin\theta(t) = g(a)\tau(t)\cos\theta(t). \tag{3}$$

By differentiating (2) with respect to t,

$$\begin{aligned}\kappa_\beta(t)\boldsymbol{N}_\beta(t) = {} & (cf(a)\alpha_\beta(t) - c\alpha(t)\cos\theta(t))\gamma(t) + (cg(a)\alpha_\beta(t) + \kappa(t)\cos\theta(t) - \tau(t)\sin\theta(t))\boldsymbol{N}(t) \\ & - \sin\theta(t)\dot{\theta}(t)\boldsymbol{T}(t) + \cos\theta(t)\dot{\theta}(t)\boldsymbol{B}(t).\end{aligned}$$

Since $\boldsymbol{\beta}$ is the Mannheim mate of γ, then

$$\boldsymbol{B}_\beta(t) = \epsilon(-cg(a)\gamma(t) + f(a)\boldsymbol{N}(t)).$$

We have known that $\boldsymbol{N}_\beta$ is orthogonal to $\boldsymbol{B}_\beta$ and $\boldsymbol{\beta}$, then

$$cf(a)\alpha_\beta(t) - c\alpha(t)\cos\theta(t) = 0, \tag{4}$$

$$cg(a)\alpha_\beta(t) + \kappa(t)\cos\theta(t) - \tau(t)\sin\theta(t) = 0. \tag{5}$$

We apply $f^2(a) + cg^2(a) = 1$ and obtain

$$\alpha_\beta(t) = f(a)\alpha(t)\cos\theta(t) - g(a)\kappa(t)\cos\theta(t) + g(a)\tau(t)\sin\theta(t), \tag{6}$$

$$(f(a)\kappa(t) + cg(a)\alpha(t))\cos\theta(t) = f(a)\tau(t)\sin\theta(t).$$

We assume $f(a)\kappa(t) + cg(a)\alpha(t) \neq 0$, and then

$$\cos\theta(t) = \frac{f(a)\tau(t)\sin\theta(t)}{f(a)\kappa(t) + cg(a)\alpha(t)}. \tag{7}$$

We put (7) in (3), then

$$\alpha(t)\kappa(t)(f^2(a) - cg^2(a)) + c\alpha^2(t)f(a)g(a) = f(a)g(a)(\kappa^2(t) + \tau^2(t)).$$

By differentiating (5), (6) and applying (7), we obtain

$$\kappa_\beta(t) = \dot{\theta}(t) = \frac{f^2(a)(\dot{\kappa}(t)\tau(t) - \kappa(t)\dot{\tau}(t)) + cg(a)f(a)(\dot{\alpha}(t)\tau(t) - \alpha(t)\dot{\tau}(t))}{f^2(a)\tau^2(t) + (f(a)\kappa(t) + cg(a)\alpha(t))^2}.$$

Since $\boldsymbol{\beta}(t)$ is a non-degenerate curve, that is $\kappa_\beta(t) \neq 0$, the proof is complete.

Conversely, for some curve in $\mathbb{M}^3(c)$, its curvature and torsion satisfy

$$f(a)\alpha(t) - g(a)\kappa(t) = \lambda(t)g(a)\tau(t), \tag{8}$$

$$\lambda(t) = \frac{f(a)\tau(t)}{f(a)\kappa(t) + cg(a)\alpha(t)}. \tag{9}$$

We define a curve $\boldsymbol{\beta}(t)$ by using $f(a)$ and $g(a)$, $\boldsymbol{\beta}(t) = f(a)\boldsymbol{\gamma}(t) + g(t)\boldsymbol{N}(t)$. We assume that

$$\boldsymbol{T}_\beta(t) = \cos\theta(t)\boldsymbol{T}(t) + \sin\theta(t)\boldsymbol{B}(t),$$

where $\lambda(t) = \cos\theta(t)/\sin\theta(t)$. By direct differentiating, we easily find that (4), (5) are satisfied. Moreover, we see that

$$\kappa_\beta(t) = \dot{\theta}(t), \ \ \boldsymbol{N}_\beta(t) = -\sin\theta(t)\boldsymbol{T}(t) + \cos\theta(t)\boldsymbol{B}(t).$$

By taking the derivative of $\boldsymbol{N}_\beta(t)$ and applying (8) and (9), we have $\boldsymbol{B}_\beta(t) = \epsilon(-cg(a)\boldsymbol{\gamma}(t) + f(a)\boldsymbol{N}(t))$ and $f(v \pm a)\boldsymbol{\gamma}(t) + g(v \pm a)\boldsymbol{N}(t) = f(v)\boldsymbol{\beta}(t) + g(v)\boldsymbol{B}_\beta(t)$. Therefore, $\boldsymbol{\gamma}$ is a Mannheim curve of Frenet type framed base curve. $\square$

Next, we will study the existence condition of Mannheim mates of Frenet type framed base curves in $\mathbb{M}^3(c)$. By the similar method used in Theorem 2, we come to the following theorem.

Theorem 3. *A Frenet type framed base curve $\boldsymbol{\beta}$ in $\mathbb{M}^3(c)$ is a Mannheim mate if and only if $\tau_\beta^2(t) - c\alpha_\beta^2(t) \neq 0$, $(f^2(a)\alpha_\beta^2(t) + g^2(a)\tau_\beta^2(t))\kappa_\beta(t) = f(a)g(a)(\tau_\beta(t)\dot{\alpha}_\beta(t) - \dot{\tau}_\beta(t)\alpha_\beta(t))$.*

Corollary 2. *Let $\boldsymbol{\gamma}$ be a Mannheim curve of a Frenet type framed base curve and $\boldsymbol{\beta}$ be the Mannheim mate of $\boldsymbol{\gamma}$, then*

$$\alpha_\beta(t) = g(a)\tau(t)\sqrt{1 + \left(\frac{f(a)\tau(t)}{f(a)\kappa(t) + cg(a)\alpha(t)}\right)^2}$$

and $\boldsymbol{\gamma}$ is a regular curve.

Therefore, we know that the torsion of a Mannheim curve of a Frenet type framed base curve vanishes at the singular point of the Mannheim curve. And there does not exist singular Mannheim curve of a Frenet type framed base curve in 3-space forms.

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