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Article

New Criteria for Meromorphic Starlikeness and Close-to-Convexity

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Abstract: The main purpose of current paper is to obtain some new criteria for meromorphic strongly starlike functions of order α and strongly close-to-convexity of order α. Furthermore, the main results presented here are compared with the previous outcomes obtained in this area.

Keywords: differential subordination; strongly close-to-convex functions; starlike functions; meromorphic strongly starlike functions

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1. Introduction and Preliminaries

For two analytic functions f and F in $\mathbb{U} := \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}$, it is stated that the function f *is subordinate to* the function F in $\mathbb{U}$, written as $f(z) \prec F(z)$, if there exists a Schwarz function ϖ, which is analytic in $\mathbb{U}$ with

$$\varpi(0) = 0 \qquad \text{and} \qquad |\varpi(z)| < 1 \qquad (z \in \mathbb{U}),$$

such that $f(z) = F\big(\varpi(z)\big)$ for all $z \in \mathbb{U}$. In particular, if F be a univalent function in $\mathbb{U}$, then we have below equivalence:

$$f(z) \prec F(z) \iff f(0) = F(0) \quad \text{and} \quad f(\mathbb{U}) \subset F(\mathbb{U}).$$

Let Σ_n denote the category of all functions analytic in the punctured open unit disk $\mathbb{U}^*$ given by

$$\mathbb{U}^* := \{z : z \in \mathbb{C} \quad \text{and} \quad 0 < |z| < 1\} = \mathbb{U} \setminus \{0\},$$

which have the form

$$f(z) = \frac{1}{z} + \sum_{k=n}^{\infty} a_{k-1} z^{k-1} \quad (n \in \mathbb{N} := \{1, 2, \cdots\}). \tag{1}$$

A function $f \in \Sigma$, where Σ is the union of Σ_n for all positive integers n, is said to be in the class $\widetilde{\mathcal{MS}}^*(\alpha)$ of meromorphic strongly starlike functions of order α if we have the condition

$$\left|\arg\left(-\frac{zf'(z)}{f(z)}\right)\right| < \frac{\alpha\pi}{2} \quad (z \in \mathbb{U}^*;\ 0 < \alpha \leqq 1).$$

In particular, $\mathcal{MS}^* := \widetilde{\mathcal{MS}}^*(1)$ is the class of meromorphic starlike functions in the open unit disk $\mathbb{U}$.

Let $\mathcal{A}_n$ be the category of all functions analytic in $\mathbb{U}$ which have the following form

$$f(z) = z^n + \sum_{k=n+1}^{\infty} a_k z^k \quad (n \in \mathbb{N}). \tag{2}$$

The class $\mathcal{A}_1$ is denoted by $\mathcal{A}$.

Let $\widetilde{\mathcal{S}}^*(\alpha)$ be the subcategory of $\mathcal{A}$ defined as follows

$$\widetilde{\mathcal{S}}^*(\alpha) := \left\{ f : f \in \mathcal{A} \quad \text{and} \quad \left| \arg\left(\frac{zf'(z)}{f(z)} \right) \right| < \frac{\alpha\pi}{2} \quad (z \in \mathbb{U};\ 0 < \alpha \leqq 1) \right\}.$$

The classes $\widetilde{\mathcal{S}}^*(\alpha)$ will be called the class of *strongly starlike functions of order* α. In particular, $\mathcal{S}^* := \widetilde{\mathcal{S}}^*(1)$ is the class of *starlike functions* in $\mathbb{U}$.

By means of the principle of subordination between analytic functions, the above definition is equivalent to

$$\widetilde{\mathcal{S}}^*(\alpha) := \left\{ f : f \in \mathcal{A} \quad \text{and} \quad \frac{zf'(z)}{f(z)} \prec \left(\frac{1+z}{1-z} \right)^{\alpha} \quad (z \in \mathbb{U};\ 0 < \alpha \leqq 1) \right\}.$$

Furthermore, let $\widetilde{\mathcal{CC}}(\alpha)$ denote the category of all functions in $\mathcal{A}$ which are *strongly close-to-convex of order* α in $\mathbb{U}$ if there exists a function $g \in \mathcal{S}^*$ such that

$$\left| \arg\left(\frac{zf'(z)}{g(z)} \right) \right| < \frac{\alpha\pi}{2} \qquad (z \in \mathbb{U};\ 0 < \alpha \leqq 1).$$

In particular, $\mathcal{CC} := \widetilde{\mathcal{CC}}(1)$ is the class of *close-to-convex functions* in $\mathbb{U}$.

In the year 1978, Miller and Mocanu [1] introduced the method of differential subordinations. Because of the interesting properties and applications possessed by the Briot-Bouquet differential subordination, there have been many attempts to extend these results. Then, in recent years, several authors obtained several applications of the method of differential subordinations in geometric function theory by using differential subordination associated with starlikeness, convexity, close-to-convexity and so on (see, for example, [2–13]). Furthermore, based on the generalized Jack lemma, the well-known lemma of Nunokawa and so on, certain sufficient conditions were derived in [14–16] considering concept of arg, real part and imaginary part for function to be p-valently starlike and convex one in the unit disk.

The aim of the current paper is to obtain some new criteria for univalence, strongly starlikeness and strongly close-to-convexity of functions in the normalized analytic function class $\mathcal{A}_n$ in the open unit disk $\mathbb{U}$ and meromorphic strongly starlikeness in the punctured open unit disk $\mathbb{U}^*$ by using a lemma given by Nunokawa (see [17,18]). Further, the current results are compared with the previous outcomes obtained in this area.

In order to prove our main results, we require the following lemma.

Lemma 1 (**see** [17,18])**.** *Let the function* $p(z)$ *given by*

$$p(z) = 1 + \sum_{n=m}^{\infty} c_n z^n \qquad (c_m \neq 0;\ m \in \mathbb{N})$$

be analytic in $\mathbb{U}$ *with*

$$p(0) = 1 \qquad \text{and} \qquad p(z) \neq 0 \qquad (z \in \mathbb{U}).$$

If there exists a point z_0 (with $|z_0| < 1$) such that

$$\left|\arg\left(p(z)\right)\right| < \frac{\gamma\pi}{2} \qquad (|z| < |z_0|)$$

and

$$\left|\arg\left(p(z_0)\right)\right| = \frac{\gamma\pi}{2}$$

for some $\gamma > 0$, then

$$\frac{z_0 p'(z_0)}{p(z_0)} = ik\gamma \qquad (i = \sqrt{-1}),$$

where

$$k \geqq \frac{m(a + a^{-1})}{2} \geqq m \quad \text{when} \quad \arg\left(p(z_0)\right) = \frac{\gamma\pi}{2} \tag{3}$$

and

$$k \leqq -\frac{m(a + a^{-1})}{2} \leqq -m \quad \text{when} \quad \arg\left(p(z_0)\right) = -\frac{\gamma\pi}{2}, \tag{4}$$

where

$$[p(z_0)]^{1/\gamma} = \pm ia \quad \text{and} \quad a > 0.$$

2. Main Results

Theorem 1. *Let p be an analytic function in $\mathbb{U}$, given by*

$$p(z) = 1 + \sum_{n=m}^{\infty} c_n z^n \qquad (c_m \neq 0;\ m \geq 2)$$

and $p(z) \neq 0$ for $z \in \mathbb{U}$. Let α_0 is the only root of the equation

$$\arctan(2m\alpha) - \pi\alpha = 0.$$

If

$$\left|\arg\left(p^2(z) - 2p(z)zp'(z)\right)\right| < \frac{\pi}{2}\left[\frac{2}{\pi}\arctan(2m\alpha) - 2\alpha\right], \tag{5}$$

where $0 < \alpha < \alpha_0$, then

$$\left|\arg\left(p(z)\right)\right| < \frac{\alpha\pi}{2} \qquad (z \in \mathbb{U}).$$

Proof. To prove our result we suppose that there exists a point $z_0 \in \mathbb{U}$ so that

$$\left|\arg\left(p(z)\right)\right| < \frac{\alpha\pi}{2} \quad \text{for} \quad |z| < |z_0|$$

and

$$\left|\arg\left(p(z_0)\right)\right| = \frac{\alpha\pi}{2}.$$

Then, Lemma 1, gives us that

$$\frac{zp'(z_0)}{p(z_0)} = ik\alpha,$$

where $[p(z_0)]^{\frac{1}{\alpha}} = \pm ia$ $(a > 0)$ and k is given by (3) or (4).

For the case $\arg\left(p(z_0)\right) = \dfrac{\alpha\pi}{2}$ when

$$[p(z_0)]^{\frac{1}{\alpha}} = ia \qquad (a > 0),$$

with $k \geqq m$, we have

$$\begin{aligned}\arg\left(p^2(z_0) - 2p(z_0)z_0p'(z_0)\right) &= \arg\left(p^2(z_0)\left(1 - 2\frac{z_0p'(z_0)}{p(z_0)}\right)\right)\\ &= \arg\left(p^2(z_0)\right) + \arg\left(1 - 2\frac{z_0p'(z_0)}{p(z_0)}\right)\\ &= 2\arg\left(p(z_0)\right) + \arg\left(1 - i2k\alpha\right)\\ &\leqq \alpha\pi - \arctan(2m\alpha)\\ &= \frac{-\pi}{2}\left(\frac{2}{\pi}\arctan(2m\alpha) - 2\alpha\right),\end{aligned}$$

which contradicts with condition (5).

Next, for the case $\arg\left(p(z_0)\right) = -\frac{\alpha\pi}{2}$ when

$$[p(z_0)]^{\frac{1}{\alpha}} = -ia \qquad (a > 0),$$

with $k \leqq -m$, applying the similar method as the above, we can get

$$\begin{aligned}\arg\left(p^2(z_0) - 2p(z_0)z_0p'(z_0)\right) &\leqq -\alpha\pi + \arctan(2m\alpha)\\ &= \frac{\pi}{2}\left(\frac{2}{\pi}\arctan(2m\alpha) - 2\alpha\right),\end{aligned}$$

which is a contradiction to (5).

Therefore, from the two mentioned contradictions, we obtain

$$\left|\arg\left(p(z)\right)\right| < \frac{\alpha\pi}{2} \qquad (z \in \mathbb{U}).$$

This completes our proof. □

Let $\psi(r, s, t; z) : \mathbb{C}^3 \times \mathbb{U} \to \mathbb{C}$ and let h be univalent in $\mathbb{U}$. If p is analytic in $\mathbb{U}$ and satisfies the (second order) differential subordination

$$\psi(p(z), zp'(z), z^2p''(z); z) \prec h(z), \tag{6}$$

then p is called a solution of the differential subordination. The univalent function q is called a dominant of the solution of the differential subordination or more simply a dominant, if $p \prec q$ for all p satisfying (6). A dominant $\tilde{q}$ satisfying $\tilde{q} \prec q$ for all dominants q of (6) is said to be the best dominant of (6). The best dominant is unique up to a rotation of $\mathbb{U}$. If $p(z) = 1 + a_nz^n + a_{n+1}z^{n+1} + \cdots$ be analytic in $\mathbb{U}$, then p will be called a $(1, n)$-solution, q a $(1, n)$-dominant, and $\tilde{q}$ the best $(1, n)$-dominant.

The following result, which is one of the types of differential subordinations was expressed in [1].

Theorem 2 ([19], Theorem 3.1e, p. 77). *Let h be convex in $\mathbb{U}$, with $h(0) = 1$ and $\operatorname{Re} h(z) > 0$. Let also $p(z) = 1 + a_nz^n + a_{n+1}z^{n+1} + \cdots$ be analytic in $\mathbb{U}$. If p satisfies*

$$p^2(z) + 2p(z)zp'(z) \prec h(z), \tag{7}$$

then

$$p(z) \prec q(z) = \sqrt{Q(z)},$$

where

$$Q(z) = \frac{1}{nz^{\frac{1}{n}}} \int_0^z h(t) t^{\frac{1}{n}-1} dt$$

and the function q is the best $(1, n)$*-dominant.*

Remark 1. *The form* (5) *cannot be used to obtain in inequality* (7)*. Therefore, Theorem* 1 *is a small extension of Theorem 2.*

For $m = 2$ in Theorem 1 we have

$$\sigma_2(\alpha) =: \frac{2}{\pi} \arctan(4\alpha) - 2\alpha > 0 \tag{8}$$

for $\alpha \in (0, \alpha_0)$ which $\alpha_0 = 1/4$ is the smallest positive root of the equation $\sigma_2(\alpha)$. So we have the following results

Remark 2. *Suppose that* $f \in \Sigma_1$ *with*

$$p(z) := -\frac{zf'(z)}{f(z)} \neq 0,$$

and $0 < \alpha < 1/4$ *satisfy the following inequality*

$$\left| \arg \left(\left(\frac{zf'(z)}{f(z)} \right)^2 \left(1 - 2 \left[1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} \right] \right) \right) \right| < \frac{\sigma_2(\alpha)\pi}{2},$$

where $\sigma_2(\alpha)$ *is given by* (8)*. Then f is meromorphic strongly starlike function of order* α*.*

Remark 3. *Suppose that* $f \in \mathcal{A}_2$ *with*

$$p(z) := \sqrt{\frac{f(z)}{z}} \neq 0,$$

and $0 < \alpha < \frac{1}{2}$ *satisfy the following inequality*

$$\left| \arg \left(\frac{2f(z)}{z} - f'(z) \right) \right| < \frac{\sigma_2(\alpha)\pi}{2},$$

where $\sigma_2(\alpha)$ *is given by* (8)*. Then*

$$\left| \arg \sqrt{\frac{f(z)}{z}} \right| < \frac{\alpha\pi}{2} \qquad (z \in \mathbb{U}).$$

Since $\sigma_2(\alpha)$ given by (8) takes its maximum value at $\alpha = \sqrt{(4-\pi)/16\pi}$, we obtain the following result.

Corollary 1. *Let p be an analytic function in* $\mathbb{U}$*, given by*

$$p(z) = 1 + \sum_{n=2}^{\infty} c_n z^n$$

and $p(z) \neq 0$ *for* $z \in \mathbb{U}$. *Let*

$$\left| \arg \left(p^2(z) - 2p(z)zp'(z) \right) \right| < \frac{\sigma_2 \left(\sqrt{\frac{4-\pi}{16\pi}} \right) \pi}{2} \simeq 0.071125,$$

then

$$\left| \arg \left(p(z) \right) \right| < \frac{\sqrt{\frac{4-\pi}{16\pi}}\,\pi}{2} \simeq 0.20528 \qquad (z \in \mathbb{U}).$$

Theorem 3. *Let* p *be an analytic function in* $\mathbb{U}$, *given by*

$$p(z) = 1 + \sum_{n=2}^{\infty} c_n z^n$$

and $p(z) \neq 0$ *for* $z \in \mathbb{U}$. *Let* α_0 *be the smallest positive root of the equation*

$$\frac{2}{\pi} \arctan \left(\frac{2\alpha \left(\frac{1-2\alpha}{1+2\alpha} \right)^{(1+2\alpha)/2} \cos(\pi\alpha)}{1 - 2\alpha - 2\alpha \left(\frac{1-2\alpha}{1+2\alpha} \right)^{(1+2\alpha)/2} \sin(\pi\alpha)} \right) - \alpha = 0. \tag{9}$$

Suppose that

$$\left| \arg \left(p(z) - \frac{zp'(z)}{[p(z)]^2} \right) \right| < \frac{\delta(\alpha)\pi}{2}, \tag{10}$$

where

$$\delta(\alpha) = \frac{2}{\pi} \arctan \left(\frac{2\alpha \left(\frac{1-2\alpha}{1+2\alpha} \right)^{(1+2\alpha)/2} \cos(\pi\alpha)}{1 - 2\alpha - 2\alpha \left(\frac{1-2\alpha}{1+2\alpha} \right)^{(1+2\alpha)/2} \sin(\pi\alpha)} \right) - \alpha \tag{11}$$

and $0 < \alpha < \alpha_0$. *Then*

$$\left| \arg \left(p(z) \right) \right| < \frac{\alpha\pi}{2} \qquad (z \in \mathbb{U}).$$

Proof. First, let us define

$$\delta(\alpha) = \frac{2}{\pi} \arctan \left(\frac{n(\alpha)}{m(\alpha)} \right) - \alpha$$

where

$$n(\alpha) = 2\alpha \left(\frac{1-2\alpha}{1+2\alpha} \right)^{(1+2\alpha)/2} \cos(\pi\alpha) \quad \text{and} \quad m(\alpha) = 1 - 2\alpha - 2\alpha \left(\frac{1-2\alpha}{1+2\alpha} \right)^{(1+2\alpha)/2} \sin(\pi\alpha),$$

then we have $\delta(0) = 0$, $\delta(\alpha)_{\alpha \to 1/2} = -1/2$, and $\delta'(0) > 0$. Therefore, there exists in $\left(0, 1/2\right)$ the smallest positive root α_0 of the equality (9), so that $\delta(\alpha) > 0$ for $\alpha \in (0, \alpha_0)$.

Now we suppose that there exists a point $z_0 \in \mathbb{U}$ such that

$$\left| \arg \left(p(z) \right) \right| < \frac{\alpha\pi}{2} \quad \text{for} \quad |z| < |z_0|$$

and

$$\left| \arg \left(p(z_0) \right) \right| = \frac{\alpha\pi}{2}.$$

Then, from Lemma 1, it follows that

$$\frac{zp'(z_0)}{p(z_0)} = ik\alpha,$$

where $[p(z_0)]^{\frac{1}{\alpha}} = \pm ia$ $(a > 0)$ and k is given by (3) or (4) for $m = 2$.

For the case $\arg\left(p(z_0)\right) = \dfrac{\alpha\pi}{2}$ when

$$[p(z_0)]^{\frac{1}{\alpha}} = ia \qquad (a > 0),$$

we have

$$\begin{aligned}\arg\left(p(z_0) - \frac{zp'(z_0)}{[p(z_0)]^2}\right) &= \arg\left(p(z_0)\left(1 - \frac{zp'(z_0)}{p(z_0)}\,\frac{1}{[p(z_0)]^2}\right)\right)\\ &= \arg\left(p(z_0)\right) + \arg\left(1 - ik\alpha\,\frac{1}{(ia)^{2\alpha}}\right)\\ &= \arg\left(p(z_0)\right) + \arg\left(1 + \frac{k\alpha}{a^{2\alpha}}\,e^{-i\pi(1+2\alpha)/2}\right).\end{aligned}$$

Since

$$\frac{k\alpha}{a^{2\alpha}} \geqq \alpha(a^{1-2\alpha} + a^{-1-2\alpha}),$$

we now define a real function g by

$$g(a) = a^{1-2\alpha} + a^{-1-2\alpha} \qquad (a > 0).$$

Then this function takes on the minimum value for a given by

$$a = \sqrt{\frac{1+2\alpha}{1-2\alpha}}.$$

Therefore, from the above inequality we obtain

$$\frac{k\alpha}{a^{2\alpha}} \geqq \alpha\left(\left(\frac{1+2\alpha}{1-2\alpha}\right)^{(1-2\alpha)/2} + \left(\frac{1+2\alpha}{1-2\alpha}\right)^{(-1-2\alpha)/2}\right) = \frac{2\alpha}{1-2\alpha}\left(\frac{1-2\alpha}{1+2\alpha}\right)^{(1+2\alpha)/2} =: l(\alpha).$$

Therefore

$$\begin{aligned}\arg\left(p(z_0) - \frac{zp'(z_0)}{[p(z_0)]^2}\right) &\leqq \frac{\alpha\pi}{2} + \arctan\left(\frac{-l(\alpha)\cos(\pi\alpha)}{1 - l(\alpha)\sin(\pi\alpha)}\right)\\ &= \frac{\alpha\pi}{2} - \arctan\left(\frac{l(\alpha)\cos(\pi\alpha)}{1 - l(\alpha)\sin(\pi\alpha)}\right)\\ &= -\frac{\delta(\alpha)\pi}{2},\end{aligned}$$

which is contradict with condition (10).

Next, for the case $\arg\left(p(z_0)\right) = -\dfrac{\alpha\pi}{2}$ when

$$[p(z_0)]^{\frac{1}{\alpha}} = -ia \qquad (a > 0),$$

with

$$\frac{k\alpha}{a^{2\alpha}} \leqq -\alpha(a^{1-2\alpha} + a^{-1-2\alpha}),$$

applying the similar method as the above, we can get

$$\begin{aligned}\arg\left(p(z_0)-\frac{zp'(z_0)}{[p(z_0)]^2}\right)&=\arg\left(p(z_0)\right)+\arg\left(1-ik\alpha\,\frac{1}{(-ia)^{2\alpha}}\right)\\&=\arg\left(p(z_0)\right)+\arg\left(1-\frac{k\alpha}{a^{2\alpha}}\,e^{i\pi(1+2\alpha)/2}\right)\\&\geqq-\frac{\alpha\pi}{2}+\arctan\left(\frac{l(\alpha)\cos(\pi\alpha)}{1-l(\alpha)\sin(\pi\alpha)}\right)\\&=\frac{\delta(\alpha)\pi}{2},\end{aligned}$$

which is a contradiction to condition (10).

Therefore, from the two mentioned contradictions, we obtain

$$\left|\arg\left(p(z)\right)\right|<\frac{\alpha\pi}{2}\qquad(z\in\mathbb{U}).$$

This completes the proof of Theorem 3. □

Theorem 4 (**[19], Corollary 3.4a.3, p. 124**). *Let β and γ be complex numbers with $\beta\neq 0$ and let p and h be analytic in $\mathbb{U}$ with $p(0)=h(0)$. If $P(z)=\beta h(z)+\gamma$ satisfies*

(i) Re $P^2(z)>0$

(ii) *P or P^{-1} is convex, then*

$$p(z)+zp'(z)\cdot[\beta p(z)+\gamma]^{-2}\prec h(z),\tag{12}$$

implies $p(z)\prec h(z)$.

The condition (10) can be written as a generalized Briot-Bouquet differential subordination. However, It is remarkable that the condition (12) among the outcomes on the generalized Briot-Bouquet differential subordination collected in ([19], Ch. 3) is not taken into account the case $\gamma=0,\ \beta=i$ which we have in (10).

Corollary 2. *Let $f\in\Sigma_2$ with*

$$p(z):=-\frac{zf'(z)}{f(z)}\neq 0$$

and $0<\alpha<\alpha_0$ satisfy the following inequality

$$\left|\arg\left(\frac{f(z)}{zf'(z)}\left(1+\frac{zf''(z)}{f'(z)}\right)-\frac{zf'(z)}{f(z)}-1\right)\right|<\frac{\delta(\alpha)\pi}{2},$$

where $\delta(\alpha)$ is given by (11). Then f is meromorphic strongly starlike function of order α.

Theorem 5. *Let p be an analytic function in $\mathbb{U}$, given by*

$$p(z)=1+\sum_{n=m}^{\infty}c_nz^n\qquad(c_m\neq 0;\ m\in\mathbb{N})$$

and $p(z)\neq 0$ for $z\in\mathbb{U}$. Let $\alpha>0$ and $\beta>0$ satisfy the inequality

$$\arctan(m\alpha)>\frac{\pi\alpha}{2\beta}.$$

Suppose that

$$\left|\arg\left(p(z)\left[1-\frac{zp'(z)}{p(z)}\right]^{\beta}\right)\right| < \frac{\pi}{2}\left[\frac{2\beta}{\pi}\arctan(m\alpha)-\alpha\right]. \tag{13}$$

Then

$$\left|\arg\left(p(z)\right)\right| < \frac{\alpha\pi}{2} \qquad (z\in\mathbb{U}).$$

Proof. Suppose that there exists a point $z_0 \in \mathbb{U}$ such that

$$\left|\arg\left(p(z)\right)\right| < \frac{\alpha\pi}{2} \quad \text{for} \quad |z|<|z_0|$$

and

$$\left|\arg\left(p(z_0)\right)\right| = \frac{\alpha\pi}{2}.$$

Then, from Lemma 1, it follows that

$$\frac{zp'(z_0)}{p(z_0)} = ik\alpha,$$

where $[p(z_0)]^{\frac{1}{\alpha}} = \pm ia$ $(a>0)$ and k is given by (3) or (4).

For the case $\arg\left(p(z_0)\right) = \frac{\alpha\pi}{2}$ when

$$[p(z_0)]^{\frac{1}{\alpha}} = ia \qquad (a>0),$$

with $k \geqq m$, we have

$$\begin{aligned}\arg\left(p(z_0)\left[1-\frac{z_0p'(z_0)}{p(z_0)}\right]^{\beta}\right) &= \arg\left(p(z_0)\right)+\beta\arg\left(1-\frac{z_0p'(z_0)}{p(z_0)}\right)\\ &= \arg\left(p(z_0)\right)+\beta\arg\left(1-ik\alpha\right)\\ &\leqq \frac{\alpha\pi}{2}-\beta\arctan(m\alpha)\\ &= \frac{-\pi}{2}\left(\frac{2\beta}{\pi}\arctan(m\alpha)-\alpha\right),\end{aligned}$$

which contradicts our hypothesis in (13).

Next, for the case $\arg\left(p(z_0)\right) = -\frac{\alpha\pi}{2}$ when

$$[p(z_0)]^{\frac{1}{\alpha}} = -ia \qquad (a>0),$$

with $k \leqq -m$, applying the similar method as the above, we can get

$$\begin{aligned}\arg\left(p(z_0)\left[1-\frac{z_0p'(z_0)}{p(z_0)}\right]^{\beta}\right) &\geqq -\frac{\alpha\pi}{2}+\beta\arctan(m\alpha)\\ &= \frac{\pi}{2}\left(\frac{2\beta}{\pi}\arctan(m\alpha)-\alpha\right),\end{aligned}$$

which is a contradiction to (13).

Therefore, from the two mentioned contradictions, we obtain

$$\left|\arg\left(p(z)\right)\right| < \frac{\alpha\pi}{2} \qquad (z\in\mathbb{U}).$$

This completes the proof of Theorem 5. □

Remark 4. *By choosing $m = 2$ and $\beta = 1$ in Theorem 5, we have the result obtained by Nunokawa and Sokół in ([11], Theorem 2.4).*

By choosing

$$p(z) := -\frac{zf'(z)}{f(z)} \neq 0,$$

in Theorem 6, we obtain a sufficient condition for strongly meromorphic starlikeness as follows.

Corollary 3. *Let $f \in \Sigma_2$ with*

$$p(z) := -\frac{zf'(z)}{f(z)} \neq 0.$$

Let $\alpha > 0$ and $\beta > 0$ satisfy the inequality

$$\arctan(2\alpha) > \frac{\pi\alpha}{2\beta}.$$

Suppose that

$$\left|\arg\left(-\frac{zf'(z)}{f(z)}\left[1+\frac{zf'(z)}{f(z)}-\left(1+\frac{zf''(z)}{f'(z)}\right)\right]^{\beta}\right)\right| < \frac{\pi}{2}\left[\frac{2\beta}{\pi}\arctan(2\alpha)-\alpha\right]. \tag{14}$$

Then f is meromorphic strongly starlike function of order α.

Theorem 6. *Let p be an analytic function in $\mathbb{U}$ with $p(0) = 1$, $p'(0) \neq 0$ and $p(z) \neq 0$ for $z \in \mathbb{U}$ that satisfies the following inequality*

$$\left|\arg\left(\frac{p(z)\left(p(z)+zp'(z)\right)}{p(z)-\beta zp'(z)}\right)\right| < \frac{\xi(\alpha)\pi}{2},$$

where

$$\xi(\alpha) = \alpha + \frac{2}{\pi}\left(\arctan\left(\alpha\right)+\arctan\left(\beta\alpha\right)\right) \quad (\alpha > 0;\ \beta \geqq 0). \tag{15}$$

Then

$$\left|\arg\left(p(z)\right)\right| < \frac{\alpha\pi}{2} \qquad (z \in \mathbb{U}).$$

Proof. To prove the result asserted by Theorem 6, we suppose that there exists a point $z_0 \in \mathbb{U}$ such that

$$\left|\arg\left(p(z)\right)\right| < \frac{\alpha\pi}{2} \quad \text{for} \quad |z| < |z_0|$$

and

$$\left|\arg\left(p(z_0)\right)\right| = \frac{\alpha\pi}{2}.$$

Then, from Lemma 1, it follows that

$$\frac{zp'(z_0)}{p(z_0)} = ik\alpha,$$

where $[p(z_0)]^{\frac{1}{\alpha}} = \pm ia$ $(a > 0)$ and k is given by (3) or (4) for $m = 1$.

For the case

$$\arg\left(p(z_0)\right) = \frac{\alpha\pi}{2},$$

where $[p(z_0)]^{\frac{1}{\alpha}} = ia \ (a > 0)$ and $k \geqq 1$, we have

$$\begin{aligned}
\arg\left(\frac{p(z_0)\left(p(z_0)+z_0p'(z_0)\right)}{p(z_0)-\beta z_0 p'(z_0)}\right) &= \arg\left(p(z_0)\left(\frac{1+\dfrac{zp'(z_0)}{p(z_0)}}{1-\beta\dfrac{zp'(z_0)}{p(z_0)}}\right)\right) \\
&= \arg\left(p(z_0)\right) + \arg\left(\frac{1+ik\alpha}{1-i\beta k\alpha}\right) \\
&= \arg\left(p(z_0)\right) + \arg\left(1+ik\alpha\right) - \arg\left(1-i\beta k\alpha\right) \\
&= \frac{\alpha\pi}{2} + \arctan\left(k\alpha\right) - \arctan\left(-\beta k\alpha\right) \\
&= \frac{\alpha\pi}{2} + \arctan\left(k\alpha\right) + \arctan\left(\beta k\alpha\right) \\
&\geqq \frac{\alpha\pi}{2} + \arctan\left(\alpha\right) + \arctan\left(\beta\alpha\right) \\
&= \frac{\xi(\alpha)\pi}{2},
\end{aligned}$$

which contradicts our hypothesis in Theorem 6.

Next, for the case

$$\arg\left(p(z_0)\right) = -\frac{\alpha\pi}{2},$$

where $[p(z_0)]^{\frac{1}{\alpha}} = -ia \ (a > 0)$ and $k \leqq -1$, applying the similar method as the above, we can get

$$\begin{aligned}
\arg\left(\frac{p(z_0)\left(p(z_0)+z_0p'(z_0)\right)}{p(z_0)-\beta z_0 p'(z_0)}\right) &= \arg\left(p(z_0)\right) + \arg\left(\frac{1+ik\alpha}{1-i\beta k\alpha}\right) \\
&= -\frac{\alpha\pi}{2} + \arctan\left(k\alpha\right) + \arctan\left(\beta k\alpha\right) \\
&\leqq -\frac{\alpha\pi}{2} - \arctan\left(\alpha\right) - \arctan\left(\beta\alpha\right) \\
&= -\frac{\xi(\alpha)\pi}{2},
\end{aligned}$$

which is a contradiction to the assumption of Theorem 6.

Therefore, from the two mentioned contradictions, we obtain

$$\left|\arg\left(p(z)\right)\right| < \frac{\alpha\pi}{2} \qquad (z \in \mathbb{U}).$$

This completes the proof of Theorem 6. □

Remark 5.

(i) *If $\beta\alpha^2 < 1$ in Theorem 6, then (15) is equal to*

$$\xi = \alpha + \frac{2}{\pi}\arctan\left(\frac{\alpha(1+\beta)}{1-\beta\alpha^2}\right).$$

(ii) *By setting $\beta = 0$ and $p(z) := f'(z) \neq 0$ in Theorem 6, we have the result obtained by Nunokawa et al. in ([20], Theorem 3).*

By setting

$$p(z) := \frac{zf'(z)}{g(z)} \neq 0,$$

in Theorem 6, we obtain a sufficient condition for strongly close-to-convexity as follows.

Corollary 4. *For $g \in \mathcal{S}^*$ and $f \in \mathcal{A}$ such that $2f''(0) \neq g''(0)$, suppose that the following inequality*

$$\left|\arg\left(\frac{2\left(f'(z)\right)^2}{f'(z)g'(z) - f''(z)g(z)} - \frac{zf'(z)}{g(z)}\right)\right| < \frac{\xi(\alpha)\pi}{2}$$

is satisfied, where

$$\xi(\alpha) = \alpha + \frac{2}{\pi}\left(\arctan(\alpha) + \arctan(\beta\alpha)\right) \qquad (\alpha > 0,\ \beta \geqq 0). \tag{16}$$

Then

$$\left|\arg\left(\frac{zf'(z)}{g(z)}\right)\right| < \frac{\alpha\pi}{2} \qquad (z \in \mathbb{U}).$$

Remark 6. *Similar to Corollary 4 by setting*

$$p(z) := \frac{zf'(z)}{f(z)} \neq 0,$$

in Theorem 6, (or $g =: f$ in Corollary 4), we can obtain a sufficient condition for strongly starlikeness.

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