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Improved Oscillation Criteria for 2nd-Order Neutral Differential Equations with Distributed Deviating Arguments

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Abstract: In this study, we establish new sufficient conditions for oscillation of solutions of second-order neutral differential equations with distributed deviating arguments. By employing a refinement of the Riccati transformations and comparison principles, we obtain new oscillation criteria that complement and improve some results reported in the literature. Examples are provided to illustrate the main results.

Keywords: deviating argument; second order; neutral differential equation; oscillation

1. Introduction

This study is concerned with creating new oscillation criteria for the second-order non-linear neutral differential equation with distributed deviating arguments

$$\left(r(t)\left(z'(t)\right)^{\alpha}\right)' + \int_a^b q(t,s)\, f\left(x\left(\sigma(t,s)\right)\right) \mathrm{d}s = 0, \tag{1}$$

where $t \geq t_0$ and

$$z(t) := x(t) + \int_c^d p(t,s)\, x\left(\tau(t,s)\right) \mathrm{d}s.$$

Throughout this paper, we assume that:

(H_1) α is a quotient of add positive integers;
(H_2) $r \in C(I,(0,\infty))$, $p \in C(I \times [c,d],[0,\infty))$, $q \in C(I \times [a,b],[0,\infty))$, $q(t,s)$ is not zero on any half line $[t_*,\infty) \times [a,b]$, $t_* \geq t_0$, $\int_c^d p(t,s)\, \mathrm{d}s < 1$ and

$$\int_{t_0}^{\infty} r^{-1/\alpha}(s)\, \mathrm{d}s = \infty; \tag{2}$$

(H_3) $\tau, \sigma \in C(I,\mathbb{R})$, $\tau(t,s) \leq t$, $\sigma(t,s) \leq t$ and $\lim_{t\to\infty} \tau(t,s) = \lim_{t\to\infty} \sigma(t,s) = \infty$;
(H_4) $f \in C(\mathbb{R},\mathbb{R})$ and there exists a constant $k > 0$ such that $f(x) \geq kx^{\alpha}$ for $x \neq 0$.

By a solution of (1), we mean a function $x \in C^1([t,\infty),\mathbb{R})$, $t_x \geq t_0$, which has the property $r(t)(z'(t))^{\alpha} \in C^1([t_0,\infty),\mathbb{R})$, and satisfies (1) on $[t_x,\infty)$. We consider only those solutions x of (1) which satisfy $\sup\{|x(t)| : t \geq t_x\} > 0$, for all $t > t_x$. If x is neither eventually positive nor eventually negative, then x is called oscillatory; otherwise it is called non-oscillatory. The equation itself is called oscillatory if all its solutions oscillate.

In a differential equation with neutral delay, the highest-order derivative appears both with and without delay. In addition to the theoretical importance, the qualitative study of neutral equations has great practical importance. In fact, the neutral equations arise in the study of vibrating masses attached to an elastic bar, in problems concerning electric networks containing lossless transmission lines (as in high-speed computers), and in the solution of variational problems with time delays, see [1,2].

Over the past decades, the issue of studying the oscillation properties for delay/neutral differential equations has been a very active research area see [1–19].

For some related works, Sun et al. [13] and Dzurina et al. [5] obtained some oscillation criteria for

$$(r(t)|x'(t)|^{\alpha-1}x'(t))' + q(t)|x[\sigma(t)]|^{\alpha-1}x[\sigma(t)] = 0. \tag{3}$$

Xu et al. [15,16] and Liu et al. [8] extended the results of [5,13] to (3) with neutral term. Sahiner [12] obtained some general oscillation criteria for neutral delay equations

$$\left(r(t)\left(x(t) + p(t)x(t-\tau_0)\right)'\right)' + q(t)f(x(\sigma(t))) = 0,$$

In [14], Wang established some general oscillation criteria for equation

$$\left(r(t)\left(x(t) + p(t)x(t-\tau_0)\right)'\right)' + \int_a^b q(t,s)\,x(\sigma(t,s))\,\mathrm{d}s = 0, \tag{4}$$

by using Riccati technique and averaging functions method. Xu and Weng [17] and Zhao and Meng [19], established some oscillation criteria for (4), which complemented and extended the results in [12,14]. In 2011, Baculikova and Dzurina [3] investigated the properties of delayed equations

$$\left(r(t)\left(\left(x(t) + p(t)x(\tau(t))\right)'\right)^{\alpha}\right)' + q(t)\,x^{\beta}(\sigma(t)) = 0. \tag{5}$$

They are provided some comparison theorems which compare the second-order (5) with the first-order differential equations.

It is known that the determination of the signs of the derivatives of the solution is necessary and significant effect before studying the oscillation of delay differential equations. The other essential thing is to establish relationships between derivatives of different orders. Depending on improving the relationship between the neutral function z and its first derivative z', we create new and improved criteria for oscillation of solutions of Equation (1). During this study, we use Riccati transformations and comparison principles to obtain the different criteria for oscillation of (1). Examples are provided to illustrate the main results.

2. Preliminary Results

For convenience, we denote that

$$U(t) := \int_a^b q(t,s)\left[1 - \int_c^d p(\sigma(t,s),v)\,\mathrm{d}v\right]^{\alpha}\mathrm{d}s,$$

$$\eta_{t_0}(t) := \int_{t_0}^{t} r^{-1/\alpha}(u)\,\mathrm{d}u,\ \widetilde{\eta}_{t_0}(t) := \eta_{t_0}(t) + \frac{k}{\alpha}\int_{t_0}^{t}\eta_{t_1}(u)\,\eta_{t_0}^{\alpha}(\sigma(u,a))\,U(u)\,\mathrm{d}u,$$

$$\widehat{\eta}(t) := \exp\left(-\alpha\int_{\sigma(t,a)}^{t}\frac{\mathrm{d}u}{\widetilde{\eta}_{t_0}(u)r^{1/\alpha}(u)}\right),$$

$$R(t) = \alpha / (r(t))^{1/\alpha}, \;\; Q(t) := kU(t)\,\widehat{\eta}(t) \;\text{ and }\; G(t) := \int_t^\infty Q(s)\,\mathrm{d}s.$$

The following lemmas mainly help us to prove the main results:

Lemma 1. *Let $g(x) = Ax - Bx^{(\alpha+1)/\alpha}$ where $A, B > 0$ are constants. Then g attains its maximum value on $\mathbb{R}$ at $x^* = (\alpha A / ((\alpha+1)B))^\alpha$ and*

$$\max_{x \in r} g = g(x^*) = \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}} \frac{A^{\alpha+1}}{B^\alpha}. \tag{6}$$

Lemma 2. *[3] If x is a positive solution of (1) on $[t_0, \infty)$, then there exists a $t_1 \geq t_0$ such that*

$$z(t) > 0,\; z'(t) > 0,\; \left(r(t)\,(z'(t))^\alpha\right)' \leq 0, \tag{7}$$

for $t \geq t_1$.

Lemma 3. *Let x be a positive solution of Equation (1). Then the function z satisfies*

$$\left(r(t)\,(z'(t))^\alpha\right)' \leq -kU(t)\,(z(\sigma(t,a)))^\alpha, \tag{8}$$

$$z(t) \geq \widetilde{\eta}_{t_1}(t)\, r^{1/\alpha}(t)\, z'(t) \tag{9}$$

and

$$\left(r(t)\,(z'(t))^\alpha\right)' \leq -kU(t)\,\widehat{\eta}(t)\, z^\alpha(t). \tag{10}$$

Proof. Assume that there exists a $t_1 \geq t_0$ such that $x(t) > 0$, $x(\tau(t,v)) > 0$ and $x(\sigma(t,s)) > 0$ for $t \geq t_1$, $v \in [c,d]$ and $s \in [a,b]$. From Lemma 2, we have (7) holds. Thus, by definition of $z(t)$, we obtain

$$\begin{aligned} x(t) &= z(t) - \int_c^d p(t,v)\, x(\tau(t,v))\,\mathrm{d}v \\ &\geq z(t) - \int_c^d p(t,v)\, z(\tau(t,v))\,\mathrm{d}v \\ &\geq z(t)\left[1 - \int_c^d p(t,v)\,\mathrm{d}v\right], \end{aligned}$$

which, with (1), implies that

$$\left(r(t)\,(z'(t))^\alpha\right)' \leq -k \int_a^b q(t,s)\, z^\alpha(\sigma(t,s)) \left[1 - \int_c^d p(\sigma(t,s), v)\,\mathrm{d}v\right]^\alpha \mathrm{d}s.$$

Since $z'(t) > 0$ and $\frac{\partial}{\partial s}\sigma(t,s) > 0$, we obtain $z(\sigma(t,s)) > z(\sigma(t,a))$ and so

$$\left(r(t)\,(z'(t))^\alpha\right)' \leq -kU(t)\, z^\alpha(\sigma(t,a)).$$

Applying the chain rule and simple computation, it is easy to see that

$$\begin{aligned} \eta_{t_1}(t)\left(r(t)\,(z'(t))^\alpha\right)' &= \alpha\left(r^{1/\alpha}(t)\, z'(t)\right)^{\alpha-1} \eta_{t_1}(t)\left(r^{1/\alpha}(t)\, z'(t)\right)' \\ &= -\alpha\left(r^{1/\alpha}(t)\, z'(t)\right)^{\alpha-1} \frac{\mathrm{d}}{\mathrm{d}t}\left(z(t) - \eta_{t_1}(t)\, r^{1/\alpha}(t)\, z'(t)\right). \end{aligned} \tag{11}$$

Combining (8) and (11), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(z(t)-\eta_{t_1}(t)\, r^{1/\alpha}(t)\, z'(t)\right) \geq \frac{k}{\alpha}\eta_{t_1}(t)\left(r^{1/\alpha}(t)\, z'(t)\right)^{1-\alpha} U(t)\, z^{\alpha}(\sigma(t,a)).$$

Integrating this inequality from t_1 to t, we have

$$z(t) \geq \eta_{t_1}(t)\, r^{1/\alpha}(t)\, z'(t) + \frac{k}{\alpha}\int_{t_1}^{t} \eta_{t_1}(u)\, U(u)\left(r^{1/\alpha}(u)\, z'(u)\right)^{1-\alpha} z^{\alpha}(\sigma(u,a))\,\mathrm{d}u. \tag{12}$$

From the monotonicity of $r^{1/\alpha}(t)\, z'(t)$, we have

$$z(t) = z(t_1) + \int_{t_1}^{t} \frac{1}{r^{1/\alpha}(u)}\left(r^{1/\alpha}(u)\, z'(u)\right)\mathrm{d}u \geq \eta_{t_1}(t)\, r^{1/\alpha}(t)\, z'(t).$$

Thus, from the fact that $\left(r^{1/\alpha}(t)\, z'(t)\right)' \leq 0$, (12) becomes

$$\begin{aligned} z(t) &\geq \eta_{t_1}(t)\, r^{1/\alpha}(t)\, z'(t) \\ &\quad + \frac{k}{\alpha}\int_{t_1}^{t} \eta_{t_1}(u)\, U(u)\left(r^{1/\alpha}(u)\, z'(u)\right)^{1-\alpha} \eta_{t_1}^{\alpha}(\sigma(u,a))\left[r(\sigma(u,a))\left(z'(\sigma(u,a))\right)^{\alpha}\right]\mathrm{d}u. \\ &\geq \eta_{t_1}(t)\, r^{1/\alpha}(t)\, z'(t) + \frac{k}{\alpha}\int_{t_1}^{t}\left(r^{1/\alpha}(u)\, z'(u)\right)^{1-\alpha} \eta_{t_1}(u)\, \eta_{t_1}^{\alpha}(\sigma(u,a))\, U(u)\left[r^{1/\alpha}(u)\, z'(u)\right]^{\alpha}\mathrm{d}u \\ &\geq r^{1/\alpha}(t)\, z'(t)\left[\eta_{t_1}(t) + \frac{k}{\alpha}\int_{t_1}^{t} \eta_{t_1}(u)\, \eta_{t_1}^{\alpha}(\sigma(u,a))\, U(u)\,\mathrm{d}u\right]. \\ &\geq \widetilde{\eta}_{t_1}(t) r^{1/\alpha}(t)\, z'(t), \end{aligned}$$

or

$$\frac{z'(t)}{z(t)} \leq \frac{1}{\widetilde{\eta}_{t_1}(t) r^{1/\alpha}(t)}.$$

Integrating the latter inequality from $\sigma(t,a)$ to t, we get

$$\frac{z(\sigma(t,a))}{z(t)} \geq \exp\left(-\int_{\sigma(t,a)}^{t} \frac{\mathrm{d}u}{\widetilde{\eta}_{t_1}(u) r^{1/\alpha}(u)}\right).$$

which with (8), gives

$$\begin{aligned} \frac{\left(r(t)\left(z'(t)\right)^{\alpha}\right)'}{z^{\alpha}(t)} &\leq -kU(t)\left(\frac{z(\sigma(t,a))}{z(t)}\right)^{\alpha} \\ &\leq -kU(t)\, \widehat{\eta}(t). \end{aligned}$$

The proof is complete. □

Lemma 4. *Let x be a positive solution of equation (1). If we define the function*

$$\Psi(t) = \phi(t)\, r(t)\left(\frac{z'(t)}{z(t)}\right)^{\alpha}, \tag{13}$$

then

$$\Psi'(t) \leq \frac{\phi'_{+}(t)}{\phi(t)}\Psi(t) - k\phi(t) U(t)\, \widehat{\eta}(t) - \frac{\alpha}{\left(\phi(t)\, r(t)\right)^{1/\alpha}}\Psi^{(\alpha+1)/\alpha}(t). \tag{14}$$

Proof. Assume that there exists a $t_1 \geq t_0$ such that $x(t) > 0$, $x(\tau(t,v)) > 0$ and $x(\sigma(t,s)) > 0$ for $t \geq t_1$, $v \in [c,d]$ and $s \in [a,b]$. From Lemma 3, we have (10) holds. Thus, from the definition of $\Psi(t)$, we obtain $\Psi(t) > 0$ for $t \geq t_1$. Differentiating (13), we arrive at

$$\Psi'(t) = \frac{\phi'(t)}{\phi(t)}\Psi(t) + \phi(t)\frac{(r(t)z'(t))'}{z^{\alpha}(t)} - \alpha\phi(t)r(t)\left(\frac{z'(t)}{z(t)}\right)^{\alpha+1}.$$

From (10) and (13), we deduce that

$$\Psi'(t) \leq \frac{\phi'_{+}(t)}{\phi(t)}\Psi(t) - k\phi(t)U(t)\widehat{\eta}(t) - \frac{\alpha}{(\phi(t)r(t))^{1/\alpha}}\Psi^{(\alpha+1)/\alpha}(t).$$

The proof is complete. □

3. Main Results

In this section, we establish the oscillation criteria for the solutions of (1).

Theorem 1. *If the first-order delay differential equation*

$$\omega'(t) + k\widetilde{\eta}^{\alpha}_{t_1}(\sigma(t,a))U(t)\omega(\sigma(t,a)) = 0 \tag{15}$$

is oscillatory, then (1) is oscillatory.

Proof. Suppose the contrary that (1) has a non-oscillatory solution x on $[t_0,\infty)$. Without loss of generality, we assume that there exists a $t_1 \geq t_0$ such that $x(t) > 0$, $x(\tau(t,v)) > 0$ and $x(\sigma(t,s)) > 0$ for $t \geq t_1$, $v \in [c,d]$ and $s \in [a,b]$. From Lemma 3, we have (8) and (9) hold. Using (8) and (9), one can see that $\omega(t) = r(t)(z'(t))^{\alpha}$ is a positive solution of the first order delay differential inequality

$$\omega'(t) + k\widetilde{\eta}^{\alpha}_{t_1}(\sigma(t,a))U(t)\omega(\sigma(t,a)) \leq 0.$$

In view of ([11] Theorem 1), the associated delay equation (15) also has a positive solution, we find a contradiction. The proof is complete. □

Corollary 1. *If*

$$\limsup_{t\to\infty}\int_{\sigma(t,a)}^{t}\widetilde{\eta}^{\alpha}_{t_1}(\sigma(u,a))U(u)\,\mathrm{d}u > \frac{1}{k},\ \frac{\partial}{\partial t}\sigma(t,s) \geq 0 \tag{16}$$

or

$$\liminf_{t\to\infty}\int_{\sigma(t,a)}^{t}\widetilde{\eta}^{\alpha}_{t_1}(\sigma(u,a))U(u)\,\mathrm{d}u > \frac{1}{k\mathrm{e}}, \tag{17}$$

then (1) is oscillatory.

Proof. It is well known that (16) or (17) ensures oscillation of (15), see ([7] Theorem 2.1.1). □

Lemma 5. *Assume that σ is strictly increasing with respect to t for all $s \in (a,b)$. Suppose for some $\delta > 0$ that*

$$\liminf_{t\to\infty}\int_{\sigma(t,a)}^{t}\widetilde{\eta}^{\alpha}_{t_1}(\sigma(u,a))U(u)\,\mathrm{d}u \geq \delta \tag{18}$$

and (1) has an eventually positive solution x. Then,

$$\frac{w(\sigma(t,a))}{w(t)} \geq \theta_n(\delta), \tag{19}$$

for every $n \geq 0$ and t large enough, where $w(t) := r(t)(z'(t))^{\alpha}$,

$$\theta_0(u) := 1 \text{ and } \theta_n(u) := \exp(\rho\theta_{n-1}(u)). \tag{20}$$

Proof. Assume that (1) has a positive solution x on $[t_0, \infty)$. Then, we can expect the existence of a $t_1 \geq t_0$ such that $x(t) > 0$, $x(\tau(t, v)) > 0$ and $x(\sigma(t, s)) > 0$ for $t \geq t_1$, $v \in [c, d]$ and $s \in [a, b]$. Proceeding as in the proof of Theorem 1, we deduce that ω is a positive solution of first order delay differential equation (15). In a similar way to that followed in proof of Lemma 1 in [18], we can prove that (19) holds. □

Theorem 2. *Assume that σ is strictly increasing with respect to t for all $s \in (a, b)$ and (18) holds for some $\delta < 0$. If there exists a function $\varphi \in C^1(I, (0, \infty))$ such that*

$$\limsup_{t\to\infty} \int_{t_1}^{t} \left(k\varphi(u)\, U(u) - \frac{(\varphi'_+(u))^{\alpha+1}\, r(\sigma(u,a))}{(\alpha+1)^{\alpha+1}\, \theta_n(\delta)\, \varphi^{\alpha}(u)\, (\sigma'(u,a))^{\alpha}} \right) = \infty, \tag{21}$$

for some sufficiently large $t \geq t_1$and for some $n \geq 0$, where $\theta_n(\delta)$ is defined as (20) and $\varphi'_+(t) = \max\{0, \varphi'(t)\}$, then (1) is oscillatory.

Proof. Suppose the contrary that (1) has a non-oscillatory solution x on $[t_0, \infty)$. Without loss of generality, we assume that there exists a $t_1 \geq t_0$ such that $x(t) > 0$, $x(\tau(t, v)) > 0$ and $x(\sigma(t, s)) > 0$ for $t \geq t_1$, $v \in [c, d]$ and $s \in [a, b]$. From Lemma 3, we have (8) holds. It follows from Lemma 5 that there exists a $t \geq t_1$ large enough such that

$$\frac{z'(\sigma(t,a))}{z'(t)} \geq \left(\frac{\theta_n(\delta)\, r(t)}{r(\sigma(t))} \right)^{1/\alpha}. \tag{22}$$

Define the function

$$\Phi(t) := \varphi(t) r(t) \left(\frac{z'(t)}{z(\sigma(t,a))} \right)^{\alpha}. \tag{23}$$

Then, $\Phi(t) > 0$ for $t \geq t_1$. Differentiating (23), we get

$$\Phi'(t) = \frac{\varphi'(t)}{\varphi(t)} \Phi(t) + \varphi(t) \frac{(r(t)(z'(t))^{\alpha})'}{z^{\alpha}(\sigma(t,a))} - \alpha\varphi(t) r(t) \left(\frac{z'(t)}{z(\sigma(t,a))} \right)^{\alpha} \left(\frac{z'(\sigma(t))}{z(\sigma(t,a))} \right) \sigma'(t,a).$$

From (8), (22) and (23), we obtain

$$\Phi'(t) \leq -k\varphi(t)\, U(t) + \frac{\varphi'_+(t)}{\varphi(t)} \Phi(t) - \frac{\alpha \theta_n^{1/\alpha}(\delta)\, \sigma'(t,a)}{(\varphi(t)\, r(\sigma(t,a)))^{1/\alpha}} \Phi^{(\alpha+1)/\alpha}(t). \tag{24}$$

Using Lemma 1 with $A = \varphi'_+(t) / \varphi(t)$ and $B = \alpha\theta_n^{1/\alpha}(\delta) / (\varphi(t)\, r(\sigma(t)))^{-1/\alpha}$, (24) yield

$$\Phi'(t) \leq -k\varphi(t)\, U(t) + \frac{\varphi'_+(t)^{\alpha+1}\, r(\sigma(t,a))}{(\alpha+1)^{\alpha+1}\, \theta_n(\delta)\, \varphi^{\alpha}(t)\, (\sigma'(t,a))^{\alpha}}.$$

Integrating this inequality from t_1 to t, we have

$$\int_{t_1}^{t} \left(k\varphi(u)\, U(u) - \frac{(\varphi'_+(u))^{\alpha+1}\, r(\sigma(u,a))}{(\alpha+1)^{\alpha+1}\, \theta_n(\delta)\, \varphi^{\alpha}(u)\, (\sigma'(u,a))^{\alpha}} \right) du \leq \Phi(t),$$

then we find a contradiction with condition (21). The proof is complete. □

Theorem 3. *Assume that there exists a function $\phi \in C^1(I, (0, \infty))$ such that*

$$\limsup_{t\to\infty} \int_{t_1}^{t} \left(k\phi(u)\, U(u)\, \widehat{\eta}(u) - \frac{r(u)\, (\phi'_+(u))^{\alpha+1}}{(\alpha+1)^{\alpha+1}\, \phi^{\alpha}(u)} \right) du = \infty. \tag{25}$$

for some sufficiently large $t \geq t_1$, where $\phi'_+(t) = \max\{0, \psi'(t)\}$, then (1) is oscillatory.

Proof. Suppose the contrary that (1) has a non-oscillatory solution x on $[t_0, \infty)$. Without loss of generality, we assume that there exists a $t_1 \geq t_0$ such that $x(t) > 0$, $x(\tau(t,v)) > 0$ and $x(\sigma(t,s)) > 0$ for $t \geq t_1$, $v \in [c,d]$ and $s \in [a,b]$. From Lemma 3, we have (8)–(10) hold. Next, using Lemma 4, we arrive at (14). Using Lemma 1 with $A = \phi'_+(t)/\phi(t)$ and $B = \alpha(\phi(t) r(t))^{-1/\alpha}$, (14) becomes

$$\Psi'(t) \leq -k\phi(t) U(t) \widehat{\eta}(t) + \frac{r(t)\left(\phi'_+(t)\right)^{\alpha+1}}{(\alpha+1)^{\alpha+1} \phi^{\alpha}(t)}.$$

Integrating this inequality from t_1 to t, we have

$$\int_{t_1}^{t} \left(k\phi(u) U(u) \widehat{\eta}(u) - \frac{r(u)\left(\phi'_+(u)^{\alpha+1}\right)}{(\alpha+1)^{\alpha+1} \phi^{\alpha}(u)} \right) \mathrm{d}u \leq \Psi(t),$$

This is the contrary with condition (25). The proof is complete. □

By different method, we establish new oscillation results for Equation (1).

Theorem 4. *Assume that*

$$\int_{t_0}^{\infty} Q(t)\, \mathrm{d}t = \infty, \tag{26}$$

then, Equation (1) is oscillatory.

Proof. Suppose the contrary that (1) has a non-oscillatory solution x on $[t_0, \infty)$. Without loss of generality, we assume that there exists a $t_1 \geq t_0$ such that $x(t) > 0$, $x(\tau(t,v)) > 0$ and $x(\sigma(t,s)) > 0$ for $t \geq t_1$, $v \in [c,d]$ and $s \in [a,b]$. Consider the function Ψ defined as in (13), it follows from Lemma 4 that (14) holds. Set $\phi(t) := 1$, (14) becomes

$$\Psi'(t) + Q(t) + R(t) \Psi^{\frac{\alpha+1}{\alpha}}(t) \leq 0 \tag{27}$$

or

$$\Psi'(t) + Q(t) \leq 0. \tag{28}$$

Integrating (28) from t_3 to t and using (26), we arrive at

$$\Psi(t) \leq \Psi(t_3) - \int_{t_3}^{t} Q(t)\, \mathrm{d}s \to \infty \quad \text{as} \quad t \to \infty,$$

which is a contradiction with the fact that $\Psi(t) > 0$ and therefore the proof is complete. □

Definition 1. *Let $\{y_n(t)\}_{n=0}^{\infty}$ be a sequence of functions defined as*

$$y_n(t) = \int_{t}^{\infty} R(s)\, y_{n-1}^{\frac{\alpha+1}{\alpha}}(s)\, \mathrm{d}s + y_0(t), \quad t \geq t_0, \quad n = 1, 2, 3, \ldots. \tag{29}$$

and

$$y_0(t) = G(t), \quad t \geq t_0,$$

where $y_n(t) \leq y_{n+1}(t)$, $t \geq t_0$.

Lemma 6. *Assume that x is a positive solution of (1). Then $\Psi(t) \geq y_n(t)$ such that $\Psi(t)$ and $y_n(t)$ are defined as in (13) and (29), respectively. Moreover, there exists a positive function $y(t)$ on $[T, \infty)$ such that $\lim_{n\to\infty} y_n(t) = y(t)$ for $t \geq T \geq t_0$ and*

$$y\left(t\right)=\int_{t}^{\infty}R\left(s\right)y^{\frac{\alpha+1}{\alpha}}\left(s\right)\mathrm{d}s+y_{0}\left(t\right),\ t\geq T. \tag{30}$$

Proof. Let x be a positive solution of (1). Proceeding as in the proof of Theorem 4, we arrive at (27). By integrating (27) from t to t', we obtain

$$\Psi\left(t'\right)-\Psi\left(t\right)+\int_{t}^{t'}Q\left(s\right)ds+\int_{t}^{t'}\Psi^{\frac{\alpha+1}{\alpha}}\left(s\right)R\left(s\right)\mathrm{d}s\leq 0.$$

This implies

$$\Psi\left(t'\right)-\Psi\left(t\right)+\int_{t}^{t'}\Psi^{\frac{\alpha+1}{\alpha}}\left(s\right)R\left(s\right)\mathrm{d}s\leq 0.$$

Then, we conclude that

$$\int_{t}^{\infty}\Psi^{\frac{\alpha+1}{\alpha}}\left(s\right)R\left(s\right)\mathrm{d}s<\infty \text{ for } t\geq T, \tag{31}$$

otherwise, $\Psi(t')\leq\Psi(t)-\int_{t}^{t'}\Psi^{\frac{\alpha+1}{\alpha}}\left(s\right)R(s)\mathrm{d}s\rightarrow-\infty$ as $t'\rightarrow\infty$, which is a contradiction with $\Psi(t)>0$. Since $\Psi(t)>0$ and $\Psi'(t)>0$, it follows from (27) that

$$\Psi\left(t\right)\geq G\left(t\right)+\int_{t}^{\infty}\Psi^{\frac{\alpha+1}{\alpha}}\left(s\right)R\left(s\right)\mathrm{d}s=y_{0}\left(t\right)+\int_{t}^{\infty}\Psi^{\frac{\alpha+1}{\alpha}}\left(s\right)R\left(s\right)\mathrm{d}s, \tag{32}$$

or

$$\Psi\left(t\right)\geq G\left(t\right):=y_{0}\left(t\right).$$

Hence, $\Psi(t)\geq y_n(t)$, $n=1,2,3,....$ Since $\{y_n(t)\}_{n=0}^{\infty}$ increasing and bounded above, we get that$y_n\rightarrow y$ as $n\rightarrow\infty$. Using Lebesgue's monotone convergence theorem, we see that (29) turns into (30) as $n\rightarrow\infty$. □

Theorem 5. *Assume that*

$$\liminf_{t\rightarrow\infty}\frac{1}{y_{0}\left(t\right)}\int_{t}^{\infty}y_{0}^{\frac{\alpha+1}{\alpha}}\left(s\right)R\left(s\right)\mathrm{d}s>\frac{\alpha}{\left(\alpha+1\right)^{\frac{\alpha+1}{\alpha}}}, \tag{33}$$

then, (1) is oscillatory.

Proof. Suppose the contrary that (1) has a non-oscillatory solution x on $[t_0,\infty)$. Without loss of generality, we assume that there exists a $t_1\geq t_0$ such that $x\left(t\right)>0$ for $t\geq t_1$. Proceeding as in the proof of Lemma 6, we arrive at (32). From (32), we find

$$\frac{\Psi(t)}{y_{0}(t)}\geq 1+\frac{1}{y_{0}(t)}\int_{t}^{\infty}y_{0}^{\frac{\alpha+1}{\alpha}}\left(s\right)R\left(s\right)\left(\frac{\Psi(s)}{y_{0}(s)}\right)^{\frac{\alpha+1}{\alpha}}\mathrm{d}s. \tag{34}$$

If we consider $\mu=inf_{t\geq T}\left(\Psi(t)/y_{0}(t)\right)$, then obviously $\mu\geq 1$. Using (33) and (34), we see that

$$\mu\geq 1+\alpha\left(\frac{\mu}{\alpha+1}\right)^{\frac{\alpha+1}{\alpha}}$$

or

$$\frac{\mu}{\alpha+1}\geq\frac{1}{\alpha+1}+\frac{\alpha}{\alpha+1}\left(\frac{\mu}{\alpha+1}\right)^{\frac{\alpha+1}{\alpha}},$$

which contradicts the expected value of μ and α, therefore, the proof is complete. □

Theorem 6. *If there exist some $y_n(t)$ such that*

$$\limsup_{t\to\infty} y_n(t)\left(\int_{t_0}^{t} r^{-\frac{1}{\alpha}}(s)\mathrm{d}s\right)^{\alpha} > 1, \tag{35}$$

then, (1) is oscillatory.

Proof. Suppose the contrary that (1) has a non-oscillatory solution x on $[t_0,\infty)$. Without loss of generality, we assume that there exists a $t_1 \geq t_0$ such that $x(t) > 0$ for $t \geq t_1$. Let $\Psi(t)$ defined as in (13). Then,

$$\begin{aligned}
\frac{1}{\Psi(t)} &= \frac{1}{r(t)}\left(\frac{z(t)}{z'(t)}\right)^{\alpha} = \frac{1}{r(t)}\left(\frac{z(T)+\int_T^t r^{-1/\alpha}(s)\, r^{1/\alpha}(s)\, z'(s)\,\mathrm{d}s}{z'(t)}\right)^{\alpha} \\
&\geq \frac{1}{r(t)}\left(\frac{r^{1/\alpha}(t)\, z'(t)\int_T^t r^{-1/\alpha}(s)\,\mathrm{d}s}{z'(t)}\right)^{\alpha} \\
&= \left(\int_T^t r^{-1/\alpha}(s)\,\mathrm{d}s\right)^{\alpha},
\end{aligned} \tag{36}$$

for $t \geq T$. Thus, it follows from (36) that

$$\Psi(t)\left(\int_{t_0}^{t} r^{-1/\alpha}(s)\,\mathrm{d}s\right)^{\alpha} \leq \left(\frac{\int_{t_0}^t r^{-1/\alpha}(s)\,\mathrm{d}s}{\int_T^t r^{-1/\alpha}(s)\,\mathrm{d}s}\right)^{\alpha},$$

and so

$$\limsup_{t\to\infty} \Psi(t)\left(\int_{t_0}^{t} r^{-\frac{1}{\alpha}}(s)\mathrm{d}s\right)^{\alpha} \leq 1,$$

which contradicts (35). The proof is complete. □

Corollary 2. *If there exist some $y_n(t)$ such that either*

$$\int_{t_0}^{\infty} Q(t)\exp\left(\int_{t_0}^{t} y_n^{\frac{1}{\alpha}}(s)R(s)\,\mathrm{d}s\right)\mathrm{d}t = \infty \tag{37}$$

or

$$\int_{t_0}^{\infty} R(t)\, y_n^{\frac{1}{\alpha}}(t) y_0(t)\exp\left(\int_{t_0}^{t} R(s)\, y_n^{\frac{1}{\alpha}}(s)\mathrm{d}s\right)\mathrm{d}t = \infty, \tag{38}$$

then (1) is oscillatory.

Proof. Suppose the contrary that (1) has a non-oscillatory solution x on $[t_0,\infty)$. Without loss of generality, we assume that there exists a $t_1 \geq t_0$ such that $x(t) > 0$ for $t \geq t_1$. From Lemma 6, we get that (30) holds. Using (30), we have

$$\begin{aligned}
y'(t) &= -R(t)\, y^{\frac{\alpha+1}{\alpha}}(t) - Q(t) \\
&\leq -R(t)\, y_n^{\frac{1}{\alpha}}(t)\, y(t) - Q(t).
\end{aligned} \tag{39}$$

Hence,

$$\int_T^t Q(s)\exp\left(\int_T^s y_n^{\frac{1}{\alpha}}(u)R(u)\,\mathrm{d}u\right)\mathrm{d}s \leq y(T) < \infty,$$

which contradicts (37).

Next, let $M(t) = \int_t^\infty R(s)\, y^{\frac{\alpha+1}{\alpha}}(s)\, \mathrm{d}s$. Then, we obtain

$$\begin{aligned} M'(t) &= -R(t)\, y^{\frac{\alpha+1}{\alpha}}(t) \\ &\leq -R(t)\, y_n^{\frac{1}{\alpha}}(t)\, y(t) \\ &= -R(t)\, y_n^{\frac{1}{\alpha}}(t)\left(M(t) + y_0(t)\right). \end{aligned}$$

Therefore, we find

$$\int_T^\infty R(t)\, y_n^{\frac{1}{\alpha}}(t) y_0(t) \exp\left(\int_T^t R(s)\, y_n^{\frac{1}{\alpha}}(s) \mathrm{d}s\right) \mathrm{d}t < \infty,$$

which contradicts (38). The proof is complete. □

4. Examples

Example 1. *Consider the differential equation*

$$\left(\left(\left(x(t) + p_0 x(\tau_0 t)\right)'\right)^\alpha\right)' + \int_\lambda^1 \frac{q_0}{t^{\alpha+1}} x^\alpha(ts)\, \mathrm{d}s = 0, \tag{40}$$

where $\lambda, \tau_0 \in (0,1)$. *It is easy to verify that*

$$U(t) = \frac{q_0}{t^{\alpha+1}}(1-\lambda)\left[1-p_0\right]^\alpha,\ \eta_{t_0}(t) = t \text{ and } \widetilde{\eta}_{t_0}(t) = Mt,$$

where

$$M := 1 + \lambda^\alpha \frac{q_0}{\alpha}(1-\lambda)\left[1-p_0\right]^\alpha.$$

Using Corollary 1, we see that (40) is oscillatory if

$$\left(M^\alpha \lambda^\alpha q_0 (1-\lambda)\left[1-p_0\right]^\alpha\right) \ln\frac{1}{\lambda} > \frac{1}{\mathrm{e}}$$

or

$$\alpha(M-1) M^\alpha \ln\frac{1}{\lambda} > \frac{1}{\mathrm{e}}. \tag{41}$$

Next, we note that $R(t) = \alpha$,

$$\widehat{\eta}_{t_1}(t) = \lambda^{1/M},\ Q(t) = \frac{N}{t^{\alpha+1}}\lambda^{\alpha/M},\ G(t) = \frac{N\lambda^{\alpha/M}}{\alpha}\frac{1}{t^{\alpha+1}},$$

where $N = q_0(1-p_0)^\alpha(1-\lambda)$. *From Theorem 5, (40) is oscillatory if*

$$\left(\frac{N}{\alpha}\lambda^{\alpha/M}\right)^{1/\alpha} > \frac{\alpha}{(\alpha+1)^{(\alpha+1)/\alpha}}.$$

Remark 1. *Consider a particular case of (40), namely,*

$$\left(x(t) + \frac{1}{2}x(\tau_0 t)\right)'' + \frac{q_0}{t^2}x(\lambda t) = 0, \tag{42}$$

From the results in Example 1, Equation (42) is oscillatory if

$$\lambda\frac{q_0}{2}\left(1 + \frac{1}{2}\lambda q_0\right) \ln\frac{1}{\lambda} > \frac{1}{\mathrm{e}}. \tag{43}$$

Applying Corollary 2 in [3], we see that (42) is oscillatory if

$$q_0 \lambda \ln \frac{1}{2\lambda} > \frac{2}{\mathrm{e}}. \tag{44}$$

Obviously, in the case where $\lambda = 1/3$, conditions (43) and (44) reduce to $q > 1.588$ and $q > 5.443$, respectively. Thus, a new criterion improve some related results in [3].

Example 2. *Consider the differential equation*

$$\left(x\left(t\right) + \int_0^1 \frac{1}{2} x\left(\frac{t-x}{3}\right) \mathrm{d}x \right)'' + \int_0^1 \left(\frac{q_0}{t^2}\right) x\left(\frac{t-s}{2}\right) \mathrm{d}s = 0, \tag{45}$$

where $q_0 > 0$. It is easy to verify that

$$U\left(t\right) = \frac{q_0}{t^2}, \eta_{t_0}\left(t\right) = t$$

and

$$\widetilde{\eta}_{t_0} = t + \frac{q_0}{4} \int_{t_0}^{t} dx = t\left(1 + \frac{q_0}{4}\right).$$

Using Corollary 1, if

$$\frac{q_0}{4}\left(1 + \frac{q_0}{4}\right) \ln 2 > \frac{1}{e},$$

then (45) is oscillatory.

5. Conclusions

The growing interest in the oscillation theory of functional differential equation is due to the many applications of this theory in many fields, see [1,2]. In this work, we used comparison principles and Riccati transformation techniques to obtain new oscillation criteria for neutral differential Equation (1). Our new criteria improved a number of related results [3,4,14]. Further, we extended and generalized the recent works [9,10].

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