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Modular Uniform Convexity in Every Direction in $L^{p(\cdot)}$ and Its Applications

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Abstract: We prove that the Lebesgue space of variable exponent $L^{p(\cdot)}(\Omega)$ is modularly uniformly convex in every direction provided the exponent p is finite a.e. and different from 1 a.e. The notion of uniform convexity in every direction was first introduced by Garkavi for the case of a norm. The contribution made in this work lies in the discovery of a modular, uniform-convexity-like structure of $L^{p(\cdot)}(\Omega)$, which holds even when the behavior of the exponent $p(\cdot)$ precludes uniform convexity of the Luxembourg norm. Specifically, we show that the modular $\rho(u) = \int_\Omega |u(x)|dx$ possesses a uniform-convexity-like structure even if the variable exponent is not bounded away from 1 or ∞. Our result is new and we present an application to fixed point theory.

Keywords: modular uniform convexity; modular vector spaces; uniform convexity; variable exponent spaces

1. Introduction

The most remote origins of the notion of variable exponent Lebesgue spaces $L^{p(\cdot)}$ can be traced back to [1], where a particular case was introduced as a generalization of the variable exponent sequence spaces. The first systematic treatment of $L^{p(\cdot)}$ spaces is to be found in [2]. Since then, due to the variety of applications in which these spaces play a role, a vast amount of research was devoted to the study of variable exponent spaces.

The applicability of the variable exponent spaces was already demonstrated by their role the description of the hydrodynamical behavior of non-Newtonian fluids; see for example [3–5] and the model of image restoration [6,7] for more details.

Early on, the issue of uniform convexity with respect to the Luxemburg norm was identified as an important problem and was studied in [8]. It was proved there that if $\Omega \subseteq \mathbb{R}^n$ has positive Lebesgue measure, then the Luxemburg norm on $L^{p(\cdot)}(\Omega)$ is uniformly convex if and only if the variable exponent is bounded away from 1 and ∞. More precisely, if

$$p_- := \operatorname*{ess\,inf}_{x\in\Omega} p(x) \quad \text{and} \quad p_+ := \operatorname*{ess\,sup}_{x\in\Omega} p(x),$$

then the uniform convexity of the Luxemburg norm is equivalent to the condition $1 < p_- \leq p_+ < \infty$. The condition $p_+ < \infty$ is, in turn, necessary and sufficient for the validity of the Δ_2 condition on the modular

$$L^{p(\cdot)}(\Omega) \ni u \to \int_\Omega |u(x)|^{p(x)}dx = \rho(u),$$

defined on $L^{p(\cdot)}(\Omega)$. In particular, the preceding modular does not fulfill the Δ_2 condition when $p_+ = \infty$. More precisely, setting $\Omega_\infty = \{x \in \Omega : p(x) = \infty\}$, the Δ_2 condition fails for ρ if $p_+ = \infty$, even if $|\Omega_\infty| = 0$; see [9]. In the sequel we will refer to this case as that of an everywhere finite, unbounded exponent. On the other hand, in view of the result in [8], the condition $p_- = 1$ also precludes the uniform convexity for the Luxemburg norm, even if $p(x) > 1$ everywhere.

In the present work we address the preceding two cases and prove that the modular ρ is uniformly convex in every direction (Definition 1) under the assumptions that $p(x) > 1$ almost everywhere and that $|\Omega_\infty| = 0$. The novelty here is that the discussed uniform convexity of ρ holds even in the limit-point cases $p_- = 1$ or $p_+ = \infty$. This notion was first introduced by Garkavi [10,11] for the case of a norm. As an application of the geometric property of ρ alluded to above, we present a fixed point result for mappings which are nonexpansive in the modular sense.

2. Preliminaries

Given a domain $\Omega \subset \mathbb{R}^n$, $\mathcal{M}(\Omega)$ will stand for the vector space of all real-valued, Borel-measurable functions defined on Ω. The subset of $\mathcal{M}(\Omega)$ consisting of admissible exponents functions

$$p : \Omega \longrightarrow [1, \infty]$$

will be denoted by $\mathcal{P}(\Omega)$. As usual, the Lebesgue measure of a subset $A \subset \mathbb{R}^n$ will be denoted by $|A|$. We say that $\rho : \mathcal{M}(\Omega) \to [0, \infty]$ is a convex regular modular function if the following hold:

(1) $\rho(\phi) = 0$ if and only if $\phi = 0$;
(2) $\rho(\alpha\phi) = \rho(\phi)$, if $|\alpha| = 1$;
(3) $\rho(\alpha\phi + (1-\alpha)\psi) \le \alpha\rho(\phi) + (1-\alpha)\rho(\psi)$, for any $\alpha \in [0, 1]$,

where $\phi, \psi \in \mathcal{M}(\Omega)$. If the inequality (3) is strict whenever $\phi \neq \psi$ and $\alpha \in (0, 1)$, ρ is called strictly convex (SC).

For each such $p \in \mathcal{P}(\Omega)$ define the modular $\rho : \mathcal{M}(\Omega) \to [0, \infty]$ by

$$\rho(u) = \int_{\Omega_0 \cup \Omega_1} |u(x)|^{p(x)} dt + \sup_{x \in \Omega_\infty} |u(x)|, \tag{1}$$

where

$$\begin{aligned} \Omega_0 &= \{x \in \Omega : \ 1 < p(x) < \infty\}, \\ \Omega_1 &= \{x \in \Omega : \ p(x) = 1\}, \\ \Omega_\infty &= \{x \in \Omega : \ p(x) = \infty\}. \end{aligned}$$

For $p \in \mathcal{P}(\Omega)$ the following notation will be used throughout this work:

$$p_- := \operatorname*{ess\,inf}_{x \in \Omega_0} p(x) \quad \text{and} \quad p_+ := \operatorname*{ess\,sup}_{x \in \Omega_0} p(x).$$

The space $L^{p(\cdot)}(\Omega)$ is defined as

$$L^{p(\cdot)}(\Omega) = \{u \in \mathcal{M}(\Omega); \text{ there exists } \lambda > 0 \text{ such that } \rho(u/\lambda) < \infty\},$$

furnished with the Luxemburg norm; namely, for $u \in L^{p(\cdot)}(\Omega)$ one will set

$$\|u\|_{p(\cdot)} = \inf\left\{\lambda > 0 : \rho\left(\frac{u}{\lambda}\right) \le 1\right\}. \tag{2}$$

It is well known [2,12] that under the above assumptions $L^{p(\cdot)}(\Omega)$ is a Banach space. Of particular importance is the question of uniform convexity of the norm (2). The matter is in fact settled: uniform convexity of the Luxemburg norm has been characterized in the following theorem:

Theorem 1. *[8] Conditions (i)–(iii) are equivalent:*

(i) $L^{p(\cdot)}(\Omega)$ *is reflexive.*

(ii) $L^{p(\cdot)}(\Omega)$ *is uniformly convex.*

(iii) $1 < p_- \leq p_+ < \infty$.

There is therefore no hope of uniform convexity of the Luxemburg norm unless the variable exponent is bounded away from 1 and ∞. At first glance, the situation when the exponent p is either unbounded or its infimum is 1 seems to be insurmountable if one insists on focusing on the norm. However, a closer examination reveals a uniform-convexity-like structure of the modular whose depth and implications justify further analysis. This is the driving motivation of the present work. We open the discussion with some preliminary results whose relevance will be apparent in the sequel.

3. Uniform Convexity in Every Direction (*UCED*)

The following technical lemma is a cornerstone in the approach that follows. For a detailed proof we refer the reader to [13].

Lemma 1. *[14] Let $p \in \mathbb{R}$, $1 < p \leq 2$. Then the following inequality holds:*

$$\left|\frac{a+b}{2}\right|^p + \frac{p(p-1)}{2}\left|\frac{a-b}{|a|+|b|}\right|^{2-p}\left|\frac{a-b}{2}\right|^p \leq \frac{1}{2}\Big(|a|^p + |b|^p\Big),$$

for any $a, b \in \mathbb{R}$ such that $|a| + |b| \neq 0$.

The next result is elementary; see [2,12,13] for more details.

Lemma 2. *Let $\Omega \subseteq \mathbb{R}^n$ be a domain and let $p \in \mathcal{P}(\Omega)$ satisfy $p_+ < \infty$. Then*

$$\|u\|_{p(\cdot)} \leq \max\left\{\left(\int_\Omega |u|^p\, dt\right)^{\frac{1}{p_-}}, \left(\int_\Omega |u|^p\, dt\right)^{\frac{1}{p_+}}\right\} \tag{3}$$

The following theorem is the simplest form of the result in [9], to which we refer the reader for a complete proof.

Theorem 2. *[9] Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and assume that $p \in \mathcal{P}(\Omega)$ satisfies $p_- > 1$ and $p(t) < \infty$ a.e. in Ω. Then for fixed $r > 0$, $0 < \varepsilon \leq 1$ and any $a \in L^{p(\cdot)}(\Omega)$, $b \in L^{p(\cdot)}(\Omega)$ such that $\rho(a) \leq r$, $\rho(b) \leq r$ and $\rho\left(\frac{a-b}{2}\right) \geq \varepsilon r$, one has the estimate*

$$\rho\left(\frac{a+b}{2}\right) \leq r\left(1 - \min\left\{\frac{\varepsilon}{2}, (p_- - 1)\frac{\varepsilon^2}{2}\right\}\right).$$

To conclude this section we present Definition 1, which captures the essential modular convexity property to be dealt with in the rest of this work.

Definition 1. *A convex modular ρ on a vector space X is said to be uniformly convex in every direction (in short (UCED)) if and only if for any $z_1 \neq z_2 \in X$ and $R > 0$, there exists $\delta = \delta(z_1, z_2, R) > 0$ such that*

$$\left\{\begin{array}{lcl} \rho(x - z_1) & \leq & R \\ \rho(x - z_2) & \leq & R \end{array}\right. \implies \rho\left(x - \frac{z_1 + z_2}{2}\right) \leq R(1 - \delta),$$

for any $x \in X$. Moreover, we will say ρ is UUCED if for any $z_1 \neq z_2$ and $R_0 > 0$, there exists $\eta(z_1, z_2, R_0) > 0$ such that

$$\delta(z_1, z_2, R) \geq \eta(z_1, z_2, R_0),$$

for any $R \leq R_0$.

The following technical result will be useful to establish some fixed point results later on.

Lemma 3. *Assume that a convex modular ρ on a vector space X is (UUCED). Let $z_1, z_2 \in X$. Assume there exist $R \geq 0$ and $x_n \in X$ such that*

$$\begin{cases} \limsup\limits_{n\to\infty} \rho(x_n - z_1) & \leq & R \\ \limsup\limits_{n\to\infty} \rho(x_n - z_2) & \leq & R, \\ \limsup\limits_{n\to\infty} \rho\left(x_n - \frac{z_1+z_2}{2}\right) & = & R. \end{cases}$$

Then $z_1 = z_2$ holds.

Proof. If $R = 0$, then

$$\limsup_{n\to\infty} \rho(x_n - z_1) = \limsup_{n\to\infty} \rho(x_n - z_2) = 0.$$

which implies that $\{x_n\}$ ρ-converges to z_1 and z_2. The uniqueness of the ρ-limit implies $z_1 = z_2$. Otherwise, assume $R > 0$ and $z_1 \neq z_2$. Fix $\varepsilon > 0$. Using the definition of the limit-sup, there exists $n_0 \geq 1$ such that

$$\rho(x_n - z_1) \leq R + \varepsilon \ \ and \ \ \rho(x_n - z_2) \leq R + \varepsilon,$$

for any $n \geq n_0$. Since ρ is *UUCED*, set $\eta = \eta(z_1, z_2, R) > 0$. Then we have $\delta(z_1, z_2, R+\varepsilon) \geq \eta$ which implies

$$\rho\left(x_n - \frac{z_1+z_2}{2}\right) \leq (R+\varepsilon)(1 - \delta(z_1, z_2, R+\varepsilon)) \leq (R+\varepsilon)(1-\eta),$$

for any $n \geq n_0$. Letting $n \to \infty$, it follows that

$$\limsup_{n\to\infty} \rho\left(x_n - \frac{z_1+z_2}{2}\right) \leq (R+\varepsilon)(1-\eta).$$

Since ε was taken arbitrarily, we get

$$R = \limsup_{n\to\infty} \rho\left(x_n - \frac{z_1+z_2}{2}\right) \leq R\,(1-\eta) < R.$$

This contradiction forces $z_1 = z_2$, as claimed. □

Note that it is easy to check that if ρ is *UUCED*, then it is strictly convex.

4. The $(UUCED)$ Property for the Variable Exponent Spaces $L^{p(\cdot)}$

We aim at proving the following Theorem:

Theorem 3. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain and $p \in \mathcal{P}(\Omega)$ be an admissible exponent. Then the following properties are equivalent*

(i) $\quad |\Omega_1| = |\Omega_\infty| = 0.$

(ii) *The modular ρ is strictly convex (SC).*
(iii) *The modular ρ is (UUCED).*

Proof. As we noted before, it is clear that (*iii*) implies (*ii*). Since $L^1(\Omega_1)$ and $L^\infty(\Omega_\infty)$ are not strictly convex, then (*ii*) implies (*i*). To complete the proof, we need to show that (*i*) implies (*iii*).

Note that the assumption (i) implies that $1 < p(x) < \infty$ a.e. Let z_1 and z_2 be in $L^{p(\cdot)}(\Omega)$ such that $z_1 \neq z_2$. Thus, the set

$$\widetilde{\Omega} = \{x \in \Omega;\ z_1(x) \neq z_2(x)\}$$

has positive measure; i.e., $|\widetilde{\Omega}| > 0$. Fix $a \in (1,2)$. We have $\widetilde{\Omega} = \widetilde{\Omega}_{1a} \cup \widetilde{\Omega}_{a\infty}$, where

$$\widetilde{\Omega}_{1a} = \{x \in \widetilde{\Omega} : 1 < p(x) < a\} \quad and \quad \widetilde{\Omega}_{a\infty} = \{x \in \widetilde{\Omega} :\ a \leq p(x)\}.$$

Fix $R > 0$ and let $u \in L^{p(\cdot)}(\Omega)$, be selected in such a way that

$$\rho(u - z_1) \leq R \quad and \quad \rho(u - z_2) \leq R.$$

It will be shown that there exists $\delta(z_1, z_2, R) > 0$ such that

$$\rho\left(u - \frac{z_1 + z_2}{2}\right) \leq R\,(1 - \delta(z_1, z_2, R)).$$

To simplify the notation, for $v \in L^{p(\cdot)}(\Omega)$ we set

$$\rho_{1a}(v) = \int_{\widetilde{\Omega}_{1a}} |v(x)|^{p(x)} dx \ \ and \ \ \rho_{1a}^c(v) = \int_{(\widetilde{\Omega}_{1a})^c} |v(x)|^{p(x)} dx.$$

The proof of Theorem 3 is split in two different scenarios, depending on whether $|\widetilde{\Omega}_{1a}| > 0$ or $|\widetilde{\Omega}_{1a}| = 0$.

Case 1: $|\widetilde{\Omega}_{1a}| > 0$

By definition and in accordance with the above terminology, one has

$$\rho\left(u - \frac{z_1 + z_2}{2}\right) = \rho_{1a}\left(u - \frac{z_1 + z_2}{2}\right) + \rho_{1a}^c\left(u - \frac{z_1 + z_2}{2}\right).$$

For $x \in \widetilde{\Omega}_{1a}$, Lemma 1 implies

$$\left|u(x) - \frac{z_1(x) + z_2(x)}{2}\right|^{p(x)} + Z(x) \leq \frac{1}{2}\Big(|u(x) - z_1(x)|^{p(x)} + |u(x) - z_2(x)|^{p(x)}\Big), \tag{4}$$

where

$$\begin{aligned} Z(x) &= \frac{p(x)(p(x)-1)}{2} \left|\frac{z_1(x) - z_2(x)}{|u(x) - z_1(x)| + |u(x) - z_2(x)|}\right|^{2-p(x)} \left|\frac{z_1(x) - z_2(x)}{2}\right|^{p(x)} \\ &= \frac{p(x)(p(x)-1)}{2^{p(x)+1}} \frac{|z_1(x) - z_2(x)|^2}{\Big||u(x) - z_1(x)| + |u(x) - z_2(x)|\Big|^{2-p(x)}}. \end{aligned}$$

To facilitate the computations, for $x \in \widetilde{\Omega}_{1a}$ set

$$\begin{aligned} \gamma(x) &= \frac{p(x)(p(x)-1)}{2^{p(x)+1}} < \frac{a(a-1)}{4} < \frac{1}{2}, \\ f(x) &= \gamma(x)|z_1(x)-z_2(x)|^2, \\ g(x) &= \frac{1}{\Big(|u(x)-z_1(x)|+|u(x)-z_2(x)|\Big)^{2-p(x)}}. \end{aligned}$$

By assumption, one has

$$\begin{aligned} \int_{\widetilde{\Omega}_{1a}} \left(\frac{1}{g(x)^{p(x)/2}}\right)^{2/(2-p(x))} dx &= \int_{\widetilde{\Omega}_{1a}} \Big(|u(x)+z_1(x)|+|u(x)-z_2(x)|\Big)^{p(x)} dx \\ &\leq \int_{\widetilde{\Omega}_{1a}} 2^{p(x)-1}\Big(|u(x)-z_1(x)|^{p(x)}+|u(x)-z_2(x)|^{p(x)}\Big)dt \\ &\leq 2\int_{\widetilde{\Omega}_{1a}} \Big(|u(x)-z_1(x)|^{p(x)}+|u(x)-z_2(x)|^{p(x)}\Big)dx \\ &\leq 2(2R)=4R. \end{aligned}$$

Next, a direct application of Hölder's inequality yields

$$\begin{aligned} \int_{\widetilde{\Omega}_{1a}} f(x)^{p(x)/2}dx &= \int_{\widetilde{\Omega}_{1a}} (f(x)g(x))^{p(x)/2}\frac{1}{g(x)^{p(x)/2}}dx \\ &\leq C_p\Big\|(f(x)g(x))^{p(x)/2}\Big\|_{2/p(x)} \Big\|\frac{1}{g(x)^{p(x)/2}}\Big\|_{2/2-p(x)}, \end{aligned}$$

where C_p is a constant that depends only on the exponent function $p(\cdot)$; in fact, C_p can be chosen to depend only on the auxiliary parameter a. Since for $t \in \widetilde{\Omega}_{1a}$, $2/p(x) \leq 2$ and $2/(2-p(x)) \leq 2/(2-a)$ hold, a straightforward application of Lemma 2 yields

$$\begin{aligned} \int_{\widetilde{\Omega}_{1a}} f(x)^{p(x)/2}dt &\leq C_p\left(\int_{\widetilde{\Omega}_{1a}} f(x)g(x)dx\right)^{\alpha}\left(\int_{\widetilde{\Omega}_{1a}} \frac{1}{g(x)^{p(x)/(2-p(x))}}dx\right)^{\beta} \\ &\leq C_p\,(4\,R)^{\beta}\left(\int_{\widetilde{\Omega}_{1a}} f(x)g(x)dx\right)^{\alpha}, \end{aligned}$$

where $\alpha \in A = \{(2/p)_+,(2/p)_-\}$ and $\beta \in B = \{(2/(2-p))_+,(2/(2-p))_-\}$. Set

$$\Delta(R,z_1,z_2,\alpha,\beta) = \frac{1}{\Big(C_p\,(4\,R)^{\beta}\Big)^{1/\alpha}}\left(\int_{\widetilde{\Omega}_{1a}} \gamma(x)^{p(x)/2}|z_1(x)-z_2(x)|^{p(x)}dx\right)^{1/\alpha} \tag{5}$$

and define

$$\Delta(R,z_1,z_2) = \min\{\Delta(R,z_1,z_2,\alpha,\beta), \alpha \in A, \beta \in B\}.$$

Since $1 < p(x) < \infty$ a.e., we have $\Delta(R,z_1,z_2) > 0$. It thus follows that

$$\int_{\widetilde{\Omega}_{1a}} f(x)g(x)dx = \int_{\widetilde{\Omega}_{1a}} \frac{\gamma(x)|z_1(x)-z_2(x)|^2}{\Big(|u(x)-z_1(x)|+|u(x)-z_2(x)|\Big)^{2-p(x)}}dx \geq \Delta(R,z_1,z_2).$$

In conclusion, on account of inequality (4), we have

$$\rho_{1a}\left(u-\frac{z_1+z_2}{2}\right)+\Delta(R,z_1,z_2)\leq\frac{\rho_{1a}(u-z_1)+\rho_{1a}(u-z_2)}{2}.$$

The convexity of ρ^c_{1a} yields

$$\rho^c_{1a}\left(u-\frac{z_1+z_2}{2}\right)\leq\frac{\rho^c_{1a}(u-z_1)+\rho^c_{1a}(u-z_2)}{2},$$

the preceding inequalities, added up, imply

$$\rho\left(u-\frac{z_1+z_2}{2}\right)+\Delta(R,z_1,z_2)\leq\frac{\rho(u-z_1)+\rho(u-z_2)}{2}\leq R.$$

Set $\delta_1(z_1,z_2,R)=\dfrac{1}{R}\Delta(R,z_1,z_2)$. Then $\delta_1(z_1,z_2,R)$ is a decreasing, positive function of R and we have

$$\rho\left(u-\frac{z_1+z_2}{2}\right)\leq R(1-\delta_1(z_1,z_2,R)).$$

Case 2: $|\widetilde{\Omega}_{1a}|=0$; i.e., $\widetilde{\Omega}=\widetilde{\Omega}_{a\infty}$

In this case, the restriction of $p(\cdot)$ to Ω_1 satisfies $p_-\geq a>1$. Let u_1, z_{11} and z_{12} be the restrictions to Ω_1, of u, z_1 and z_2 respectively. For $v\in L^{p(\cdot)}(\Omega)$, write

$$\rho_{1a}(v)=\int\limits_{\widetilde{\Omega}}|v(x)|^{p(x)}dx\ \ and\ \ \rho^c_{1a}(v)=\int\limits_{(\widetilde{\Omega})^c}|v(x)|^{p(x)}dx.$$

It is clear from the fact that $z_1=z_2$ on $(\widetilde{\Omega})^c$ that

$$\rho^c_{1a}(u-z_1)=\rho^c_{1a}(u-z_2)=\rho^c_{1a}\left(u-\frac{z_1+z_2}{2}\right)=R_u\leq R. \tag{6}$$

For the same reason,

$$\rho_{1a}\left(\frac{z_1-z_2}{2}\right)=\rho\left(\frac{z_1-z_2}{2}\right)>0.$$

Set $\varepsilon=\rho((z_1-z_2)/2)/R$. Hence

$$R\,\varepsilon=\rho_{1a}\left(\frac{z_1-z_2}{2}\right)\leq\frac{\rho_{1a}(u-z_1)+\rho_{1a}(u-z_2)}{2}\leq R-R_u\leq R.$$

Thus,

$$\begin{cases}\rho_{1a}\left(\dfrac{z_1-z_2}{2}\right) & \geq\ (R-R_u)\,\varepsilon,\\ \rho_{1a}(u-z_1) & \leq\ R-R_u,\\ \rho_{1a}(u-z_2) & \leq\ R-R_u,\end{cases}$$

It follows from $|\Omega_\infty|=0$ and the application of Theorem 2 to the modular ρ_{1a} on $L^{p(\cdot)}(\Omega_1)$ with $r=R-R_u$ that

$$\rho_{1a}\left(u-\frac{z_1+z_2}{2}\right)\leq(R-R_u)\,\Big(1-\delta_2(\varepsilon)\Big).$$

where

$$\delta_2(\varepsilon)=\min\left\{\frac{\varepsilon}{2},(p_--1)\,\frac{\varepsilon^2}{2}\right\}.$$

Hence

$$\begin{aligned}
\rho\left(u-\frac{z_1+z_2}{2}\right) &= \rho_{1a}\left(u-\frac{z_1+z_2}{2}\right)+\rho_{1a}^{c}\left(u-\frac{z_1+z_2}{2}\right)\\
&= \rho_{1a}\left(u-\frac{z_1+z_2}{2}\right)+R_u\\
&\le (R-R_u)\left(1-\delta_2(\varepsilon)\right)+R_u\\
&= R\left(1-\frac{R-R_u}{R}\delta_2(\varepsilon)\right)\\
&\le R\left(1-\varepsilon\,\delta_2(\varepsilon)\right),
\end{aligned}$$

since $R-R_u\ge R\varepsilon$. Set

$$\delta(z_1,z_2,R)=\min\left\{\delta_1(z_1,z_2,R),\frac{\rho((z_1-z_2)/2)}{R}\,\delta_2\left(\frac{\rho((z_1-z_2)/2)}{R}\right)\right\}.$$

Then $\delta(z_1,z_2,R)$ is a decreasing positive function of R and we have

$$\rho\left(u-\frac{z_1+z_2}{2}\right)\le R\,(1-\delta(z_1,z_2,R)),$$

which completes the proof of our claim. □

5. Example

We present an example, originally given in ([15], Example 2.15), to illustrate the relevance of our main result. Consider the set $B=\{\mathbf{u}\in L^{p(x)}((0,\infty)) : \|\mathbf{u}\|_\infty\le\frac{1}{2}$, where $p(x)=x+1\}$, let τ_h be the translation operator on $L^{x+1}((0,\infty))$ (i.e., $\tau_h(\mathbf{u})(x)=\mathbf{u}(x-h)$) and ρ be the modular on $L^{x+1}((0,\infty))$ defined as

$$\rho(\mathbf{u})=e^{-2}\int_0^\infty|\mathbf{u}(x)|^{x+1}dx. \tag{7}$$

Consider the operator

$$\begin{aligned}
J : L^{x+1}((0,\infty)) &\to L^{x+1}((0,\infty))\\
\mathbf{u} &\to \chi_{[1,\infty]}\tau_{-1}\mathbf{u},
\end{aligned}$$

observe that

$$\chi_{[1,\infty]}\tau_{-1}\mathbf{u}(x) = \begin{cases}\mathbf{u}(x-1), & \text{if } x\ge 1,\\ 0, & \text{if } 0\le x<1.\end{cases}$$

It is a trivial matter to verify that $J(B)\subseteq B$. Observe that for $\mathbf{u},\mathbf{v}\in B$,

$$\begin{aligned}
\rho(J(\mathbf{u})-J(\mathbf{v})) &= e^{-2}\int_0^\infty|J(\mathbf{u})(x)-J(\mathbf{v})(x)|^{x+1}dx\\
&= e^{-2}\int_1^\infty|\mathbf{u}(x-1)-\mathbf{v}(x-1)|^{x+1}dx\\
&= e^{-2}\int_0^\infty|\mathbf{u}(x)-\mathbf{v}(x)|^{x+1}|\mathbf{u}(x)-\mathbf{v}(x)|dx\\
&\le \rho(\mathbf{u}-\mathbf{v}).
\end{aligned}$$

Hence, J is ρ-nonexpansive. However,

$$\rho(e\chi_{[0,1]}) = e^{-2}\int_0^1 e^{x+1}dx = e^{-1}(e-1) < 1,$$

whereas

$$\rho(J(e\chi_{[0,1]})) = e^{-2}\int_1^2 e^{x+1}dx = e-1 > 1.$$

Denote the Luxemburg norm associated to the modular ρ given in (7) on $L^{x+1}((0,\infty))$ by $\|\cdot\|_{x+1}$. It follows from the preceding two inequalities that $\|2^{-1}e\chi_{[0,1]}\|_{x+1} \leq 2^{-1}$, whereas $\|J(2^{-1}e\chi_{[0,1]})\|_{x+1} > 2^{-1}$; in other words, J is not $\|\cdot\|_{x+1}$-nonexpansive on B.

6. Application

Next, we discuss an application of the modular *UUCED* to the fixed point problem for ρ-nonexpansive mappings. The following technical lemma will be useful.

Lemma 4. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain and $p \in \mathcal{P}(\Omega)$ be an admissible exponent, such that $|\Omega_1| = |\Omega_\infty| = 0$. Let $W \subset L^{p(\cdot)}(\Omega)$ be a convex nonempty subset and $\{u_n\}$ be a sequence in $L^{p(\cdot)}(\Omega)$. Define $\Theta : W \to [0,\infty]$ by*

$$\Theta(w) = \limsup_{n\to\infty} \rho(u_n - w).$$

Assume $\inf_{w\in W}\Theta(w) < \infty$. Then Θ has at most one minimum point.

Proof. Assume there exist $v^*, w^* \in W$ such that

$$\Theta(v^*) = \Theta(w^*) = \inf_{w\in W}\Theta(w) = \inf_{w\in W}\left(\limsup_{n\to\infty}\rho(u_n - w)\right).$$

Since ρ is convex and W is a convex subset, it follows that

$$\inf_{w\in W}\Theta(w) \leq \Theta\left(\frac{w^*+v^*}{2}\right) \leq \frac{\Theta(w^*)+\Theta(v^*)}{2} = \inf_{w\in W}\Theta(w),$$

which implies

$$\Theta(w^*) = \Theta(v^*) = \Theta(\frac{w^*+v^*}{2}).$$

Theorem 3 and Lemma 3, with $R = \inf_{w\in W}\Theta(w)$, yield $v^* = w^*$. $\square$

Next, we discuss the existence of the minimum point. This is not a sure thing even in the case of a norm in a Banach space. We will start our discussion with the case of a uniformly continuous modular.

Definition 2. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain and $p \in \mathcal{P}(\Omega)$ be an admissible exponent. ρ is said to be uniformly continuous if for every $\varepsilon > 0$ and $L > 0$, there exists $\delta > 0$ such that*

$$|\rho(x+y) - \rho(x)| \leq \varepsilon,$$

whenever $\rho(y) \leq \delta$ and $\rho(x) \leq L$, for any $x, y \in L^{p(\cdot)}(\Omega)$.

As shown in the works by Chen [16] and Kaminska [17], the modular ρ in $L^{p(\cdot)}(\Omega)$ is uniformly continuous if and only if $p_+ < \infty$. Lemma 5.1 in [18] implies the following

Lemma 5. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain and $p \in \mathcal{P}(\Omega)$ be an admissible exponent. Assume that ρ is uniformly continuous; then the function $\Theta : W \to [0, \infty]$ defined by*

$$\Theta(w) = \limsup_{n\to\infty} \rho(u_n - w),$$

where $\{u_n\}$ is a sequence in $L^{p(\cdot)}(\Omega)$, is ρ-lower semicontinuous.

Recall that the modular ρ has the property (R) [18] if and only if every nonincreasing sequence $\{C_n\}$ of nonempty, ρ-bounded, ρ-closed, convex subsets of $L^{p(\cdot)}(\Omega)$ has nonempty a intersection. For concrete examples of $L^{p(\cdot)}(\Omega)$ spaces for which ρ has the property (R), the reader may consult [9].

Lemma 6. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain and $p \in \mathcal{P}(\Omega)$ be an admissible exponent, such that $|\Omega_1| = 0$ and $p_+ < \infty$. Assume ρ has the property (R). Let $W \subset L^{p(\cdot)}(\Omega)$ be a ρ-bounded, ρ-closed, convex nonempty subset and $\{u_n\}$ be a sequence in $L^{p(\cdot)}(\Omega)$. Define $\Theta : W \to [0, \infty]$ by*

$$\Theta(w) = \limsup_{n\to\infty} \rho(u_n - w).$$

Assume $\inf_{w\in W} \Theta(w) < \infty$. Then Θ has a minimum point in W.

Proof. Under the above assumptions, Θ is ρ-lower semicontinuous. Set

$$W_n = \left\{ w \in W, \Theta(w) \leq \inf_{w\in W} \Theta(w) + \frac{1}{n} \right\},$$

for $n \geq 1$. Clearly W_n is a nonempty ρ-closed and convex subset of W, for any $n \geq 1$. The property (R) implies $\bigcap_{n\geq 1} W_n \neq \emptyset$. Clearly, we have $\Theta(w^*) = \inf_{w\in W} \Theta(w)$, for any $w^* \in \bigcap_{n\geq 1} W_n$; i.e., Θ has a minimum point in W as claimed. □

Now we are ready to state the main application of this work.

Theorem 4. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain and $p \in \mathcal{P}(\Omega)$ be an admissible exponent, such that $|\Omega_1| = 0$ and $p_+ < \infty$. Assume ρ has the property (R). Let $W \subset L^{p(\cdot)}(\Omega)$ be ρ-bounded, ρ-closed, convex nonempty subset. Let $T : W \to W$ be ρ-nonexpansive; i.e.,*

$$\rho(T(x) - T(y)) \leq \rho(x - y), \quad x, y \in W.$$

Then T has a fixed point.

Proof. Fix $x_0 \in W$. Consider $\Theta : W \to [0, \infty]$ defined by

$$\Theta(w) = \limsup_{n\to\infty} \rho(T^n(x_0) - w).$$

Since W is ρ-bounded, it readily follows that $\Theta(w) \leq \sup_{w_1,w_2\in W} \rho(w_1 - w_2) < \infty$, for any $w \in W$, which implies $\inf_{w\in W} \Theta(w) < \infty$. Lemmas 4 and 6 imply that Θ has a unique minimum point $z \in W$. Since

$$\begin{aligned} \Theta(T(z)) &= \limsup_{n\to\infty} \rho(T^n(x_0) - T(z)) \\ &\leq \limsup_{n\to\infty} \rho(T^{n-1}(x_0) - z) \\ &= \Theta(z), \end{aligned}$$

we conclude that $T(z)$ is also a minimum point of Θ. The uniqueness of the minimum proved in Lemma 4 forces $T(z) = z$; i.e., z is a fixed point of T. $\square$

7. Conclusions

In this paper we were able to shed light on the uniform-convexity structure ((*SE*), (*UCED*) and (*UUCED*)) of the modular ρ under the optimal assumptions $|\Omega_1| = |\Omega_\infty| = 0$. We underline the fact that our result applies to the end-point cases

$$p_- := \operatorname*{ess\,inf}_{x \in \Omega_0} p(x) = 1 \quad \text{and} \quad p_+ = \operatorname*{ess\,sup}_{x \in \Omega_0} p(x) := \infty.$$

and that it is the first of its kind in the literature. Theorem 3 cannot be improved. In fact, it follows by definition that the modular ρ coincides with the L^1 norm on any function supported in Ω_1 and with the L^∞ norm on any function supported on Ω_∞. Since neither L^1 or L^∞ is even strictly convex, it follows that our result cannot be extended to the case when either Ω_1 or Ω_∞ has positive measure; i.e., Theorem 3 is optimal.

Theorem 3 has also opened a new direction in the applications of the estimate for variable exponent spaces; see [12,19]. Theorem 4 is an example of a result that was hitherto unknown for the end point case $p_- = 1$.

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