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Some Diophantine Problems Related to k-Fibonacci Numbers

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Abstract: Let $k \geq 1$ be an integer and denote $(F_{k,n})_n$ as the k-Fibonacci sequence whose terms satisfy the recurrence relation $F_{k,n} = kF_{k,n-1} + F_{k,n-2}$, with initial conditions $F_{k,0} = 0$ and $F_{k,1} = 1$. In the same way, the k-Lucas sequence $(L_{k,n})_n$ is defined by satisfying the same recursive relation with initial values $L_{k,0} = 2$ and $L_{k,1} = k$. The sequences $(F_{k,n})_{n\geq 0}$ and $(L_{k,n})_{n\geq 0}$ were introduced by Falcon and Plaza, who derived many of their properties. In particular, they proved that $F_{k,n}^2 + F_{k,n+1}^2 = F_{k,2n+1}$ and $F_{k,n+1}^2 - F_{k,n-1}^2 = kF_{k,2n}$, for all $k \geq 1$ and $n \geq 0$. In this paper, we shall prove that if $k > 1$ and $F_{k,n}^s + F_{k,n+1}^s \in (F_{k,m})_{m\geq 1}$ for infinitely many positive integers n, then $s = 2$. Similarly, that if $F_{k,n+1}^s - F_{k,n-1}^s \in (kF_{k,m})_{m\geq 1}$ holds for infinitely many positive integers n, then $s = 1$ or $s = 2$. This generalizes a Marques and Togbé result related to the case $k = 1$. Furthermore, we shall solve the Diophantine equations $F_{k,n} = L_{k,m}$, $F_{k,n} = F_{n,k}$ and $L_{k,n} = L_{n,k}$.

Keywords: k-Fibonacci number; k-Lucas number; Galois theory; Diophantine equation

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1. Introduction

Let $(F_n)_n$ be the Fibonacci sequence given by $F_{n+2} = F_{n+1} + F_n$, for $n \geq 0$, where $F_0 = 0$ and $F_1 = 1$. Some terms of this sequence are

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987.$$

It is widely known that these numbers have many very interesting properties (consult ([1], pp. 53–56) and [2,3] and references therein). For instance, the ratio of two consecutive of these numbers converges to the Golden section $\alpha = (1 + \sqrt{5})/2$ (the applications of Golden ratio appear in many research areas, particularly in physics, engineering, architecture, nature and art (see e.g., [1,4]). Among, the many identities related to Fibonacci numbers, we cite

$$F_n^2 + F_{n+1}^2 = F_{2n+1}, \text{ for all } n \geq 0.$$

In particular, this naive identity asserts that the sum of the square of two consecutive Fibonacci numbers is always a Fibonacci number. In a recent paper, Marques and Togbé [5] searched for similar identities in higher powers. However, they proved that if $F_n^s + F_{n+1}^s$ is a Fibonacci number for all sufficiently large n, then $s = 1$ or $s = 2$.

The Fibonacci sequence was generalized in many different ways, some of them you can find in [6–18]. By keeping its order (which is 2), we have a general Lucas sequence $(C_n)_n = (C_n(a, b))_n$

which is defined by the recurrence $C_n = aC_{n-1} + bC_{n-2}$, for $n \geq 2$, and with $C_i = i$, for $i \in \{0,1\}$ (moreover, the integer parameters a and b must be such that if $x^2 - ax - b = (x-r)(x-s)$, then r/s is not a root of unity). In particular, $(C(1,1))_n$ is the Fibonacci sequence. Recently, Falcon and Plaza [11,19,20] studied the case $(C(k,1))_n$ which was called the k-Fibonacci sequence and denoted by $F^{(k)} = (F_{k,n})_n$. So, this sequence satisfies the recurrence relation

$$F_{k,n} = kF_{k,n-1} + F_{k,n-2}, \tag{1}$$

with initial conditions $F_{k,0} = 0$ and $F_{k,1} = 1$. In the same way, the companion k-Lucas sequence $L^{(k)} = (L_{k,n})_n$ is defined by satisfying the same recursive relation with initial values $L_{k,0} = 2$ and $L_{k,1} = k$. In particular, they proved that the following identities hold for all $n \geq 0$

$$F_{k,n}^2 + F_{k,n+1}^2 = F_{k,2n+1}\,, \tag{2}$$

and

$$F_{k,n+1}^2 - F_{k,n-1}^2 = kF_{k,2n}\,. \tag{3}$$

In this paper, we shall work on some Diophantine problems related to these sequences. Our first result concerns the search for higher power identities related to (2) and (3) in the spirit of the Marques and Togbé paper ([5]). More precisely, we prove that

Theorem 1. *Let $k \geq 2$ be any integer. If $s \geq 1$ satisfies that*

$$F_{k,n}^s + F_{k,n+1}^s \in (F_{k,m})_{m\geq 1}$$

for infinitely many positive integers n, then $s = 2$.

Theorem 2. *Let $k \geq 1$ be any integer. If $s \geq 1$ satisfies that*

$$F_{k,n+1}^s - F_{k,n-1}^s \in (kF_{k,m})_{m\geq 1}$$

for infinitely many positive integers n, then $s = 1$ or $s = 2$.

Several problems in number theory are actually questions about the intersection of two known sequences (or sets). Before giving examples, let us recall some terminology: let $F := (F_n)_{n\geq 0}$ be the Fibonacci sequence, $\mathbb{P} := \{p : p \text{ prime}\}$, $\mathcal{P} := \{y^t : y,t \in \mathbb{Z},\ t > 1\}$ (the perfect powers), $\mathcal{F} := \{n! : n \in \mathbb{Z},\ n \geq 0\}$, $\mathcal{R} := \{a(10^n - 1)/9 : 1 \leq a \leq 9,\ n \in \mathbb{Z},\ n > 0\}$ (the repdigits or unidigital numbers). Below, we cite some results about the intersection of these sets:

- Erdös and Selfridge [21] proved that $\mathcal{F} \cap \mathcal{P} = \{1\}$.
- In 2000, Luca [22] proved that $F \cap \mathcal{R} = \{0,1,1,2,3,5,8,55\}$.
- Luca [23] also proved that $F \cap \mathcal{F} = \{1,2\}$.
- In 2003, Bugeaud et al. [24] showed that $F \cap \mathcal{P} = \{0,1,8,144\}$ (see [25] for a generalization).

However, some related questions are still open problems, as for instance, it is unknown if the sets $\mathbb{P} \cap F$ and $\mathbb{P} \cap \mathcal{R}$ are infinite. The usual method is to solve some special Diophantine equations (see e.g., [26–31]).

In the next results, we shall find the intersection $F^{(k)} \cap L^{(k)}$ as well as we shall solve the symmetric equations $F_{k,n} = F_{n,k}$ and $L_{k,n} = L_{n,k}$. More precisely,

Theorem 3. *The only solutions of the Diophantine equation*

$$F_{k,n} = L_{k,m}$$

in positive integers k, m and n are $(k,m,n) = (k,1,2)$ and $(1,2,4)$.

Theorem 4. *The only solutions of the Diophantine equation*

$$F_{k,n} = F_{n,k}$$

in distinct positive integers k and n are $(n,k) = (1,2)$ and $(2,1)$.

Theorem 5. *There is no solution for the Diophantine equation*

$$L_{k,n} = L_{n,k}$$

in distinct positive integers k and n.

Now, we give a brief overview of the methods which will be used here. For the proof of Theorems 1 and 2, we shall apply the same approach as in [32] (i.e., asymptotic results together with Galois' theory). For Theorems 3–5, we shall use a plenty of inequalities together with some (combinatorial) identities for $F_{k,n}$ and $L_{k,n}$.

2. Auxiliary Results

In this section, we shall provide some useful tools which will be very useful in order to prove our theorems.

First, we observe that, similarly to the Fibonacci and Lucas sequences, their k versions also satisfy the Binet's formulas

$$F_{k,n} = \frac{\sigma_1^n - \sigma_2^n}{\sigma_1 - \sigma_2} \quad \text{and} \quad L_{k,n} = \sigma_1^n + \sigma_2^n, \tag{4}$$

where $\sigma_1 = (k + \sqrt{k^2+4})/2$ and $\sigma_2 = (k - \sqrt{k^2+4})/2$. Moreover, note that

$$\sigma_1^2 = k\sigma_1 + 1 \text{ and } \sigma_1\sigma_2 = -1.$$

By using Binet's formula of k-Fibonacci numbers, it is a simple matter to prove that

$$\lim_{n\to\infty} \frac{F_{k,n+\ell}}{F_{k,n}} = \sigma_1^\ell.$$

In addition, we shall need some lower and upper bounds for the terms of the k-Fibonacci and k-Lucas sequences.

Lemma 1 (Lemma 2.3 of [32])**.** *Let k, s be any positive integers. Then*

$$L_{k,s} \le (k^2+2)^{s/2}.$$

Lemma 2. *For all $k \ge 1$ and $n \ge 1$, it holds that*

$$\sigma_1^{n-2} \le F_{k,n} \le \sigma_1^{n-1} \quad \textit{and} \quad \sigma_1^{n-1} < L_{k,n} < \sigma_1^{n+1}.$$

The proof of this lemma can be found in ([32], Lemma 2.2).

Lemma 3. *Let l, m be any positive integers. Then*

$$(l+2)^{m+2} \le l(l+4)^{m+1} + 2 \tag{5}$$

Proof. For $m = 1$ we can rewrite inequality (5) as $4 \le (l+1)^2$, thus (5) holds for every positive integer l. Now, we show that the following inequality, slightly stronger than (5),

$$(l+2)^{m+2} \le l(l+4)^{m+1} \tag{6}$$

holds for every $m \geq 2$ and every positive integer l. Indeed, this inequality follows from the fact that $m \geq 2/l$ and so

$$\left(\frac{l+4}{l+2}\right)^{m+1} = \left(1+\frac{2}{l+2}\right)^{m+1} > 1+\frac{2(m+1)}{l+2} > 1+\frac{2}{l} = \frac{l+2}{l}.$$

Thus, $(l+2)^{m+2} \leq l(l+4)^{m+1}$ as desired. □

Now, we are ready to deal with the proof of theorems.

3. The Proof of the Theorems

3.1. Proof of Theorem 1

Suppose that $F_{k,n}^s + F_{k,n+1}^s = F_{k,t(n)}$ for infinitely many positive integers n (say n belonging to an infinity set $\mathcal{S}$), where $t(n)$ is an arithmetic function.

By using the estimates in (2), we have

$$\sigma_1^{t(n)-2} \leq F_{k,t(n)} = F_{k,n}^s + F_{k,n+1}^s \leq \sigma_1^{(n-1)s} + \sigma_1^{ns} = \sigma_1^{(n-1)s}(1+\sigma_1^s) < \sigma_1^{ns+1}$$

and

$$\sigma_1^{t(n)-1} \geq F_{k,t(n)} = F_{k,n}^s + F_{k,n+1}^s \geq \sigma_1^{(n-2)s} + \sigma_1^{(n-1)s} = \sigma_1^{(n-2)s}(1+\sigma_1^s) > \sigma_1^{ns-s}.$$

Thus, $ns - s + 1 < t(n) < ns + 3$ for all $n \in \mathcal{S}$. Therefore, $t(n) = ns + t$, for all $n \in \mathcal{S}'$ (for some infinite set $\mathcal{S}' \subseteq \mathcal{S}$), where t is a constant (depending only on s).

Then, consider the equation $F_{k,n}^s + F_{k,n+1}^s = F_{k,ns+t}$ and by dividing through by $F_{k,n}^s$, we obtain

$$1 + \left(\frac{F_{k,n+1}}{F_{k,n}}\right)^s = \frac{F_{k,ns+t}}{F_{k,n}^s}.$$

The left-hand side above tends to $1+\sigma_1^s$ as $n \to \infty$ (in $\mathcal{S}'$). For the right-hand side, we have by the binomial theorem that $F_{k,n}^s = (\sigma_1 - \sigma_2)^{-s}(\sigma_1^{sn} + O(\sigma_1^{(s-1)n}))$. Thus

$$\frac{F_{k,sn+t}}{F_{k,n}^s} = (\sigma_1 - \sigma_2)^{s-1}\left(\frac{\sigma_1^t - \sigma_2^{ns+t}/\sigma_1^{ns}}{1+O\left(\sigma_1^{n(s-1)}\right)/\sigma_1^{ns}}\right). \tag{7}$$

When $n \to \infty$, one has $\sigma_2^{ns+t}/\sigma_1^{ns} \to 0$ (since $\sigma_1 > 1$ and $|\sigma_2| < 1/2$ for $k \geq 2$) and $O(\sigma_1^{n(s-1)})/\sigma_1^{ns} \to 0$. Thus,

$$\lim_{n\to\infty, n\in\mathcal{S}'} \frac{F_{k,ns+t}}{F_{k,n}^s} = \left(\sqrt{k^2+4}\right)^{s-1}\sigma_1^t.$$

Thus, we obtain the Diophantine equation

$$1+\sigma_1^s = \left(\sqrt{k^2+4}\right)^{s-1}\sigma_1^t. \tag{8}$$

Now, we shall use a little taste of algebraic and Galois theory.

Now, note that $\sigma_1, \sigma_2 \in \mathbb{K} := \mathbb{Q}(\sqrt{k^2+4})$, then by conjugating relation (12) by the automorphism $\psi : \sigma_1 \mapsto \sigma_2$ of $\mathrm{Gal}(\mathbb{K}/\mathbb{Q})$, we obtain

$$1+\sigma_2^s = \left(-\sqrt{k^2+4}\right)^{s-1}\sigma_2^t. \tag{9}$$

Multiplying the identities (8) and (9) and by using that $\sigma_1\sigma_2 = -1$, we obtain

$$(-1)^{s-1+t}(k^2+4)^{s-1} = 1 + (-1)^s + \sigma_1^s + \sigma_2^s. \tag{10}$$

By taking absolute values, by using $\sigma_1 < k+1$, $|\sigma_2| < 1/2$, and the triangle inequality, one has

$$(k^2+4)^{s-1} < 5/2 + (k+1)^s. \tag{11}$$

Thus,

$$k^{2(s-1)} < (k^2+4)^{s-1} < 5/2 + (k+1)^s < (k+1)^{s+1} < k^{1.6(s+1)},$$

where we used that $k+1 < k^{1.6}$, for $k \geq 2$. Thus, $2(s-1) < 1.6(s+1)$ and then $1 \leq s \leq 9$. If $s = 1$, then from (10) we have $(-1)^t = \sigma_1 + \sigma_2 = k \geq 2$ which is an absurdity. For $s = 2$ we have the identity in (2). For $3 \leq s \leq 9$, we shall prove that $(k^2+4)^{s-1} > 5/2 + (k+1)^s$ contradicting (11). Indeed, we have that

$$\left(\frac{k^2+4}{k+1}\right)^{s-1} = \left(1 + \frac{k^2-k+3}{k+1}\right)^{s-1} > 1 + \frac{2(k^2-k+3)}{k+1} > \frac{5}{2(k+1)^2} + k + 1 \geq \frac{5}{2(k+1)^{s-1}} + k + 1.$$

The last inequality follows because $2(k^2-k+3) > 5/(2k+2) + k(k+1)$ (since $k^2 \geq 3k - 19/4$). The proof is then complete.

3.2. *Proof of Theorem 2*

The proof of Theorem 2 is similar to the proof of Theorem 1, so we will not formulate it completely, but we will only state the steps in which we use different identities.

Suppose that $F_{k,n+1}^s - F_{k,n-1}^s = kF_{k,t(n)}$ for infinitely many positive integers n (say n belonging to an infinite set $\mathcal{S}$), where $t(n)$ is an arithmetic function.

By using the estimates in Lemma 2, we have

$$\begin{aligned}
\sigma_1^{t(n)-3} &\leq \sigma_1^{t(n)-2}(-\sigma_2) \leq \sigma_1^{t(n)-2}(\sigma_1 - k) \\
&\leq \sigma_1^{t(n)-2}(\sigma_1 - 1) \leq \sigma_1^{t(n)-1} - \sigma_1^{t(n)-2} \\
&\leq F_{k,t(n)+1} - F_{k,t(n)-1} = kF_{k,t(n)} \\
&= F_{k,n+1}^s - F_{k,n-1}^s \leq F_{k,n+1}^s \leq \sigma^{sn}
\end{aligned}$$

and

$$\begin{aligned}
\sigma^{t(n)} &\geq F_{k,t(n)+1} \geq F_{k,t(n)+1} - F_{k,t(n)-1} = kF_{k,t(n)} \\
&= F_{k,n+1}^s - F_{k,n-1}^s \geq \sigma_1^{s(n-1)} - \sigma_1^{s(n-2)} \\
&\geq \sigma_1^{s(n-1)}(1 - \sigma_1^{-s})\sigma_1^{s(n-1)} \frac{\sigma_1^s - 1}{\sigma_1^s} \\
&\geq \sigma_1^{s(n-1)} \frac{1}{\sigma_1^s} \geq \sigma_1^{s(n-2)}.
\end{aligned}$$

Thus, $sn - 2s < t(n) < sn + 3$ for all $n \in \mathcal{S}$. Therefore, $t(n) = ns + t$, for all $n \in \mathcal{S}' \subseteq \mathcal{S}$, where $\mathcal{S}'$ is an infinite set and t is a constant, which depends only on s. Then, we consider the equation

$$F_{k,n+1}^s - F_{k,n-1}^s = kF_{k,ns+t}$$

and divide it by $F_{k,ns+t}$. Therefore,

$$\frac{F_{k,n+1}^s}{F_{k,ns+t}} - \frac{F_{k,n-1}^s}{F_{k,ns+t}} = k.$$

Now we consider $n \to \infty$ (in $\mathcal{S}'$) in the previous equality. By the similar procedure as in (7) we have

$$\begin{aligned} \lim_{n\to\infty, s\in S} \frac{F^s_{k,n+1}}{F_{k,ns+t}} &= (\sigma_1 - \sigma_2)^{1-s}\, \frac{1}{\sigma_1^{t-s}}, \\ \lim_{n\to\infty, s\in S} \frac{F^s_{k,n-1}}{F_{k,ns+t}} &= (\sigma_1 - \sigma_2)^{1-s}\, \frac{1}{\sigma_1^{t+s}}. \end{aligned}$$

Hence, we obtain the Diophantine equation

$$\sigma_1^{2s} - 1 = k\left(\sqrt{k^2+4}\right)^{s-1} \sigma_1^{t+s}. \tag{12}$$

Again, we shall use a little of Galois theory. Note that $\sigma_1, \sigma_2 \in \mathbb{K} := \mathbb{Q}(\sqrt{k^2+4})$, then by conjugating the relation (12) by the automorphism $\psi : \sigma_1 \mapsto \sigma_2$ of $\mathrm{Gal}(\mathbb{K}/\mathbb{Q})$, we obtain

$$\sigma_2^{2s} - 1 = k\left(-\sqrt{k^2+4}\right)^{s-1} \sigma_2^{t+s}. \tag{13}$$

By multiplying (12) by (13) and by using that $\sigma_1\sigma_2 = -1$, we obtain

$$1 + (-1)^{2s} - \sigma_1^{2s} - \sigma_2^{2s} = k^2\left(k^2+4\right)^{s-1} (-1)^{s-1}(-1)^{t+s}, \tag{14}$$

which can be rewritten, by using Binet's formula for k-Lucas numbers and using a clear condition that t has to be even, by the following way

$$L_{k,2s} = k^2\left(k^2+4\right)^{s-1} + 2\,. \tag{15}$$

For $s = 1$ and $s = 2$ we have $L_{k,2} = k^2\left(k^2+4\right)^0 + 2$ or $L_{k,4} = k^2\left(k^2+4\right)^1 + 2$, thus the last equality holds for any k. For $s \geq 3$, we deduce from Lemmas 1 and 3 that there is no solution. The proof is then complete.

3.3. *Proof of Theorem 3*

By using the estimates in Lemma 2, we get

$$\sigma_1^{n-2} \leq F_{k,n} = L_{k,m} < \sigma_1^{m+1}$$

and

$$\sigma_1^{n-1} \geq F_{k,n} = L_{k,m} > \sigma_1^{m-1}.$$

Then $m < n < m+3$ which yields $n \in \{m+1, m+2\}$. Now, we have two cases to consider:

Case 1. $n = m+1$. In this case, we have the equation $F_{k,m+1} = L_{k,m}$ and by using Binet's formulas, we obtain

$$\sigma_1^{m+1} - \sigma_2^{m+1} = (\sigma_1 - \sigma_2)(\sigma_1^m + \sigma_1^m)$$

and after a straightforward calculation, we get $\sigma_1^{m-1} = \sigma_2^{m-1}$. If $m = 1$, then we have the family of solutions $F_{k,2} = L_{k,1} = k$, for all $k \geq 1$. If $m > 1$, then $k < \sigma_1^{m-1} = \sigma_2^{m-1} < 1$ arriving at an absurdity.

Case 2. $n = m+2$. In this case, we have the equation $F_{k,m+2} = L_{k,m}$ and by using Binet's formulas, we obtain

$$\sigma_1^{m+2} - \sigma_2^{m+2} = (\sigma_1 - \sigma_2)(\sigma_1^m + \sigma_2^m)$$

and after a straightforward calculation, we get

$$\sigma_1^{m+1}(\sigma_1 - 1) = \sigma_2^{m+2} - \sigma_1^m\sigma_2 + \sigma_2^m\sigma_1 - \sigma_2^{m+1}.$$

Now, by multiplying this equality by σ_2^m and by using $\sigma_1\sigma_2 = -1$, we obtain

$$(-1)^m\sigma_1(\sigma_1 - 1) = \sigma_2^{2m+2} - (-1)^m\sigma_2 - \sigma_2^{2m-1} - \sigma_2^{2m+1}.$$

Now, we apply absolute values and the triangle inequality to obtain $\sigma_1(\sigma_1 - 1) < 2$ (here we used that $|\sigma_2| < 1/2$). Since $\sigma_1 = \left(k + \sqrt{k^2+4}\right)/2$, then the inequality $\sigma_1(\sigma_1 - 1) < 2$ implies $k = 1$. In this case, we have the usual Fibonacci and Lucas sequences. Therefore the equation $F_{m+2} = L_m = F_{m+1} + F_{m-1}$ yields $F_m = F_{m-1}$ and so $m = 2$.

3.4. Proof of Theorems 4 and 5

3.4.1. Proof of Theorem 4

Suppose, without loss of generality, that $n > k$ (because the symmetry of the equation). If $k = 1$, then $F_{1,n} = F_n = 1 = F_{n,1}$, only for $n = 2$, so we have the solution $(n,k) = (2,1)$. If $k = 2$, then $F_{2,n} > 2^{n-2} > n = F_{n,2}$, for all $n \geq 5$. For $n = 1,2,3$ and 4, we do not have any solution. For $k \geq 3$, we have the following stronger result

Proposition 1. *If $n > k \geq 3$, then $F_{k,n} > F_{n,k}$.*

Proof. For proving this, we shall use the following combinatorial formula (see ([19], Proposition 7)):

$$F_{k,n} = \sum_{i=0}^{\lfloor (n-1)/2 \rfloor} \binom{n-1-i}{i} k^{n-1-2i}$$

and then

$$F_{n,k} = \sum_{i=0}^{\lfloor (k-1)/2 \rfloor} \binom{k-1-i}{i} n^{k-1-2i}.$$

Since $n > k$, then the number of terms of the sum in $F_{k,n}$ is bigger than or equal to the one in $F_{n,k}$ (since $\lfloor (n-1)/2 \rfloor \geq \lfloor (k-1)/2 \rfloor$). Thus, it suffices to prove that

$$\binom{n-1-i}{i} k^{n-1-2i} > \binom{k-1-i}{i} n^{k-1-2i},$$

for all $0 \leq i \leq \lfloor (k-1)/2 \rfloor$. To prove this, it is enough to show that

$$(n-i-1)\cdots(n-2i)k^{n-2i-1} > (k-i-1)\cdots(k-2i)n^{k-2i-1}.$$

Since $n > k$, then $(n-i-1)\cdots(n-2i) > (k-i-1)\cdots(k-2i)$ and so, we only need to prove that $k^{n-2i-1} > n^{k-2i-1}$. In fact, we shall prove a stronger fact by showing that for any $\theta > 0$, the function $f(x) := (x-\theta)/\log x$ is increasing, for $x > e$. For this, we observe that the its derivative is

$$f'(x) = \frac{\log x - (x-\theta)/x}{(\log x)^2}.$$

Since $(x-\theta)/x = 1 - \theta/x < 1 < \log x$ for $x > e$, then $f'(x) > 0$ as desired. Now, observe that for $n > k \geq 3$, then $f(n) > f(k)$ (for $\theta = 2i+1$) and after a straightforward calculation we obtain $k^{n-2i-1} > n^{k-2i-1}$ which completes the proof. $\square$

3.4.2. Proof of Theorem 5

In fact, after a straightforward computation (as done previously), we can deduce that there is no solution for $L_{k,n} = L_{n,k}$, when $n > k$ and $1 \le k \le 5$ (we can use Lemma 2, for example). So, it suffices to prove that

Proposition 2. *If $n > k \ge 6$, then $L_{k,n} > L_{n,k}$.*

Proof. We have that

$$L_{k,n} = \frac{1}{2^{n-1}} \sum_{i=0}^{\lfloor n/2 \rfloor} \binom{n}{2i} k^{n-2i}(k^2+4)^i$$

which follows from (4) together with the Binomial theorem. Thus (as in the proof of Proposition 1) in order to prove that $L_{k,n} > L_{n,k}$, it is enough to show that

$$2^{k-1}k^{n-2i}(k^2+4)^i > 2^{n-1}n^{k-2i}(n^2+4)^i, \tag{16}$$

for $n > k \ge 5$ and $1 \le i \le \lfloor k/2 \rfloor$. This is equivalent (by applying the log function) to prove that the function

$$g(x) = (k-x)\log 2 + 2i(\log x - \log k) + i(\log(k^2+4) - \log(x^2+4)) + x\log k - k\log x$$

is positive, for all $x \ge k+1$ (since $n > k$) and $1 \le i \le \lfloor k/2 \rfloor$. Since $g(k) = 0$, we only need to show that g is an increasing function (i.e., that $g'(x) > 0$, for all $x \ge k+1$). Indeed, the derivative of $g(x)$ is

$$\begin{aligned} g'(x) &= \log k + \frac{2i}{x} - \frac{2xi}{x^2+4} - \frac{k}{x} - \log 2 \\ &= \log k - \frac{(x^2+4)k - 8i}{x^3+4x} - \log 2. \end{aligned}$$

Since $i \ge 1$ and $x - 1 \ge k$, then

$$\begin{aligned} g'(x) &= \log k - \frac{(x^2+4)k - 8i}{x^3+4x} - \log 2 \\ &\ge \log k - \frac{(x^2+4)(x-1) - 8i}{x^3+4x} - \log 2 \\ &\ge \log k - \frac{x^3 - x^2 + 4x - 12}{x^3+4x} - \log 2 \\ &> \log 6 - 1 - \log 2 > 0.09. \end{aligned}$$

In conclusion, g is increasing for $x \ge k+1$. In particular, $g(n) > g(k) = 0$ and so (16) holds. This finishes the proof. □

4. Conclusions

In this paper, we study some Diophantine problems related to two special generalizations of Fibonacci and Lucas numbers. Indeed, for a positive integer k, the k-Fibonacci and k-Lucas sequences $(F_{k,n})_n$ and $(L_{k,n})_n$ are defined by the same recurrence, namely $C_n = kC_{n-1} + C_{n-2}$, with initial terms $F_{k,i} = i$ (for $i \in \{0,1\}$) and $L_{k,0} = 2$ and $L_{1,k} = k$. The first kind of problem concerns the search for higher order identities similar to $F_{k,n}^2 + F_{k,n+1}^2 = F_{k,2n+1}$ and $F_{k,n+1}^2 - F_{k,n-1}^2 = kF_{k,2n}$. In this case, we use some analytic and algebraic tools to conclude that if $F_{k,n}^s + F_{k,n+1}^s$ (resp., $(F_{k,n+1}^2 - F_{k,n-1}^2)/k$) is a k-Fibonacci number for infinitely many positive integers n, then $s = 2$ (resp., $s = 1$ or $s = 2$). The second part of the work is devoted to problems related to the intersection between k-Fibonacci and k-Lucas sequences. More precisely, we make use of analytic, algebraic and combinatorial tools to solve completely the Diophantine equations $F_{k,n} = L_{k,m}$, $F_{k,n} = F_{n,k}$ and $L_{k,n} = L_{n,k}$.

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