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On the Nonlinear Stability and Instability of the Boussinesq System for Magnetohydrodynamics Convection

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Abstract: This paper is concerned with the nonlinear stability and instability of the two-dimensional (2D) Boussinesq-MHD equations around the equilibrium state ($\overline{u} = 0$, $\overline{B} = 0$, $\overline{\theta} = \theta_0(y)$) with the temperature-dependent fluid viscosity, thermal diffusivity and electrical conductivity in a channel. We prove that if $a_+ \geq a_-$, and $\frac{d^2}{dy^2}\kappa(\theta_0(y)) \leq 0$ or $0 < \frac{d^2}{dy^2}\kappa(\theta_0(y)) \leq \beta_0$, with $\beta_0 > 0$ small enough constant, and then this equilibrium state is nonlinearly asymptotically stable, and if $a_+ < a_-$, this equilibrium state is nonlinearly unstable. Here, a_+ and a_- are the values of the equilibrium temperature $\theta_0(y)$ on the upper and lower boundary.

Keywords: Boussinesq-MHD system; asymptotic stability; nonlinear instability

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1. Introduction

1.1. Background

In this paper, we consider the Boussinesq equations for Magnetohydrodynamics (MHD) convection in a channel Ω

$$\begin{cases} \partial_t\theta + u\cdot\nabla\theta - \operatorname{div}(\kappa(\theta)\nabla\theta) = 0, \\ \partial_t u + u\cdot\nabla u - \operatorname{div}(\mu(\theta)\nabla u) + \nabla\Pi = g\,\theta e_2 + B\cdot\nabla B, \\ \partial_t B + u\cdot\nabla B - \nabla^{\perp}(\gamma(\theta)\nabla^{\perp}\cdot B) = B\cdot\nabla u, \\ \operatorname{div} u = 0, \quad \operatorname{div} B = 0, \\ (\theta, u, B)|_{t=0} = (\theta_0,\, u_0,\, B_0). \end{cases} \tag{1}$$

Here, $e_2 = (0,1)$, $\nabla^{\perp} = (-\partial_2, \partial_1)$, the domain $\Omega = \mathbb{R}\times(-\ell,\ell)$ with the constant $0 < \ell < \infty$. The unknowns are the temperature θ, the velocity field $u = (u_1, u_2)$, the magnetic field $B = (B_1, B_2)$, and the pressure Π. This system can be used to model the large scale cosmic magnetic fields that are maintained by hydromagnetic dynamos. Physically, the first equation of (1) means that the temperature provides the convective drive of the system. The second equation describes the conservation law of the momentum with the effect of the buoyancy force $g\theta e_2$. The third equation shows that the electromagnetic field is governed by the Maxwell equation. Here, we denote $\sigma(\theta)$ by the electrical conductivity, $\mu(\theta)$ the fluid viscosity, and $\kappa(\theta)$ the thermal diffusivity. Additionally, assume that they are positive and smooth in θ

$$c \leq \mu(\theta), \gamma(\theta), \kappa(\theta) \leq C, \tag{2}$$

with c and C two positive constants.

We may refer to [1–3] and the references therein to learn more about physics details and numerical simulations. Hereafter, we will use the system as the Boussinesq-MHD system, MHD-Boussinesq, or BMHD for short.

When the fluid is not affected by the temperature, the system (1) reduces to the MHD system. Many physicists and mathematicians considered this model. For example, Duvaut and Lions [4] established the local well-posedness in $H^s(\mathbb{R}^d)$ with $s \geq d$, and got global existence of small solutions. Sermange and Temam [5] studied these solutions' properties and proved that the two-dimensional (2D) local strong solution is global and unique. Recently, many authors studied the global regularity of the MHD system with partial dissipation (see, e.g., [6–9]).

When the fluid is not affected by the magnetic field, the system (1) becomes the classical Boussinesq system. Many authors studied the Cauchy problem of the 2D Boussinesq system in the presence of full viscosity. Cannon and Dibenedetto [10] and Wang and Zhang [11] proved the global well-posedness for the full viscosity and smooth initial data case. Lately, many works are devoted to studying the Boussinesq system with partial constant viscosity. For example, the global well-posedness of the system in the absence of diffusion has been independently proven by Chae [12] and Hou and Li [13] , and Chae [12] also studied the case $\mu = 0$. The global well-posedness of the system in the critical spaces in the absence of diffusion has been established by Abidi and Hmidi [14], and the global well-posedness of the system in the critical spaces in the case $\mu = 0$ has been proved by Hmidi and Keraani [15]. For the bounded domain case, the global well-posedness for the system in the absence of diffusion has been proved by Lai, Pan and Zhao [16]. Lately, the well-posedness of global strong solutions for the system in the presence of temperature-dependent diffusion with large initial data in Sobolev spaces has been shown by Li and Xu [17].

For the three-dimensional (3D) Boussinesq and MHD system, the existence of global weak solution for L^2 initial data and the well-posedness for global small smooth data for the Cauchy problem of the Boussinesq system has been proved by Danchin and Paicu [18]. The global well-posedness for the Cauchy problem of the 3D axisymmetric Boussinesq system in the absence of swirl has been shown by Hmidi and Rousset [19,20]. The partial regularity of the weak solutions for the 3D Boussinesq system has been studied by Fang, Liu, and Qian[21]. For the 3D MHD equations, the well-posedness of global axially symmetric solutions in the presence of full fluid viscosity and magnetic diffusion has been studied by Lei [22]. The partial regularity of the weak solutions for the 3D MHD system has been established by Cao and Wu [23], He and Xin [24,25], and Kang and Lee [26]. The well-posedness of global small solution has been proven by Cai and Lei [27] and He, Xu and Yu [28]. For the 3D MHD system in the presence of a nonlinear damping term, the existence of global weak solutions and well-posedness of global smooth solutions for large initial data has been recently proven by Titi and Trabelsi [29].

For the full BMHD system, Bian *et al* [30–32] and Yu and Pei [33] studied the global existence and uniqueness of the solution to the 2D BMHD system without smallness assumptions on the initial data. For the 3D case, Larios-Pei [34] proved the local well-posedness in the Sobolev space H^3. Zhai and Chen [35] considered the Cauchy problem for BMHD system in Besov space. Bian *et al* [36,37] proved the global well-posedness result for the axisymmetric BMHD system without magnetic diffusion and heat convection, and global well-posedness result for the BMHD system with a nonlinear damping term, the constant fluid viscosity, and the constant electrical conductivity. Li [38] obtained the global weak solution for the inviscid BMHD system with the constant thermal diffusivity and constant electrical conductivity. However, it is not known whether nonlinear stability and instability for the full system (1) holds with the fluid viscosity, electrical conductivity, as well as thermal diffusivity dependent on temperature around the equilibrium state $(\overline{\theta}, \overline{u}, \overline{B}) = (\theta_0(y), 0, 0)$. In this paper, we will give the precise answers to the above question for the full system (1).

1.2. Steady State and Main Results

For notational simplicity, we denote the gravity unit $g = 1$. In this paper, we assume the boundary conditions

$$u|_{\partial\Omega} = 0, \quad \theta|_{y=\pm\ell} = a_{\pm}, \quad B \cdot n|_{\partial\Omega} = 0, \ (\nabla \times B) \times n|_{\partial\Omega} = 0. \tag{3}$$

Let $\theta_0(y)$ be a smooth function on $[-\ell, \ell]$. Then the functions $(\overline{\theta}, \overline{u}, \overline{B}) = (\theta_0(y), 0, 0)$ define a equilibrium state to (1), provided

$$\nabla\Pi_0 = \theta_0(y)e_2, \quad \frac{d}{dy}\left[\kappa(\theta_0(y))\frac{d}{dy}\theta_0(y)\right] = 0,$$

which gives that $\Pi_0 = \Pi_0(y)$ and

$$\frac{d}{dy}\Pi_0 = \theta_0(y), \ \frac{d}{dy}\theta_0(y) = \frac{m_0}{\kappa(\theta_0)},$$

for some constant m_0.

Integrating the above equation about the temperature, it follows from the condition (2) and the boundary conditions (3) that

$$a_+ > a_- \Leftrightarrow m_0 > 0,$$

and in this case, we set $m_0 = 1$ for notational simplicity. Similarly, one has

$$a_+ < a_- \Leftrightarrow m_0 < 0,$$

and, for this case, we set $m_0 = -1$ for notational simplicity.

Now, define the perturbation to be

$$\sigma = \theta - \theta_0, \ u = u - \overline{u}, \ b = B - \overline{B} = B, \ p = \Pi - \Pi_0,$$

which satisfies the system:

$$\begin{cases} \partial_t\sigma + u \cdot \nabla\sigma - \mathrm{div}(\kappa(\theta_0 + \sigma)\nabla\sigma) - \partial_y(\kappa(\theta_0 + \sigma)\frac{d}{dy}\theta_0)) + u_2\frac{d}{dy}\theta_0(y) = 0, \\ \partial_t u + u \cdot \nabla u - \mathrm{div}(\mu(\theta_0 + \sigma)\nabla u) + \nabla p = \sigma e_2 + b \cdot \nabla b, \\ \partial_t b + u \cdot \nabla b - \nabla^\perp(\gamma(\theta_0 + \sigma)\nabla^\perp \cdot b) = b \cdot \nabla u, \\ \mathrm{div}\, u = 0, \ \mathrm{div}\, b = 0. \end{cases} \tag{4}$$

From the physical point of view, the sign of $\frac{d}{dy}\theta_0(y)$ that appears in the equation for the temperature σ is critical (cf. [39]). For the case $\frac{d}{dy}\theta_0(y) < 0$, the situation is unstable. While, for the case $\frac{d}{dy}\theta_0(y) > 0$ in the fluid with homogeneous thermal diffusivity, the density decreases with height and the heavier fluid is below lighter fluid. This is the situation of stable stratification, and the quantity $\mathcal{N}(y) \stackrel{\text{def}}{=} \sqrt{\frac{d}{dy}\theta_0(y)}$ is called the buoyancy or Brunt-Väisärä frequency (stratification-parameter) [39,40].

The situation for the case $\frac{d}{dy}\theta_0(y) < 0$ is closely related to the Rayleigh–Taylor instability according to the well-known Boussinesq approximation, where the temperature difference is directly proportional to the density difference between the bottom and top of the layer of fluid. The Rayleigh–Taylor instability appears when a heavy fluid is on top of a light one. The linear instability for the incompressible fluid was first established by Rayleigh in 1883 [41] and Chandrasekhar in 1981 [42]. Grenier [43] gave some examples of nonlinearly unstable solutions of Euler equations and proved an instability result for Prandtl equations. Recently, Hwang and Guo [44] obtained the nonlinear Rayleigh–Taylor instability for the inviscid incompressible fluid. Guo and Tice [45,46] proved the linear Rayleigh–Taylor instability for inviscid and viscous compressible fluids by introducing a new variational method. Later on, using the new variational method, many authors considered the effects of magnetic field in the fluid equations, see Jiang-Jiang [47–49].

This paper is concerned with the nonlinear stability and instability for the full BMHD system with the temperature-dependent fluid viscosity, thermal diffusivity, and electrical conductivity. Our results are as follows.

Theorem 1. *Assume that the function $\theta_0(y) \in C^4([-\ell,\ell])$ satisfies the boundary condition $\theta_0(y=\pm\ell) = a_\pm$, and the three functions κ, μ and γ satisfy (2). Subsequently, we have*

(i) If $a_+ < a_-$, then the equilibrium state $(\theta_0(y), 0, 0)$ in (1) is nonlinearly unstable, which is, there exists $\varepsilon_0 > 0$, such that, for any small $\delta > 0$, there exists a family of classical solutions $(\theta^\delta, u^\delta, B^\delta)$ to (1), such that

$$\|\theta_0^\delta - \theta_0\|_{H^2(\Omega)} + \|(u_0^\delta, B_0^\delta)\|_{H^2(\Omega)} \leq \delta,$$

, but for $T^\delta = O(|\ln\delta|)$,

$$\sup_{0\leq t\leq T^\delta} \{\|\theta^\delta - \theta_0\|_{L^2(\Omega))} + \|(u^\delta, B^\delta)\|_{L^2(\Omega)}\} \geq \varepsilon_0.$$

(ii) If $a_+ > a_-$, and $\frac{d^2}{dy^2}\kappa(\theta_0(y)) \leq 0$ or $0 < \frac{d^2}{dy^2}\kappa(\theta_0(y)) \leq \beta_0$, with $\beta_0 > 0$ small enough constant, then the equilibrium state $(\theta_0(y), 0, 0)$ in (1) is nonlinearly asymptotically stable, that is, there exists $\delta_0 > 0$, such that, for any $\delta \in (0, \delta_0)$, if

$$\|(\theta_0^\delta - \theta_0,\ u_0^\delta, B_0^\delta)\|_{H^2(\Omega)} \leq \delta,$$

then $(\theta_0^\delta, u_0^\delta, B_0^\delta)$ generates a global unique solution $(\theta^\delta, u^\delta, B^\delta)$ to the system (1). Moreover, it holds

$$\lim_{t\to+\infty} \|(\theta^\delta - \theta_0,\ u^\delta, B^\delta)\|_{H^2(\Omega)} = 0. \tag{5}$$

Remark 1. *For the case $a_+ = a_-$, the equilibirum state $(\theta_0(y), 0, 0)$ is stable. In fact, if $a_+ = a_-$, then $m_0 = 0$, that is, $\theta_0 = C$, we set $\theta_0 = 1$, which do not change the result in our analysis. Thus, our perturbation problem can be reformulated in the following:*

$$\begin{cases} \partial_t\sigma + u\cdot\nabla\sigma - \nabla\cdot(\kappa(\sigma+1)\nabla\sigma) = 0, \\ \partial_t u + u\cdot\nabla u - \nabla\cdot(\mu(\sigma+1)\nabla u) + \nabla p = \sigma e_2 + b\cdot\nabla b, \\ \partial_t b + u\cdot\nabla b - \nabla^\perp(\gamma(\sigma+1)\nabla^\perp\cdot b) = b\cdot\nabla u, \\ \nabla\cdot u = 0,\ \nabla\cdot b = 0, \end{cases} \tag{6}$$

where $\sigma = \theta - 1$, $u = u$, $b = B$, $p = \Pi - \Pi_0$ is the perturbation.

From (6), repeating the process of stability in Section 6, we only need to control $\int_\Omega \sigma e_2 u dxdy$. Note that

$$\int_\Omega \sigma^2 dxdy + \int_0^t \int_\Omega \kappa(\sigma+1)|\nabla\sigma|^2 dxdyds = \int_\Omega \sigma_0^2 dxdy.$$

By the assumption in (ii) of Theorem 1 and Poincare inequality for a strip, we know that

$$\|\sigma\|_2 \leq \delta,\ \int_\Omega |u_2|^2 dxdy \leq C\ell \int_\Omega |\partial_y u_2|^2 dydx.$$

Hence, choose δ small enough, $\int_\Omega \sigma e_2 u dxdy$ can be controlled by $\int_\Omega \mu(\sigma+1)|\nabla u|^2 dxdy$.

Remark 2. *Our result also holds for the incompressible 3-D BMHD system.*

Notations: The space H^1_{div} is defined as $H^1_{div} = \{u \in H^1 : \text{div}\, u = 0\}$ and this rule of definition is applied to the sapce H^m_{div} with $m \geq 0$. We define $L^\infty_t(H^1)$ by $\|u\|_{L^\infty_t(H^1)} = \sup_{0\leq s\leq t} \|u(s)\|_{H^1(\Omega)}$.

The remainder of the paper is organized, as follows. In Section 2, we construct the growing solutions to the linearized Boussinesq-MHD system for the case $a_+ < a_-$. With these precise growth rate λ, we construct an approximation solution with higher order growing modes in Section 3. In Section 4, after obtaining the crucial estimates of the linearized system, we present nonlinear energy estimates of the original perturbed equations for the case $a_+ < a_-$. In Section 5, we will prove Theorem 1, which concludes the nonlinear instability and stability. Finally, in Section 6, we give the conlusions of this paper.

2. Variational Method for the Case $a_+ < a_-$

In this section, we prove that, if $a_+ < a_-$, then there exists a smooth linear growing mode of the forms (8) with the eigenvalue $\Lambda > 0$.

We first linearize (4) around $\sigma = 0$, $u = b = \mathbf{0}$, $p = 0$ as

$$\begin{cases} \partial_t \sigma + u_2 \frac{d}{dy}\theta_0(y) - \mathrm{div}(\kappa(\theta_0)\nabla\sigma) = \partial_y(\sigma \frac{d}{dy}\kappa(\theta_0)), \\ \partial_t u - \mathrm{div}(\mu(\theta_0)\nabla u) + \nabla p = \sigma e_2, \\ \partial_t b - \nabla^\perp(\gamma(\theta_0)\nabla^\perp \cdot b) = 0, \\ \mathrm{div}\, u = 0,\ \mathrm{div}\, b = 0. \end{cases} \tag{7}$$

We want to find a dominate eigenvalue of the linearized equations (7), with its corresponding growing normal mode, which takes the form:

$$u = \tilde{u}(x,y)e^{\lambda t},\ \sigma = \tilde{\sigma}(x,y)e^{\lambda t},\ b = \tilde{b}(x,y)e^{\lambda t},\ p = \tilde{p}(x,y)e^{\lambda t}, \tag{8}$$

where $(\tilde{\sigma}(x,y), \tilde{u}(x,y), \tilde{b}(x,y), \nabla\tilde{p}(x,y)) \in H^1 \times H^1_{div} \times H^1_{div} \times L^2(\Omega)$, and satisfies the boundary condition (3) in the sense of the trace.

When $a_+ < a_-$, that is, $\frac{d}{dy}\theta_0(y) = \frac{-1}{\kappa(\theta_0)}$, plugging (8) into (7) we can easily get the following equivalent system.

Lemma 1. *When $a_+ < a_-$. Assume (8), then (7) takes the form of*

$$\begin{cases} \lambda\tilde{u} + \nabla\tilde{p} - \nabla\cdot(\mu(\theta_0)\nabla\tilde{u}) = \tilde{\sigma}e_2, \\ \lambda\kappa(\theta_0)\tilde{\sigma} - \tilde{u}_2 - \kappa(\theta_0)\nabla\cdot(\kappa(\theta_0)\nabla\tilde{\sigma}) = \kappa(\theta_0)\partial_y(\tilde{\sigma}\frac{d}{dy}\kappa(\theta_0)), \\ \lambda\tilde{b} - \nabla^\perp(\gamma(\theta_0)\nabla^\perp\cdot\tilde{b}) = 0, \\ \nabla\cdot\tilde{u} = 0,\ \nabla\cdot\tilde{b} = 0. \end{cases}$$

We define

$$\begin{aligned} I_1(\tilde{\sigma},\tilde{u},\tilde{b}) \overset{\text{def}}{=} & \int_\Omega \left[2\tilde{\sigma}\tilde{u}_2 + \tilde{\sigma}^2\kappa(\theta_0)\frac{d^2}{dy^2}(\kappa(\theta_0))\right]dxdy \\ & - \int_\Omega \left[\mu(\theta_0)|\nabla\tilde{u}|^2 + \kappa^2(\theta_0)|\nabla\tilde{\sigma}|^2 + \gamma(\theta_0)|\nabla^\perp\cdot\tilde{b}|^2\right]dxdy, \end{aligned}$$

$$J_1(\tilde{\sigma},\tilde{u},\tilde{b}) \overset{\text{def}}{=} \int_\Omega (|\tilde{u}|^2 + |\tilde{b}|^2 + \kappa(\theta_0)|\tilde{\sigma}|^2)dxdy.$$

It is easy to check that $I_1(\tilde{\sigma},\tilde{u},\tilde{b})$ and $J_1(\tilde{\sigma},\tilde{u},\tilde{b})$ are well define on the space $H^1 \times H^1_{div} \times H^1_{div}$.

Define the admissible set

$$\mathcal{A} \overset{\text{def}}{=} \{(\tilde{\sigma},\tilde{u},\tilde{b}) \in H^1 \times H^1_{div} \times H^1_{div} : J_1(\tilde{\sigma},\tilde{u},\tilde{b}) = 1 \text{ with } \tilde{u}|_{\partial\Omega} = 0,\ \tilde{\sigma}|_{\partial\Omega} = 0,\ \tilde{b}\cdot n|_{\partial\Omega} = 0\}.$$

We know that $I_1(\tilde{\sigma},\tilde{u},\tilde{b})$ has a upper bound on the set $\mathcal{A}$. We are now in a position to prove that there exists a growing mode of (8) with the eigenvalue $\Lambda > 0$.

Lemma 2. *Assume that the equilibrium temperature profile* $\overline{\theta} = \theta_0(y)$ *satisfies* $\frac{d}{dy}\theta_0(y) = \frac{-1}{\kappa(\theta_0)}$, $\Lambda \overset{def}{=} \sup_{(\tilde{\sigma},\tilde{u},\tilde{b})\in\mathcal{A}} I_1(\tilde{\sigma},\tilde{u},\tilde{b})$*, then it holds that*

(a) $I_1(\tilde{\sigma},\tilde{u},\tilde{b})$ *achieves its supremum on the admissible set* $\mathcal{A}$*,*

(b) Let $(\tilde{\sigma}_0,\tilde{u}_0,\tilde{b}_0) \in \mathcal{A}$ *be a maximizer, then there exists a* $\tilde{p}_0$*, such that* $(\tilde{\sigma}_0,\tilde{u}_0,\tilde{b}_0,\tilde{p}_0,\Lambda)$ *solves the Sturm-Liouville problem*

$$\begin{cases} \Lambda\tilde{u}_0 + \nabla\tilde{p}_0 - \nabla\cdot(\mu(\theta_0)\nabla\tilde{u}_0) = \tilde{\sigma}_0 e_2, \\ \Lambda\kappa(\theta_0)\tilde{\sigma}_0 - \tilde{u}_{02} - \kappa(\theta_0)\nabla\cdot(\kappa(\theta_0)\nabla\tilde{\sigma}_0) = \kappa(\theta_0)\partial_y(\tilde{\sigma}_0\frac{d}{dy}\kappa(\theta_0)), \\ \Lambda\tilde{b}_0 - \nabla^{\perp}(\gamma(\theta_0)\nabla^{\perp}\cdot\tilde{b}_0) = 0, \\ \nabla\cdot\tilde{u}_0 = 0,\ \nabla\cdot\tilde{b}_0 = 0, \end{cases} \tag{9}$$

with the boundary condition

$$(\nabla\times\tilde{b}_0)\times n|_{\partial\Omega} = 0. \tag{10}$$

Moreover, we have $(\tilde{\sigma}_0,\ \tilde{u}_0,\ \tilde{b}_0,\ \tilde{p}_0) \in H^1\times H^2_{div}\times H^2_{div}\times H^1$ *and* $\Lambda > 0$*.*

Proof. (a) We first choose a maximizing sequence $(\tilde{\sigma}_n,\tilde{u}_n,\tilde{b}_n) \in \mathcal{A}$ of the variational problem

$$\max_{(\tilde{\sigma},\tilde{u},\tilde{b})\in\mathcal{A}} I_1(\tilde{\sigma},\tilde{u},\tilde{b})$$

such that

$$J_1(\tilde{\sigma}_n,\tilde{u}_n,\tilde{b}_n) = 1 \quad \text{and} \quad I_1(\tilde{\sigma}_n,\tilde{u}_n,\tilde{b}_n) \to \Lambda \quad \text{as} \quad n\to+\infty,$$

which implies that $\{I_1(\tilde{\sigma}_n,\tilde{u}_n,\tilde{b}_n)\}_{n\in\mathbb{Z}}$ is bounded. It follows from this and Hölder's inequality that

$$\begin{aligned} &\int_\Omega \left[\mu(\theta_0)|\nabla\tilde{u}_n|^2 + \kappa^2(\theta_0)|\nabla\tilde{\sigma}_n|^2 + \gamma(\theta_0)|\nabla^{\perp}\cdot\tilde{b}_n|^2\right]dxdy \\ &= \int_\Omega \left[2\tilde{\sigma}_n\tilde{u}_{n2} + \tilde{\sigma}_n^2\kappa(\theta_0)\frac{d^2}{dy^2}(\kappa(\theta_0))\right]dxdy - I_1(\tilde{\sigma}_n,\tilde{u}_n,\tilde{b}_n) \\ &\le CJ_1(\tilde{\sigma}_n,\tilde{u}_n,\tilde{b}_n) - I_1(\tilde{\sigma}_n,\tilde{u}_n,\tilde{b}_n) \le C. \end{aligned}$$

Hence, $\{(\tilde{\sigma}_n,\tilde{u}_n,\tilde{b}_n)\}_{n\in\mathbb{N}}$ is uniformly bounded in $H^1\times H^1_{div}\times H^1_{div}$, which implies that there exists function $(\tilde{\sigma}_0,\tilde{u}_0,\tilde{b}_0) \in \mathcal{A}$ and subsequence of $\{(\tilde{\sigma}_n,\tilde{u}_n,\tilde{b}_n)\}_{n\in\mathbb{N}}$ (we still write $\{(\tilde{\sigma}_n,\tilde{u}_n,\tilde{b}_n)\}_{n\in\mathbb{N}}$ for notational simplicity), such that $(\tilde{\sigma}_n,\tilde{u}_n,\tilde{b}_n) \to (\tilde{\sigma}_0,\tilde{u}_0,\tilde{b}_0)$ weakly in $H^1\times H^1_{div}\times H^1_{div}$ and strongly in $L^2_{loc}\times L^2_{loc,div}\times L^2_{loc,div}$.

Therefore, one gets that, for $n\to+\infty$

$$\begin{aligned} 1 = J_1(\tilde{\sigma}_n,\tilde{u}_n,\tilde{b}_n) &= \int_\Omega(|\tilde{u}_n|^2 + |\tilde{b}_n|^2 + \kappa(\theta_0)|\tilde{\sigma}_n|^2)dxdy \\ &\to \int_\Omega(|\tilde{u}_0|^2 + |\tilde{b}_0|^2 + \kappa(\theta_0)|\tilde{\sigma}_0|^2)dxdy = J_1(\tilde{\sigma}_0,\tilde{u}_0,\tilde{b}_0), \end{aligned}$$

which implies that

$$J_1(\tilde{\sigma}_0,\tilde{u}_0,\tilde{b}_0) = 1. \tag{11}$$

On the other hand, thanks to the lower semi-continuity, one can get

$$
\begin{aligned}
&\sup_{(\tilde{\sigma},\tilde{u},\tilde{b})\in\mathcal{A}} I_1(\tilde{\sigma},\tilde{u},\tilde{b}) = \lim_{n\to+\infty}\sup I_1(\tilde{\sigma}_n,\tilde{u}_n,\tilde{b}_n)\\
&= \lim_{n\to+\infty}\int_\Omega \left[2\tilde{\sigma}_n\,\tilde{u}_{n2} + \tilde{\sigma}_n^2\kappa(\theta_0)\frac{d^2}{dy^2}(\kappa(\theta_0))\right]dxdy\\
&- \lim\inf_{n\to+\infty}\int_\Omega \left[\mu(\theta_0)|\nabla\tilde{u}_n|^2 + \kappa^2(\theta_0)|\nabla\tilde{\sigma}_n|^2 + \gamma(\theta_0)|\nabla^\perp\cdot\tilde{b}_n|^2\right]dxdy\\
&\le I_1(\tilde{\sigma}_0,\tilde{u}_0,\tilde{b}_0) \le \sup_{(\tilde{\sigma},\tilde{u},\tilde{b})\in\mathcal{A}} I_1(\tilde{\sigma},\tilde{u},\tilde{b}),
\end{aligned}
$$

which, along with (11), implies that $I_1(\tilde{\sigma},\tilde{u},\tilde{b})$ achieves its supremum on the admissible set $\mathcal{A}$, and

$$
\Lambda = \sup_{(\tilde{\sigma},\tilde{u},\tilde{b})\in\mathcal{A}} I_1(\tilde{\sigma},\tilde{u},\tilde{b}) = I_1(\tilde{\sigma}_0,\tilde{u}_0,\tilde{b}_0). \tag{12}
$$

(b) It remains to verify that the maximizer $(\tilde{\sigma}_0,\tilde{u}_0,\tilde{b}_0)$ obtained above solves the problem (9) and the boundary condition (10). In fact, for $\tau\in\mathbb{R}$, $\beta\in\mathbb{R}$ and any $(\sigma,u,b)\in H^1\times H^1_{div}\times H^1_{div}$ with boundary conditions $\sigma|_{\partial\Omega}=0$, $u|_{\partial\Omega}=0$ and $n\cdot b|_{\partial\Omega}=0$, we define that

$$
j(\tau,\beta) \triangleq J_1(\tilde{\sigma}_0+\tau\sigma+\beta\tilde{\sigma}_0,\tilde{u}_0+\tau u+\beta\tilde{u}_0,\tilde{b}_0+\tau b+\beta\tilde{b}_0)-1.
$$

Because $\partial_\beta j(\tau,\beta)|_{(0,0)} = 2\neq 0$, by the implicit function existence theorem, we get that there exists a unique function $\beta=\beta(\tau)$, defined on $\{\tau\in\mathbb{R}:|\tau|\le h\}$ with some positive constant h, such that $j(\tau,\beta(\tau))=0$, $0=\beta(0)$. Therefore,

$$
J_1(\tilde{\sigma}_0+\tau\sigma+\beta(\tau)\tilde{\sigma}_0,\tilde{u}_0+\tau u+\beta(\tau)\tilde{u}_0,\tilde{b}_0+\tau b+\beta(\tau)\tilde{b}_0)-1=0,\quad \forall\ |\tau|\le h, \tag{13}
$$

and

$$
\beta'(0) = -\frac{\partial_\tau j(\tau,\beta)|_{(0,0)}}{\partial_\beta j(\tau,\beta)|_{(0,0)}} = -\int_\Omega(\tilde{u}_0\cdot u+\tilde{b}_0\cdot b+\kappa(\theta_0)\tilde{\sigma}_0\sigma)dxdy.
$$

Define that $i(\tau)\triangleq I_1(\tilde{\sigma}_0+\tau\sigma+\beta(\tau)\tilde{\sigma}_0,\tilde{u}_0+\tau u+\beta(\tau)\tilde{u}_0,\tilde{b}_0+\tau b+\beta(\tau)\tilde{b}_0)$. Then $i(0)=I_1(\tilde{\sigma}_0,\tilde{u}_0,\tilde{b}_0)$, and $i(\tau)\le i(0)$ for any $-h\le\tau\le h$, which implies that

$$
\begin{aligned}
0 = i'(0) =& 2\beta'(0)\int_\Omega\left[2\tilde{\sigma}_0\tilde{u}_{02}+\tilde{\sigma}_0^2\kappa(\theta_0)\frac{d^2}{dy^2}(\kappa(\theta_0))-\mu(\theta_0)|\nabla\tilde{u}_0|^2-\kappa^2(\theta_0)|\nabla\tilde{\sigma}_0|^2\right.\\
&\left.-\gamma(\theta_0)|\nabla^\perp\cdot\tilde{b}_0|^2\right]dxdy\\
&+2\int_\Omega\left[\tilde{u}_{02}\sigma+\tilde{\sigma}_0u_2+\tilde{\sigma}_0\sigma\kappa(\theta_0)\frac{d^2}{dy^2}(\kappa(\theta_0))-\mu(\theta_0)\nabla\tilde{u}_0\cdot\nabla u\right.\\
&\left.-\kappa^2(\theta_0)\nabla\tilde{\sigma}_0\cdot\nabla\sigma-\nabla^\perp\cdot b\gamma(\theta_0)\nabla^\perp\cdot\tilde{b}_0\right]dxdy.
\end{aligned}
$$

Hence, it follows from (12) and (13) that

$$
\begin{aligned}
0 =& \int_\Omega\left[\tilde{u}_{02}\sigma+\tilde{\sigma}_0u_2+\tilde{\sigma}_0\sigma\kappa(\theta_0)\frac{d^2}{dy^2}(\kappa(\theta_0))-\mu(\theta_0)\nabla\tilde{u}_0\cdot\nabla u-\kappa^2(\theta_0)\nabla\tilde{\sigma}_0\cdot\nabla\sigma\right.\\
&\left.-\nabla^\perp\cdot b\gamma(\theta_0)\nabla^\perp\cdot\tilde{b}_0\right]dxdy-\Lambda\int_\Omega(\tilde{u}_0\cdot u+\tilde{b}_0\cdot b+\kappa(\theta_0)\tilde{\sigma}_0\sigma)dxdy,
\end{aligned}
$$

which implies that

$$0 = \int_{\Omega}\Big[-\Lambda\tilde{u}_0 + \tilde{\sigma}_0 e_2 + \nabla\cdot(\mu(\theta_0)\nabla\tilde{u}_0)\Big]\cdot u dxdy + \int_{\Omega}\Big[-\Lambda\tilde{b}_0 + \nabla^{\perp}(\gamma(\theta_0)\nabla^{\perp}\cdot\tilde{b}_0)\Big]\cdot b dxdy$$
$$+\int_{\partial\Omega}\gamma(\theta_0)(\nabla\times\tilde{b}_0)\times n\cdot b dxdy + \int_{\Omega}\Big[-\Lambda\kappa(\theta_0)\tilde{\sigma}_0 + \tilde{u}_{02} + \tilde{\sigma}_0\kappa(\theta_0)\frac{d^2}{dy^2}(\kappa(\theta_0))$$
$$+\nabla\cdot(\kappa^2(\theta_0)\nabla\tilde{\sigma}_0)\Big]\sigma dxdy = 0$$

for all $(\sigma, u, b)\in H^1\times H^1_{div}\times H^1_{div}$. Therefore, there exists a $\tilde{p}_0$ such that $(\tilde{\sigma}_0, \tilde{u}_0, \tilde{b}_0, \tilde{p}_0, \Lambda)$ satisfies the Sturm–Liouville problem (9) and the boundary condition (10) holds in the weak sense. By a standard regularity argument, one can show that $(\tilde{\sigma}_0, \tilde{u}_0, \tilde{b}_0, \nabla\tilde{p}_0)\in H^1\times H^2_{div}\times H^2_{div}\times L^2$ and $\Lambda > 0$, which ends the proof. □

Remark 3. *From Lemma 2, we know that the Sturm–Liouville problem (9) at least has a solution* $(\tilde{\sigma}_0, \tilde{u}_0, \tilde{b}_0, \tilde{p}_0, \Lambda)$.

3. The Exponential Growth Rate Λ

The goal of this section is to prove that the eigenvalue Λ in Section 2 is the sharp exponential growth rate for the linearized BMHD equations (7). The results are as follows.

Lemma 3. *Let* $m\in\mathbb{N}$*, and assume that* $(\sigma_0, u_0, b_0)\in H^m(\Omega)\times H^m_{div}(\Omega)\times H^m_{div}(\Omega)$ *and assume the boundary conditions for* $|\alpha|\le m-1$

$$\partial^{\alpha}\sigma|_{\partial\Omega}=0,\quad \partial^{\alpha}u|_{\partial\Omega}=0,\quad \partial^{\alpha}b\cdot n|_{\partial\Omega}=0,\ (\nabla\times\partial^{\alpha}b)\times n|_{\partial\Omega}=0.$$

Subsequently, there exists a global unique solution $(\sigma,\ u,\ b)$*, satisfying*

$$(\sigma,\ u,\ b)\in C(\mathbb{R}^+; H^m(\Omega))\times C(\mathbb{R}^+; H^m_{div}(\Omega))\times C(\mathbb{R}^+; H^m_{div}(\Omega))$$

to the linearized BMHD system (7). Moreover, for any $m_1\in\mathbb{N}$ *with* $m_1\le m$*, and* $t>0$*, it holds that*

$$\sum_{j=0}^{m_1}\Big[\|(\sqrt{\kappa}\nabla^j\sigma, \nabla^j u, \nabla^j b)(t)\|^2_{L^2} + \int_0^t \|(\kappa\nabla^{j+1}\sigma, \sqrt{\mu}\nabla^{j+1}u, \sqrt{\gamma}\nabla^j\nabla^{\perp}\cdot b)(\tau)\|^2_{L^2}\, d\tau\Big]$$
$$\le C\sum_{j=0}^{m_1}\|(\sqrt{\kappa}\nabla^j\sigma_0, \nabla^j u_0, \nabla^j b_0)\|^2_{L^2} + Ce^{2\Lambda t}\|(\sqrt{\kappa}\sigma_0, u_0, b_0)\|^2_{L^2}, \tag{14}$$

$$\sum_{j=0}^{m-2}\Big[\|(\partial_t\nabla^j u, \nabla^{j+1}p)(t)\|^2_{L^2} + \int_0^t \|\nabla^{j+1}\partial_t u(\tau)\|^2_{L^2}\, d\tau\Big]$$
$$\le C\sum_{j=0}^{m}\|(\sqrt{\kappa}\nabla^j\sigma_0, \nabla^j u_0, \nabla^j b_0)\|^2_{L^2} + Ce^{2\Lambda t}\|(\sqrt{\kappa}\sigma_0, u_0, b_0)\|^2_{L^2}. \tag{15}$$

Proof. Both the existence and uniqueness of a solution to (7) essentially follow from some *a priori* estimates. We now establish the estimates.

For the L^2 estimate, multiplying the first equation of (7) by $\kappa(\theta_0)\sigma$, and then integrating the result equation with respect to $(x,y)\in\Omega$, one obtains

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega}\kappa(\theta_0)|\sigma|^2\, dxdy - \int_{\Omega}u_2\,\sigma\, dxdy$$
$$= \int_{\Omega}\kappa(\theta_0)\mathrm{div}(\kappa(\theta_0)\nabla\sigma)\sigma\, dxdy + \int_{\Omega}\kappa(\theta_0)\partial_y\Big(\sigma\frac{\mathrm{d}}{\mathrm{d}y}\kappa(\theta_0)\Big)\sigma\, dxdy.$$

Thanks to integration by parts again, one has

$$-\int_\Omega \kappa(\theta_0)\nabla\sigma\cdot\nabla(\kappa(\theta_0)\sigma)dxdy = -\int_\Omega \kappa^2(\theta_0)|\nabla\sigma|^2 dxdy + \frac{1}{4}\int_\Omega \sigma^2\frac{\mathrm{d}^2}{\mathrm{d}y^2}(\kappa^2(\theta_0))dxdy,$$

$$\int_\Omega \kappa(\theta_0)\sigma\partial_y\Big(\sigma\frac{\mathrm{d}}{\mathrm{d}y}\kappa(\theta_0)\Big)dxdy = -\int_\Omega \sigma^2\Big(\frac{\mathrm{d}}{\mathrm{d}y}\kappa(\theta_0)\Big)^2 dxdy + \frac{1}{4}\int_\Omega \sigma^2\frac{\mathrm{d}^2}{\mathrm{d}y^2}(\kappa^2(\theta_0))dxdy,$$

which implies that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \kappa(\theta_0)|\sigma|^2\,dxdy + \int_\Omega \kappa^2(\theta_0)|\nabla\sigma|^2\,dxdy = \int_\Omega \Big(u_2\,\sigma + \sigma^2\kappa(\theta_0)\frac{d^2}{dy^2}\kappa(\theta_0)\Big)\,dxdy. \tag{16}$$

Multiplying the second and third equation in (7) with u and b, respectively, and using the fact that

$$\begin{aligned}&\int_\Omega \nabla^\perp(\gamma(\theta_0)\nabla^\perp\cdot b)\cdot bdxdy = \int_\Omega \nabla\times(\gamma(\theta)\nabla\times b)\cdot bdxdy\\ &= -\int_{\partial\Omega}\gamma(\theta)(\nabla\times b)\times n\cdot bdxdy + \int_\Omega \gamma(\theta)|\nabla\times b|^2 dx = \int_\Omega \gamma(\theta)|\nabla\times b|^2 dxdy,\end{aligned}$$

integrating by parts, we have

$$\frac{1}{2}\frac{d}{dt}\int_\Omega(|u|^2+|b|^2)\,dxdy + \int_\Omega(\mu(\theta_0)|\nabla u|^2 + \gamma(\theta_0)|\nabla^\perp\cdot b|^2)\,dxdy = \int_\Omega u_2\,\sigma\,dxdy.$$

This, together with (16), implies

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_\Omega(|(\sqrt{\kappa(\theta_0)}\sigma,u,b)|^2)\,dxdy + \int_\Omega[|(\kappa(\theta_0)\nabla\sigma,\sqrt{\mu(\theta_0)}\nabla u,\sqrt{\gamma(\theta_0)}\nabla^\perp\cdot b)|^2]\,dxdy\\ &= 2\int_\Omega u_2\,\sigma\,dxdy + \int_\Omega \sigma^2\kappa(\theta_0)\frac{d^2}{dy^2}\kappa(\theta_0)\,dxdy.\end{aligned} \tag{17}$$

From the definition of Λ in Lemma 2, we get from (17) that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega(\kappa(\theta_0)|\sigma|^2+|u|^2+|b|^2)\,dxdy \le \Lambda\int_\Omega(\kappa(\theta_0)|\sigma|^2+|u|^2+|b|^2)\,dxdy,$$

which implies that, for any $t \ge 0$

$$\int_\Omega(\kappa(\theta_0)|\sigma|^2+|u|^2+|b|^2)(t)\,dxdy \le e^{2\Lambda t}\int_\Omega(\kappa(\theta_0)|\sigma_0|^2+|u_0|^2+|b_0|^2)\,dxdy. \tag{18}$$

Notice that the quantities $\kappa(\theta_0)$, $\mu(\theta_0)$ and $\gamma(\theta_0)$ have positive lower bounds, and all of their derivatives are bounded in $\mathbb{R}$. It follows from (17) that

$$\begin{aligned}&\int_\Omega[\kappa^2(\theta_0)|\nabla\sigma|^2 + \mu(\theta_0)|\nabla u|^2 + \gamma(\theta_0)|\nabla^\perp\cdot b|^2]\,dxdy\\ &\le -\frac{1}{2}\frac{d}{dt}\int_\Omega(\kappa(\theta_0)|\sigma|^2+|u|^2+|b|^2)\,dxdy + C\int_\Omega(\kappa(\theta_0)|\sigma|^2+|u|^2+|b|^2)\,dxdy,\end{aligned}$$

which together with (18) implies that, for all $t > 0$

$$\int_0^t \|(\kappa(\theta_0)\nabla\sigma,\sqrt{\mu(\theta_0)}\nabla u,\sqrt{\gamma(\theta_0)}\nabla^\perp\cdot b)(\tau)\|_{L^2}^2\,d\tau \le Ce^{2\Lambda t}\|(\sqrt{\kappa(\theta_0)}\sigma_0,u_0,b_0)\|_{L^2}^2. \tag{19}$$

Therefore, combining (19) with (18), we can show that, for all $t \geq 0$

$$
\begin{aligned}
&\|(\sqrt{\kappa(\theta_0)}\sigma, u, b)(t)\|_{L^2}^2 + \int_0^t \|(\kappa(\theta_0)\nabla\sigma, \sqrt{\mu(\theta_0)}\nabla u, \sqrt{\gamma(\theta_0)}\nabla^{\perp}\cdot b)(\tau)\|_{L^2}^2\, d\tau \\
&\leq Ce^{2\Lambda t}\|(\sqrt{\kappa(\theta_0)}\sigma_0, u_0, b_0)\|_{L^2}^2.
\end{aligned}
\tag{20}
$$

In order to get the H^1 estimate, applying the operator ∂_i (with $i = 1, 2$ and $\partial_1 = \partial_x,\ \ \partial_2 = \partial_y$) to the equations (7), we obtain

$$
\begin{cases}
\partial_t\partial_i\sigma - \partial_i\left[\frac{1}{\kappa(\theta_0)}u_2\right] - \partial_i \mathrm{div}(\kappa(\theta_0)\nabla\sigma) = \partial_i\partial_y\left(\sigma\frac{d}{dy}\kappa(\theta_0)\right), \\
\partial_t\partial_i u - \partial_i \mathrm{div}(\mu(\theta_0)\nabla u) + \nabla\partial_i p = \partial_i\sigma e_2, \\
\partial_t\partial_i b - \nabla^{\perp}(\gamma(\theta_0)\nabla^{\perp}\cdot\partial_i b) = \nabla^{\perp}(\partial_i\gamma(\theta_0)\nabla^{\perp}\cdot b),
\end{cases}
\tag{21}
$$

which is equivalent to

$$
\begin{cases}
\partial_t\partial_i\sigma - \frac{1}{\kappa(\theta_0)}\partial_i u_2 - \frac{d}{dy}(\frac{1}{\kappa(\theta_0)})u_2\delta_2^i - \mathrm{div}(\kappa(\theta_0)\nabla\partial_i\sigma) - \mathrm{div}\left(\frac{d}{dy}\kappa(\theta_0)\nabla\sigma\right)\delta_2^i \\
\qquad = \partial_y\left(\partial_i\sigma\frac{d}{dy}\kappa(\theta_0)\right) + \partial_y\left(\sigma\frac{d^2}{dy^2}\kappa(\theta_0)\right)\delta_2^i, \\
\partial_t\partial_i u - \mathrm{div}(\mu(\theta_0)\nabla\partial_i u) - \mathrm{div}\left(\frac{d}{dy}\mu(\theta_0)\nabla u\right)\delta_2^i + \nabla\partial_i p = \partial_i\sigma e_2, \\
\partial_t\partial_i b - \nabla^{\perp}(\gamma(\theta_0)\nabla^{\perp}\cdot\partial_i b) = \nabla^{\perp}\left(\frac{d}{dy}\gamma(\theta_0)\nabla^{\perp}\cdot b\right)\delta_2^i,
\end{cases}
\tag{22}
$$

where $\delta_2^i = 0$ if $i = 1$, and $\delta_2^i = 1$ if $i = 2$. Taking the L^2 inner product of the first equation in (22) with $\kappa(\theta_0)\partial_i\sigma$, and then using integration by parts, one gets

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\Omega}\kappa(\theta_0)|\nabla\sigma|^2\, dxdy + \int_{\Omega}\left[\kappa^2(\theta_0)|\nabla^2\sigma|^2\right] dxdy = \int_{\Omega}\nabla u_2\cdot\nabla\sigma\, dxdy \\
&-\int_{\Omega} u_2\,\partial_y\sigma\,\frac{d}{dy}\log\kappa(\theta_0)\, dxdy + \int_{\Omega}\left(\frac{3}{4}\frac{d^2}{dy^2}\kappa^2(\theta_0) - \left(\frac{d}{dy}\kappa(\theta_0)\right)^2\right)|\nabla\sigma|^2\, dxdy \\
&-\frac{1}{2}\int_{\Omega}\sigma^2\frac{d}{dy}\left[\kappa(\theta_0)\frac{d^3}{dy^3}\kappa(\theta_0)\right] dxdy + \int_{\Omega}(\partial_y\sigma)^2\left[\kappa(\theta_0)\frac{d^2}{dy^2}\kappa(\theta_0) - \left(\frac{d}{dy}\kappa(\theta_0)\right)^2\right] dxdy.
\end{aligned}
\tag{23}
$$

Similarly, taking the L^2 inner product of the second and third equations in (21) with $\partial_i u$ and $\partial_i b$, respectively, and then using integration by parts, we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\Omega}|(\nabla u, \nabla b)|^2\, dxdy + \int_{\Omega}(\mu(\theta_0)|\nabla^2 u|^2 + \sum_{i=1}^{2}\gamma(\theta_0)|\nabla^{\perp}\cdot\partial_i b|^2)\, dxdy \\
&= \int_{\Omega}\nabla u_2\cdot\nabla\sigma\, dxdy + \frac{1}{2}\int_{\Omega}|\nabla u|^2\frac{d^2}{dy^2}\mu(\theta_0)\, dxdy + \frac{1}{2}\int_{\Omega}|\nabla^{\perp}\cdot b|^2\frac{d^2}{dy^2}\gamma(\theta_0)\, dxdy,
\end{aligned}
$$

which, along with (23), gives

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\Omega}(|(\sqrt{\kappa}\nabla\sigma, \nabla u, \nabla b)|^2)\, dxdy + \int_{\Omega}\left[|(\kappa\nabla^2\sigma, \sqrt{\mu}\nabla^2 u, \sqrt{\gamma}\nabla(\nabla^{\perp}\cdot b)|^2\right] dxdy \\
&\leq C\|(\kappa\nabla\sigma, \sqrt{\mu}\nabla u, \sqrt{\gamma}\nabla^{\perp}\cdot b)\|_{L^2}^2 + C\|(\sqrt{\kappa(\theta_0)}\sigma, u, b)\|_{L^2}^2.
\end{aligned}
$$

This, together with (20), implies that for all $t > 0$

$$
\begin{aligned}
&\|(\sqrt{\kappa(\theta_0)}\nabla\sigma, \nabla u,, \nabla b)(t)\|_{L^2}^2 + \int_0^t \|(\kappa\nabla^2\sigma, \sqrt{\mu}\nabla^2 u, \sqrt{\gamma}\nabla(\nabla^{\perp}\cdot b)(\tau)\|_{L^2}^2\, d\tau \\
&\leq \|(\sqrt{\kappa(\theta_0)}\nabla\sigma_0, \nabla u_0, \nabla b_0)\|_{L^2}^2 + Ce^{2\Lambda t}\|(\sqrt{\kappa(\theta_0)}\sigma_0, u_0, b_0)\|_{L^2}^2.
\end{aligned}
$$

By an induction argument, we can get (14) and (15), which completes the proof. □

4. Nonlinear Energy Estimates for the Case $a_+ < a_-$

In this section, we prove the nonlinear estimates for the nonlinear perturbation (4) for the case $a_+ < a_-$.

Lemma 4. *Assume that $\sigma_0 \in H^2(\Omega)$, $(u_0, b_0) \in H^2_{div}(\Omega) \times H^2_{div}(\Omega)$ and assume the boundary conditions*

$$\partial\sigma|_{\partial\Omega} = 0, \quad \partial u|_{\partial\Omega} = 0, \quad \partial b \cdot n|_{\partial\Omega} = 0, \ (\nabla \times \partial b) \times n|_{\partial\Omega} = 0.$$

Subsequently, there exists a unique global solution $(\sigma,\ u,\ b)$ satisfying

$$(\sigma,\ u,\ b) \in C_{loc}(\mathbb{R}^+; H^2(\Omega)) \times C_{loc}(\mathbb{R}^+; H^2_{div}(\Omega)) \times C_{loc}(\mathbb{R}^+; H^2_{div}(\Omega))$$

to the perturbed BMHD (4) with initial data $(\sigma_0,\ u_0,\ b_0)$ and the corresponding boundary conditions. Moreover, there are two positive constants $\overline{\delta} \in (0,1]$ and $T > 0$, such that, for any $t \in [0,T]$ and $\|(\sigma,u,b)(t)\|_{H^2} \leq \overline{\delta}$, it holds that

$$\|(\sigma,u,b)\|^2_{L^\infty_t(H^2)} + \int_0^t \|(\nabla\sigma,\nabla u,\nabla b)\|^2_{H^2}\, d\tau \leq C\|(\sigma_0,u_0,b_0)\|^2_{H^2} + C\int_0^t \|(\sigma,u)\|^2_{L^2}\, d\tau, \tag{24}$$

where the constant C only depends on $\overline{\theta}$ and Ω.

Proof. Similar to the proof of Lemma 3, we just need to present some necessary *a priori* estimates for sufficiently smooth solutions to (4).

Multiplying the three equations in (4) by σ, u, and b, respectively, and then integrating by parts, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega |(\sigma,u,b)|^2\, dxdy + \int_\Omega (\kappa(\theta_0+\sigma)|\nabla\sigma|^2 + \mu(\theta_0+\sigma)|\nabla u|^2 + \gamma(\theta_0+\sigma)|\nabla^\perp\cdot b|^2)\, dxdy \\
&= \int_\Omega \Big(1 + \frac{1}{\kappa(\theta_0)}\Big) u_2\,\sigma\, dxdy + \int_\Omega \frac{\kappa(\theta_0+\sigma)-\kappa(\theta_0)}{\kappa(\theta_0)}\partial_y\sigma\, dxdy,
\end{aligned}$$

which implies

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|(\sigma,u,b)\|^2_{L^2} + \int_\Omega (\kappa(\theta_0+\sigma)|\nabla\sigma|^2 + \mu(\theta_0+\sigma)|\nabla u|^2 + \gamma(\theta_0+\sigma)|\nabla^\perp\cdot b|^2)\, dxdy \\
&\lesssim \|\sigma\|^2_{L^2} + \|u\|^2_{L^2} + \|\kappa(\theta_0+\sigma)-\kappa(\theta_0)\|_{L^2}\|\partial_y\sigma\|_{L^2}.
\end{aligned}$$

Therefore, applying Young's inequality, we can show

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega (|\sigma|^2 + |u|^2 + |b|^2)\, dxdy \\
&\qquad + \int_\Omega [\kappa(\theta_0+\sigma)|\nabla\sigma|^2 + \mu(\theta_0+\sigma)|\nabla u|^2 + \gamma(\theta_0+\sigma)|\nabla^\perp\cdot b|^2]\, dxdy \\
&\lesssim \|\sigma\|^2_{L^2} + \|u\|^2_{L^2} + \|\kappa'\|^2_{L^\infty}\|\sigma\|^2_{L^2} \lesssim \|\sigma\|^2_{L^2} + \|u\|^2_{L^2}.
\end{aligned}$$

Integrating the above inequality gives that for all $t < T$

$$\begin{aligned}
&\|(\sigma,u,b)\|^2_{L^\infty_T(L^2)} + c_0\|(\nabla\sigma,\nabla u,\nabla^\perp\cdot b)\|^2_{L^2_T(L^2)} \\
&\leq \|(\sigma_0,u_0,b_0)\|^2_{L^2} + C\int_0^t (\|\sigma\|^2_{L^2} + \|u\|^2_{L^2})\, d\tau.
\end{aligned} \tag{25}$$

In order to get the H^1 estimate, applying the operator ∂_i (with $i = 1, 2$ and $\partial_1 = \partial_x,\ \ \partial_2 = \partial_y$) to the equations (4), we have

$$\begin{cases} \partial_t\partial_i\sigma + u\cdot\nabla\partial_i\sigma - \mathrm{div}(\kappa(\theta_0+\sigma)\nabla\partial_i\sigma) - \mathrm{div}(\nabla\sigma\partial_i\kappa(\theta_0+\sigma)) \\ \quad = \partial_i\Big(\frac{1}{\kappa(\theta_0)}\Big)u_2 + \frac{1}{\kappa(\theta_0)}\partial_i u_2 - \partial_y\partial_i\Big(\frac{\kappa(\theta_0+\sigma)}{\kappa(\theta_0)}\Big) - \partial_i u\cdot\nabla\sigma, \\ \partial_t\partial_i u + u\cdot\nabla\partial_i u - \mathrm{div}(\mu(\theta_0+\sigma)\nabla\partial_i u) + \nabla\partial_i p \\ \quad = \partial_i\sigma e_2 - \partial_i u\cdot\nabla u + \mathrm{div}(\nabla u\partial_i\mu(\theta_0+\sigma)) + b\cdot\nabla\partial_i b + \partial_i b\cdot\nabla b, \\ \partial_t\partial_i b + u\cdot\nabla\partial_i b - \nabla^\perp(\gamma(\theta_0+\sigma)\partial_i\nabla^\perp\cdot b) \\ \quad = b\cdot\nabla\partial_i u + \partial_i b\cdot\nabla u - \partial_i u\cdot\nabla b + \nabla^\perp(\partial_i\gamma(\theta_0+\sigma)\nabla^\perp\cdot b). \end{cases} \tag{26}$$

Multiplying the three equations in (26) by $\partial_i\sigma$, $\partial_i u$ and $\partial_i b$, respectively, and then using integration by parts, we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_\Omega(|\nabla\sigma|^2 + |\nabla u|^2 + |\nabla b|^2)\,dxdy \\ &+ \int_\Omega\big[\kappa(\theta_0+\sigma)|\nabla^2\sigma|^2 + \mu(\theta_0+\sigma)|\nabla^2 u|^2 + \gamma(\theta_0+\sigma)|\nabla\nabla^\perp\cdot b|^2\big]\,dxdy = F \end{aligned} \tag{27}$$

with

$$\begin{aligned} F \overset{\text{def}}{=} \int_\Omega\sum_{i=1}^{2}\Big[&\frac{d}{dy}\Big(\frac{1}{\kappa(\theta_0)}\Big)u_2\partial_y\sigma + \frac{1}{\kappa(\theta_0)}\partial_i u_2\partial_i\sigma + \partial_i\sigma\partial_i u_2 \\ &+ \frac{\kappa(\theta_0)\partial_y[\kappa(\theta_0+\sigma)-\kappa(\theta_0)] - \partial_y\kappa(\theta_0)[\kappa(\theta_0+\sigma)-\kappa(\theta_0)]}{\kappa^2(\theta_0)}\partial_i^2\sigma \\ &- \partial_i u\cdot\nabla\sigma\partial_i\sigma + (\partial_i b\cdot\nabla b - \partial_i u\cdot\nabla u)\cdot\partial_i u + (\partial_i b\cdot\nabla u - \partial_i u\cdot\nabla b)\cdot\partial_i b \\ &+ \frac{1}{2}\partial_i^2\mu(\theta_0+\sigma)|\nabla u|^2 + \frac{1}{2}\partial_i^2\gamma(\theta_0+\sigma)|\nabla^\perp\cdot b|^2 + \frac{1}{2}\partial_i^2\kappa(\theta_0+\sigma)|\nabla\sigma|^2\Big]\,dxdy =: \sum_{j=1}^{4}F_j. \end{aligned}$$

Thanks to the boundness of $\kappa(\theta_0)$ and its derivatives, it follows from Holder's inequality that

$$|F_1| \lesssim \|u_2\|_{L^2}\|\partial_y\sigma\|_{L^2} + \|\partial_i u_2\|_{L^2}\|\partial_i\sigma\|_{L^2} \lesssim (\|u_2\|_{L^2} + \|\nabla u\|_{L^2})\|\nabla\sigma\|_{L^2}$$

and

$$\begin{aligned} |F_2| &\lesssim \|\kappa(\theta_0+\sigma)-\kappa(\theta_0)\|_{H^1}\|\partial_i^2\sigma\|_{L^2} \lesssim (\|\kappa'\|_{L^\infty} + \|\kappa''\|_{L^\infty})\|\sigma\|_{H^1}\|\nabla^2\sigma\|_{L^2} \\ &\lesssim (\|\sigma\|_{L^2} + \|\nabla\sigma\|_{L^2})\|\nabla^2\sigma\|_{L^2}. \end{aligned}$$

Similarly, by Poincare inequality, we can estimate F_3 and F_4, as follows

$$\begin{aligned} |F_3| &\lesssim \|(\nabla\sigma,\nabla u,\nabla^\perp\cdot b)\|_{L^2}^2 + \|(\nabla\sigma,\nabla u,\nabla^\perp\cdot b)\|_{L^4}^4 \\ &\lesssim \|(\nabla\sigma,\nabla u,\nabla^\perp\cdot b)\|_{L^2}^2 + \|(\nabla\sigma,\nabla u,\nabla b)\|_{L^2}^2)\|(\nabla^2\sigma,\nabla^2 u,\nabla\nabla^\perp\cdot b)\|_{L^2}^2, \end{aligned}$$

$$\begin{aligned} |F_4| &\lesssim \|(\nabla\sigma,\nabla u,\nabla^\perp\cdot b)\|_{L^4}^4 + \|\nabla^2\sigma\|_{L^2}\|(\nabla\sigma,\nabla u,\nabla^\perp\cdot b)\|_{L^4}^2 \\ &\lesssim (\eta + C_\eta\|(\nabla\sigma,\nabla u,\nabla b)\|_{L^2}^2)\|(\nabla^2\sigma,\nabla^2 u,\nabla\nabla^\perp\cdot b)\|_{L^2}^2, \end{aligned}$$

which together with (27) implies that

$$\begin{aligned} &\frac{d}{dt}\int_\Omega(|(\nabla\sigma,\nabla u,\nabla b)|^2)\,dxdy \\ &\quad + 2\int_\Omega\big[\kappa(\theta_0+\sigma)|\nabla^2\sigma|^2 + \mu(\theta_0+\sigma)|\nabla^2 u|^2 + \gamma(\theta_0+\sigma)|\nabla\nabla^\perp\cdot b|^2\big]\,dxdy \\ &\lesssim \|(\sigma,u)\|_{L^2}^2 + \|(\nabla\sigma,\nabla u,\nabla^\perp\cdot b)\|_{L^2}^2 + (\eta + C_\eta\|(\sigma,u,b)\|_{H^1}^2)\|(\nabla^2\sigma,\nabla^2 u,\nabla\nabla^\perp\cdot b)\|_{L^2}^2. \end{aligned} \tag{28}$$

Therefore, it follows from (28) that, for some positive constant c_0 and any $t > 0$,

$$\begin{aligned}
&\|(\nabla\sigma, \nabla u, \nabla b)(t)\|_{L^2}^2 + c_0 \int_0^t \|(\nabla^2\sigma, \nabla^2 u, \nabla\nabla^\perp \cdot b)\|_{L^2}^2 \, d\tau \\
&\le \|(\nabla\sigma, \nabla u, \nabla b)(0)\|_{L^2}^2 + C \int_0^t \|(\sigma, u)\|_{L^2}^2 \, d\tau + C \int_0^t \|(\nabla\sigma, \nabla u, \nabla^\perp \cdot b)\|_{L^2}^2 \, d\tau \\
&\quad + C \int_0^t (\eta + C_\eta \|(\sigma, u, b)\|_{H^1}^2) \|(\nabla^2\sigma, \nabla^2 u, \nabla\nabla^\perp \cdot b)\|_{L^2}^2 \, d\tau,
\end{aligned}$$

which, together with (25), leads to

$$\begin{aligned}
&\|(\sigma, u, b)\|_{L_t^\infty(H^1)}^2 + c_0 \int_0^t \|(\nabla\sigma, \nabla u, \nabla^\perp \cdot b)\|_{H^1}^2 \, d\tau \le C \|(\sigma_0, u_0, b_0)\|_{H^1}^2 \\
&+ C \int_0^t \|(\sigma, u)\|_{L^2}^2 \, d\tau + C \int_0^t (\eta + C_\eta \|(\sigma, u, b)\|_{H^1}^2) \|(\nabla^2\sigma, \nabla^2 u, \nabla\nabla^\perp \cdot b)\|_{L^2}^2 \, d\tau.
\end{aligned}$$

Then we have

$$\|(\sigma, u, b)\|_{L_t^\infty(H^1)}^2 + \int_0^t \|(\nabla\sigma, \nabla u, \nabla b)\|_{H^1}^2 \, d\tau \le C \|(\sigma_0, u_0, b_0)\|_{H^1}^2 + C \int_0^t \|(\sigma, u)\|_{L^2}^2 \, d\tau,$$

if $\eta > 0$ is sufficiently small and $\|(\sigma, u, b)(t)\|_{H^1} \le \bar{\delta}$ with $\bar{\delta}$ small enough and $t \in [0, T]$ for some positive time T.

Similarly, one can obtain that

$$\begin{aligned}
&\|(\sigma, u, b)\|_{L_t^\infty(H^2)}^2 + c_0 \int_0^t \|(\nabla\sigma, \nabla u, \nabla^\perp \cdot b)\|_{H^2}^2 \, d\tau \\
&\le C \|(\sigma_0, u_0, b_0)\|_{H^2}^2 + C \int_0^t \|(\sigma, u)\|_{L^2}^2 \, d\tau \\
&\quad + C \int_0^t (\eta + C_\eta \|(\sigma, u, b)\|_{H^2}^2) \|(\nabla^3\sigma, \nabla^3 u, \nabla^2\nabla^\perp \cdot b)\|_{L^2}^2 \, d\tau,
\end{aligned}$$

which implies (24). □

5. Proof of Theorem 1

The goal of this section is to prove Theorem 1.

Proof. (i) First, from Section 3, one can construct a solution of the form

$$(\sigma^1, u^1, b^1) \stackrel{\text{def}}{=} e^{\Lambda t}(\tilde{\sigma}_0, \tilde{u}_0, \tilde{b}_0)$$

to the system (7) with the initial data $(\tilde{\sigma}_0, \tilde{u}_0, \tilde{b}_0) \in H^2(\Omega) \times H^2_{div}(\Omega) \times H^2_{div}(\Omega)$ and the corresponding boundary conditions. Moreover, these initial data can be assumed to satisfy

$$\|(\tilde{\sigma}_0, \tilde{u}_0, \tilde{b}_0)\|_{H^2} = 1$$

by a standard normalization argument.

For any $\delta \in (0, \delta_0)$, take $(\sigma_0^\delta, u_0^\delta, b_0^\delta) = \delta(\tilde{\sigma}_0, \tilde{u}_0, \tilde{b}_0)$, and $(\sigma^\delta, u^\delta, b^\delta)$ is the solution to (4) with initial data $(\sigma_0^\delta, u_0^\delta, b_0^\delta)$ and $\varepsilon_0 > 0$ sufficiently small (to be determined later), and define $T^\delta > 0$, such that

$$\delta e^{\Lambda T^\delta} \stackrel{\text{def}}{=} 2\varepsilon_0,$$

$$T^* \stackrel{\text{def}}{=} \sup\{t > 0 : \|(\sigma^\delta, u^\delta, b^\delta)(t)\|_{H^2} \le \delta_0\} \tag{29}$$

and

$$T^{**} \stackrel{\text{def}}{=} \sup\{t > 0 : \|(\sigma^\delta, u^\delta, b^\delta)(t)\|_{L^2} \le 2\delta\|(\tilde{\sigma}_0, \tilde{u}_0, \tilde{b}_0)\|_{L^2} e^{\Lambda t}\}. \tag{30}$$

Subsequently, for all $t \le \min(T^\delta, T^*, T^{**})$, using the estimate (24) and the definitions of T^* and T^{**}, we obtain

$$\begin{aligned} &\|(\sigma^\delta, u^\delta, b^\delta)(t)\|_{H^2}^2 + \int_0^t \|(\nabla\sigma^\delta, \nabla u^\delta, \nabla^\perp \cdot b^\delta)\|_{H^2}^2 \, d\tau \\ &\le C\delta^2 + C\int_0^t \|(\sigma^\delta, u^\delta)\|_{L^2}^2 \, d\tau \le C\delta^2 + 2C_1^2\delta^2 C e^{2\Lambda t}/\Lambda \le C_2\delta^2 e^{2\Lambda t} \end{aligned} \tag{31}$$

for some constant $C_2 > 0$ independent of δ.

Let $(\sigma^d, u^d, b^d) \stackrel{\text{def}}{=} (\sigma^\delta, u^\delta, b^\delta) - \delta(\sigma^1, u^1, b^1)$. Noting that $(\sigma^a_\delta, u^a_\delta, b^a_\delta) \stackrel{\text{def}}{=} \delta(\sigma^1, u^1, b^1)$ is also a solution to (7) with the initial data $(\sigma^\delta_0, u^\delta_0, b^\delta_0)$, so (σ^d, u^d, b^d) solves the following problem:

$$\begin{cases} \partial_t\sigma^d - \frac{1}{\kappa(\theta_0)} u_2^d - \operatorname{div}(\kappa(\theta_0)\nabla\sigma^d) - \partial_y\left(\sigma^d \frac{d}{dy}\kappa(\theta_0)\right) = F, \\ \partial_t u^d - \operatorname{div}(\mu(\theta_0)\nabla u^d) + \nabla p^d - \sigma^d e_2 = G, \\ \partial_t b^d - \nabla^\perp(\gamma(\theta_0)\nabla^\perp \cdot b^d) = H, \\ \operatorname{div} u^d = 0, \ \operatorname{div} b^d = 0, \ (\sigma^d, u^d, b^d)|_{t=0} = 0, \\ (\sigma^d, u^d)|_{\partial\Omega} = 0, \ b^d \cdot n|_{\partial\Omega} = 0, \ (\nabla \times b^d) \times n|_{\partial\Omega} = 0, \end{cases}$$

with

$$\begin{cases} F = -u^\delta \cdot \nabla\sigma^\delta + \operatorname{div}([\kappa(\theta_0 + \sigma^\delta) - \kappa(\theta_0)]\nabla\sigma^\delta) - \partial_y\left(\frac{\kappa(\theta_0+\sigma^\delta) - \sigma^\delta\kappa'(\theta_0)}{\kappa(\theta_0)}\right), \\ G = b^\delta \cdot \nabla b^\delta - u^\delta \cdot \nabla u^\delta + \operatorname{div}([\mu(\theta_0 + \sigma^\delta) - \mu(\theta_0)]\nabla u^\delta), \\ H = b^\delta \cdot \nabla u^\delta - u^\delta \cdot \nabla b^\delta + \nabla^\perp([\gamma(\theta_0 + \sigma^\delta) - \gamma(\theta_0)]\nabla^\perp \cdot b^\delta), \end{cases}$$

namely,

$$\begin{cases} \kappa(\theta_0)\partial_t\sigma^d - u_2^d - \kappa(\theta_0)\operatorname{div}(\kappa(\theta_0)\nabla\sigma^d) - \kappa(\theta_0)\partial_y(\sigma^d \frac{d}{dy}\kappa(\theta_0)) = \kappa(\theta_0)F, \\ \partial_t u^d - \operatorname{div}(\mu(\theta_0)\nabla u^d) + \nabla p^d - \sigma^d e_2 = G, \\ \partial_t b^d - \nabla^\perp(\gamma(\theta_0)\nabla^\perp \cdot b^d) = H, \\ \operatorname{div} u^d = 0, \ \operatorname{div} b^d = 0, \ (\sigma^d, u^d, b^d)|_{t=0} = 0, \\ (\sigma^d, u^d)|_{\partial\Omega} = 0, \ b^d \cdot n|_{\partial\Omega} = 0, \ (\nabla \times b^d) \times n|_{\partial\Omega} = 0. \end{cases} \tag{32}$$

Similar to the proof of (17), one can get from (32) that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_\Omega (\kappa(\theta_0)|\sigma^d|^2 + |u^d|^2 + |b^d|^2) \, dxdy \\ &\quad + \int_\Omega [\kappa^2(\theta_0)|\nabla\sigma^d|^2 + \mu(\theta_0)|\nabla u^d|^2 + \gamma(\theta_0)|\nabla^\perp \cdot b^d|^2] \, dxdy - 2\int_\Omega u_2^d \sigma^d \, dxdy \\ &= \int_\Omega (\sigma^d)^2 \kappa(\theta_0)\frac{d^2}{dy^2}\kappa(\theta_0) \, dxdy + \int_\Omega (\kappa(\theta_0)\, F\sigma^d + G \cdot u^d + H \cdot b^d) \, dxdy. \end{aligned}$$

Hence, by the definition of Λ, one has

$$\begin{aligned} &\frac{d}{dt}\int_\Omega (\kappa(\theta_0)|\sigma^d|^2 + |u^d|^2 + |b^d|^2) \, dxdy \\ &\le 2\Lambda\int_\Omega (\kappa(\theta_0)|\sigma^d|^2 + |u^d|^2 + |b^d|^2) \, dxdy + 2\int_\Omega (\kappa(\theta_0)\, F\sigma^d + G \cdot u^d + H \cdot b^d) \, dxdy. \end{aligned} \tag{33}$$

For the remainder terms on the right-hand-side of (33), one first can deduce that

$$
\begin{aligned}
|\int_\Omega \kappa(\theta_0)\sigma^d u^\delta \cdot \nabla\sigma^\delta \,dxdy| &\le C\|\sigma^d\|_{L^4}\|u^\delta\|_{L^4}\|\nabla\sigma^\delta\|_{L^2} \\
&\le C\|\sigma^d\|_{L^2}^{\frac12}\|u^\delta\|_{L^2}^{\frac12}\|\nabla\sigma^d\|_{L^2}^{\frac12}\|\nabla u^\delta\|_{L^2}^{\frac12}\|\nabla\sigma^\delta\|_{L^2} \\
&\le C\|\sigma^d\|_{H^1}\|u^\delta\|_{H^1}\|\nabla\sigma^\delta\|_{L^2} \\
&\le C(\|\sigma^\delta\|_{H^1}+\delta\|\sigma^1\|_{H^1})\|u^\delta\|_{H^1}\|\nabla\sigma^\delta\|_{L^2}.
\end{aligned} \tag{34}
$$

Similarly, it holds that

$$
\begin{aligned}
&|\int_\Omega u^d\cdot(b^\delta\cdot\nabla b^\delta - u^\delta\cdot\nabla u^\delta)\,dxdy| \\
&\le (\|u^\delta\|_{H^1}+\delta\|u^1\|_{H^1})\|(u^\delta,b^\delta)\|_{H^1}\|(\nabla u^\delta,\nabla b^\delta)\|_{L^2},
\end{aligned} \tag{35}
$$

and

$$
|\int_\Omega b^d\cdot(b^\delta\cdot\nabla u^\delta - u^\delta\cdot\nabla b^\delta)\,dxdy| \le (\|b^\delta\|_{H^1}+\delta\|b^1\|_{H^1})\|(u^\delta,b^\delta)\|_{H^1}\|(\nabla u^\delta,\nabla b^\delta)\|_{L^2}. \tag{36}
$$

By integration by parts, one can control the term $\Big|\int_\Omega \sigma^d\kappa(\theta_0)\mathrm{div}([\kappa(\theta_0+\sigma^\delta)-\kappa(\theta_0)]\nabla\sigma^\delta)\,dxdy\Big|$ by $\Big|\int_\Omega\Big([\kappa(\theta_0+\sigma^\delta)-\kappa(\theta_0)]\nabla\sigma^\delta\cdot\nabla\sigma^d\kappa(\theta_0)+[\kappa(\theta_0+\sigma^\delta)-\kappa(\theta_0)]\partial_y\sigma^\delta\sigma^d\frac{d}{dy}\kappa(\theta_0)\Big)\,dxdy\Big|$, which implies that

$$
\begin{aligned}
&\Big|\int_\Omega \sigma^d\kappa(\theta_0)\mathrm{div}([\kappa(\theta_0+\sigma^\delta)-\kappa(\theta_0)]\nabla\sigma^\delta)\,dxdy\Big| \\
&\lesssim \|\kappa(\theta_0+\sigma^\delta)-\kappa(\theta_0)\|_{L^4}(\|\nabla\sigma^\delta\|_{L^4}\|\nabla\sigma^d\|_{L^2}+\|\partial_y\sigma^\delta\|_{L^2}\|\sigma^d\|_{L^4}) \\
&\lesssim \|\sigma^\delta\|_{H^1}\|\nabla\sigma^\delta\|_{H^1}(\|\nabla\sigma^\delta\|_{L^2}+\delta\|\nabla\sigma^1\|_{L^2})+\|\sigma^\delta\|_{H^1}\|\nabla\sigma^\delta\|_{L^2}(\|\sigma^\delta\|_{H^1}+\delta\|\sigma^1\|_{H^1}) \\
&\lesssim \|\sigma^\delta\|_{H^1}\|\nabla\sigma^\delta\|_{H^1}(\|\sigma^\delta\|_{H^1}+\delta\|\sigma^1\|_{H^1}).
\end{aligned}
$$

Similarly, one can show that

$$
\begin{aligned}
&\int_\Omega |\mathrm{div}([\mu(\theta_0+\sigma^\delta)-\mu(\theta_0)]\nabla u^\delta])\cdot u^d| + |\nabla^\perp([\gamma(\theta_0+\sigma^\delta)-\gamma(\theta_0)]\nabla^\perp\cdot b^\delta])\cdot b^d|\,dxdy \\
&\lesssim \|\sigma^\delta\|_{H^1}\|(\nabla u^\delta,\nabla b^\delta)\|_{H^1}(\|(\nabla u^\delta,\nabla b^\delta)\|_{L^2}+\delta\|(\nabla u^1,\nabla b^1)\|_{L^2}).
\end{aligned} \tag{37}
$$

Noticing that

$$
\begin{aligned}
&\Big|\int_\Omega \sigma^d\kappa(\theta_0)\partial_y\Big(\frac{\kappa(\theta_0+\sigma^\delta)-\sigma^\delta\kappa'(\theta_0)}{\kappa(\theta_0)}\Big)\,dxdy\Big| \\
&\le \Big|\int_\Omega [\kappa(\theta_0+\sigma^\delta)-\kappa(\theta_0)-\sigma^\delta\kappa'(\theta_0)]\partial_y\sigma^d\,dxdy\Big| \\
&\quad + \Big|\int_\Omega \sigma^d\frac{d}{dy}\kappa(\theta_0)\frac{\kappa(\theta_0+\sigma^\delta)-\kappa(\theta_0)-\sigma^\delta\kappa'(\theta_0)}{\kappa(\theta_0)}\,dxdy\Big|,
\end{aligned}
$$

one has

$$
\begin{aligned}
&\Big|\int_\Omega \sigma^d\kappa(\theta_0)\partial_y\Big(\frac{\kappa(\theta_0+\sigma^\delta)-\sigma^\delta\kappa'(\theta_0)}{\kappa(\theta_0)}\Big)\,dxdy\Big| \\
&\lesssim \|\kappa(\theta_0+\sigma^\delta)-\kappa(\theta_0)-\sigma^\delta\kappa'(\theta_0)\|_{L^2}(\|\nabla\sigma^d\|_{L^2}+\|\sigma^d\|_{L^2}) \\
&\lesssim \|\kappa''\|_{L^\infty}\|\sigma^\delta\|_{L^4}^2\|\sigma^d\|_{H^1} \lesssim \|\sigma^\delta\|_{L^2}\|\nabla\sigma^\delta\|_{L^2}(\|\sigma^\delta\|_{H^1}+\delta\|\sigma^1\|_{H^1}).
\end{aligned} \tag{38}
$$

Substituting (34)-(38) into (33), yields

$$
\begin{aligned}
&\frac{d}{dt}\int_{\Omega}(\kappa(\theta_0)|\sigma^d|^2+|u^d|^2+|b^d|^2)\,dxdy\\
&\le 2\Lambda\int_{\Omega}(\kappa(\theta_0)|\sigma^d|^2+|u^d|^2+|b^d|^2)\,dxdy\\
&\quad + C\|(\sigma^\delta,u^\delta,b^\delta)\|_{H^1}\|(\nabla\sigma^\delta,\nabla u^\delta,\nabla b^\delta)\|_{H^1}\|(\sigma^\delta,u^\delta,b^\delta,\delta\sigma^1,\delta u^1,\delta b^1)\|_{H^1},
\end{aligned}
$$

which, along with the Gronwall's inequality, gives rise to

$$
\begin{aligned}
&\|(\sigma^d,u^d,b^d)\|^2_{L^\infty_t(L^2)}\\
&\le C\int_0^t e^{2\Lambda(t-\tau)}\|(\sigma^\delta,u^\delta,b^\delta)\|_{H^1}\|(\nabla\sigma^\delta,\nabla u^\delta,\nabla b^\delta)\|_{H^1}\|(\sigma^\delta,u^\delta,b^\delta,\delta\sigma^1,\delta u^1,\delta b^1)\|_{H^1}\,d\tau\\
&\le Ce^{\Lambda t}\Big(\int_0^t e^{2\Lambda(t-\tau)}\|(\sigma^\delta,u^\delta,b^\delta)\|^2_{H^1}\,d\tau\Big)^{\frac12}\|(\nabla\sigma^\delta,\nabla u^\delta,\nabla b^\delta)\|_{L^2_t(H^1)}\\
&\quad\times\|(\sigma^\delta,u^\delta,b^\delta,\delta\sigma^1,\delta u^1,\delta b^1)\|_{L^\infty_t(H^1)}.
\end{aligned}
\tag{39}
$$

It follows from (31) and (39) that, for all $t\le\min(T^\delta,\,T^*,\,T^{**})$, it holds that

$$
\|(\sigma^d,u^d,b^d)\|^2_{L^\infty_t(L^2)}\le\Big(1+\frac{1}{\sqrt{\Lambda}}\Big)C\delta^3e^{3\Lambda t}
$$

which yields that

$$
\|(\sigma^d,u^d,b^d)\|_{L^\infty_t(L^2)}\le C_3\delta^{3/2}e^{\frac32\Lambda t}
\tag{40}
$$

for some positive constant C_3 independent of δ.

Now, we claim that

$$
T^\delta=\min(T^\delta,\,T^*,\,T^{**})
\tag{41}
$$

if ε_0 is taken to be so small that

$$
0<\varepsilon_0\le\min\Big(\frac{\delta_0}{4\sqrt{C_2}},\,\frac{\|(\tilde\sigma_0,\tilde u_0,\tilde b_0)\|^2_{L^2}}{8C_3^2}\Big).
\tag{42}
$$

In fact, if $T^*=\min(T^\delta,\,T^*,\,T^{**})<T^\delta$, then (31) implies that

$$
\|(\sigma^\delta,u^\delta,b^\delta)(T^*)\|_{H^2}\le\sqrt{C_2}\delta e^{\Lambda T^*}<2\sqrt{C_2}\varepsilon_0\le\frac12\delta_0
$$

which contradicts with the definition of T^* in (29).

On the other hand, if $T^{**}=\min(T^\delta,\,T^*,\,T^{**})<T^\delta$, then it follows from (31), that

$$
\begin{aligned}
&\|(\sigma^\delta,u^\delta,b^\delta)(T^{**})\|_{L^2}\le\|(\sigma^a_\delta,u^a_\delta,b^a_\delta)(T^{**})\|_{L^2}+\|(\sigma^d,u^d,b^d)(T^{**})\|_{L^2}\\
&\le\delta\|(\sigma^1,u^1,b^1)(T^{**})\|_{L^2}+C_3\delta^{3/2}e^{\frac32\Lambda T^{**}}\le\delta\|(\tilde\sigma_0,\tilde u_0,\tilde b_0)\|_{L^2}e^{\Lambda T^{**}}+C_3\delta^{3/2}e^{\frac32\Lambda T^{**}}\\
&\le\delta e^{\Lambda T^{**}}(\|(\tilde\sigma_0,\tilde u_0,\tilde b_0)\|_{L^2}+C_3\sqrt{2\varepsilon_0})\le\frac32\|(\tilde\sigma_0,\tilde u_0,\tilde b_0)\|_{L^2}\delta e^{\Lambda T^{**}}.
\end{aligned}
$$

which contradicts with the definition of T^{**} in (30), and, thus, (41) holds. Therefore, thanks to the fact that

$$
\|(\sigma^a_\delta,u^a_\delta,b^a_\delta)(T^\delta)\|_{L^2}=\delta\|(\tilde\sigma_0,\tilde u_0,\tilde b_0)\|_{L^2}e^{\Lambda T^\delta}=2\varepsilon_0\|(\tilde\sigma_0,\tilde u_0,\tilde b_0)\|_{L^2},
$$

we get from (40) and (42) that

$$
\begin{aligned}
&\|(\sigma^\delta, u^\delta, b^\delta)(T^\delta)\|_{L^2} \geq \|(\sigma^a_\delta, u^a_\delta, b^a_\delta)(T^\delta)\|_{L^2} - \|(\sigma^d, u^d, b^d)(T^\delta)\|_{L^2} \\
&\geq 2\varepsilon_0\|(\tilde{\sigma}_0, \tilde{u}_0, \tilde{b}_0)\|_{L^2} - C_3\delta^{3/2}e^{\frac{3}{2}\Lambda t} \\
&\geq 2\varepsilon_0\|(\tilde{\sigma}_0, \tilde{u}_0, \tilde{b}_0)\|_{L^2} - 2\sqrt{2}C_3\varepsilon_0^{3/2} \geq \varepsilon_0\|(\tilde{\sigma}_0, \tilde{u}_0, \tilde{b}_0)\|_{L^2},
\end{aligned}
$$

which completes the proof of the case (i).

(ii) Now, we consider the case $a_+ > a_-$ and $\frac{d^2}{dy^2}\kappa(\theta_0(y)) \leq 0$ or $0 < \frac{d^2}{dy^2}\kappa(\theta_0(y)) \leq \beta_0$, with $\beta_0 > 0$ small enough constant. We will prove that the equilibrium state $(\theta_0(y), 0, 0)$ is nonlinearly asymptotically stable.

First, it follows from (4) that

$$
\begin{cases}
\kappa(\theta_0)\partial_t\sigma^\delta + \kappa(\theta_0)u^\delta\cdot\nabla\sigma^\delta - \kappa(\theta_0)\mathrm{div}(\kappa(\theta_0)\nabla\sigma^\delta) \\
\qquad\qquad = \kappa(\theta_0)\partial_y\left(\sigma^\delta\frac{d}{dy}\kappa(\theta_0)\right) - u_2^\delta + \kappa(\theta_0)F^\delta, \\
\partial_t u^\delta + u^\delta\cdot\nabla u^\delta - \mathrm{div}(\mu(\theta_0)\nabla u^\delta) + \nabla p^\delta = \sigma^\delta e_2 + b^\delta\cdot\nabla b^\delta + G^\delta, \\
\partial_t b^\delta + u^\delta\cdot\nabla b^\delta - \nabla^\perp(\gamma(\theta_0)\nabla^\perp\cdot b^\delta) = b^\delta\cdot\nabla u^\delta + H^\delta, \\
\mathrm{div}\, u^\delta = 0,\ \mathrm{div}\, b^\delta = 0
\end{cases}
\tag{43}
$$

with

$$
\begin{cases}
F^\delta = \mathrm{div}([\kappa(\theta_0+\sigma^\delta) - \kappa(\theta_0)]\nabla\sigma^\delta) + \partial_y\left(\frac{\kappa(\theta_0+\sigma^\delta)-\kappa(\theta_0)-\sigma^\delta\kappa'(\theta_0)}{\kappa(\theta_0)}\right), \\
G^\delta = \mathrm{div}([\mu(\theta_0+\sigma^\delta) - \mu(\theta_0)]\nabla u^\delta), \\
H^\delta = \nabla^\perp([\gamma(\theta_0+\sigma^\delta) - \gamma(\theta_0)]\nabla^\perp\cdot b^\delta).
\end{cases}
$$

Multiplying the three equations in (43) by σ^δ, u^δ, and b^δ, respectively, then using integration by parts, we can get that

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega(\kappa(\theta_0)|\sigma^\delta|^2 + |u^\delta|^2 + |b^\delta|^2)\,dxdy + \int_\Omega[\kappa^2(\theta_0)|\nabla\sigma^\delta|^2 + \mu(\theta_0)|\nabla u^\delta|^2 \\
&\quad + \gamma(\theta_0)|\nabla^\perp\cdot b^\delta|^2]\,dxdy \\
&= \int_\Omega(\sigma^\delta)^2\,\kappa(\theta_0)\frac{d^2}{dy^2}(\kappa(\theta_0))\,dxdy + \int_\Omega\kappa(\theta_0)F^\delta\,\sigma^\delta\,dxdy + \int_\Omega G^\delta\cdot u^\delta\,dxdy \\
&\quad + \int_\Omega H^\delta\cdot b^\delta\,dxdy - \int_\Omega\kappa(\theta_0)\sigma^\delta u^\delta\cdot\nabla\sigma^\delta\,dxdy - \int_\Omega\Big[u^\delta\cdot(u^\delta\cdot\nabla u^\delta) \\
&\quad - u^\delta\cdot(b^\delta\cdot\nabla b^\delta) + b^\delta\cdot(u^\delta\cdot\nabla b^\delta) - b^\delta\cdot(b^\delta\cdot\nabla u^\delta)\Big]\,dxdy.
\end{aligned}
\tag{44}
$$

Notice that

$$
\begin{aligned}
|\int_\Omega\kappa(\theta_0)\sigma^\delta u^\delta\cdot\nabla\sigma^\delta\,dxdy| &\leq C\|\sigma^\delta\|_{L^4}\|u^\delta\|_{L^4}\|\nabla\sigma^\delta\|_{L^2} \\
&\leq C\|\sigma^\delta\|_{L^2}^{\frac{1}{2}}\|u^\delta\|_{L^2}^{\frac{1}{2}}\|\nabla\sigma^\delta\|_{L^2}^{\frac{1}{2}}\|\nabla u^\delta\|_{L^2}^{\frac{1}{2}}\|\nabla\sigma^\delta\|_{L^2},
\end{aligned}
$$

we get

$$
|\int_\Omega\kappa(\theta_0)\sigma^\delta u^\delta\cdot\nabla\sigma^\delta\,dxdy| \lesssim \|(\sigma^\delta, u^\delta)\|_{L^2}\|(\nabla\sigma^\delta, \nabla u^\delta)\|_{L^2}^2. \tag{45}
$$

Thanks to the fact that $\mathrm{div}\, u^\delta = \mathrm{div}\, b^\delta = 0$, we have

$$
\int_\Omega[u^\delta\cdot(u^\delta\cdot\nabla u^\delta) - u^\delta\cdot(b^\delta\cdot\nabla b^\delta) + b^\delta\cdot(u^\delta\cdot\nabla b^\delta) - b^\delta\cdot(b^\delta\cdot\nabla u^\delta)]\,dxdy = 0. \tag{46}
$$

By using integration by parts, one can control the term

$$\Big|\int_\Omega \sigma^\delta \kappa(\theta_0)\mathrm{div}([\kappa(\theta_0+\sigma^\delta)-\kappa(\theta_0)]\nabla\sigma^\delta)\,dxdy\Big|$$

by

$$\Big|\int_\Omega \Big([\kappa(\theta_0+\sigma^\delta)-\kappa(\theta_0)]\nabla\sigma^\delta\cdot\nabla\sigma^\delta\kappa(\theta_0)+[\kappa(\theta_0+\sigma^\delta)-\kappa(\theta_0)]\partial_y\sigma^\delta\sigma^\delta\frac{d}{dy}\kappa(\theta_0)\Big)\,dxdy\Big|,$$

which implies that

$$\begin{aligned}
&|\int_\Omega \sigma^\delta \kappa(\theta_0)\mathrm{div}([\kappa(\theta_0+\sigma^\delta)-\kappa(\theta_0)]\nabla\sigma^\delta)\,dxdy|\\
&\lesssim \|\kappa(\theta_0+\sigma^\delta)-\kappa(\theta_0)\|_{L^4}(\|\nabla\sigma^\delta\|_{L^4}\|\nabla\sigma^\delta\|_{L^2}+\|\partial_y\sigma^\delta\|_{L^2}\|\sigma^\delta\|_{L^4})\\
&\lesssim \|\sigma^\delta\|_{L^4}\|\nabla\sigma^\delta\|_{L^4}\|\nabla\sigma^\delta\|_{L^2}+\|\partial_y\sigma^\delta\|_{L^2}\|\sigma^\delta\|_{L^4}^2\\
&\lesssim C_\eta\|\sigma^\delta\|_{H^1}\|\nabla\sigma^\delta\|_{L^2}^2+\eta\|\nabla^2\sigma^\delta\|_{L^2}^2+C_\eta\|\sigma^\delta\|_{L^2}^2\|\nabla\sigma^\delta\|_{L^2}^2.
\end{aligned} \tag{47}$$

Similarly, it follows from integration by parts that

$$\begin{aligned}
&|\int_\Omega \mathrm{div}([\mu(\theta_0+\sigma^\delta)-\mu(\theta_0)]\nabla u^\delta])\cdot u^\delta\,dxdy| \le \|\mu(\theta_0+\sigma^\delta)-\mu(\theta_0)\|_{L^2}\|\nabla u^\delta\|_{L^4}^2\\
&\lesssim \|\sigma^\delta\|_{L^2}\|\nabla u^\delta\|_{L^2}\|\nabla^2 u^\delta\|_{L^2}\lesssim \eta\|\nabla^2u^\delta\|_{L^2}^2+C_\eta\|\sigma^\delta\|_{L^2}^2\|\nabla u^\delta\|_{L^2}^2,
\end{aligned} \tag{48}$$

and

$$\begin{aligned}
&|\int_\Omega \nabla^\perp([\gamma(\theta_0+\sigma^\delta)-\gamma(\theta_0)]\nabla^\perp\cdot b^\delta])\cdot b^\delta\,dxdy| \le \|\gamma(\theta_0+\sigma^\delta)-\gamma(\theta_0)\|_{L^2}\|\nabla^\perp\cdot b^\delta\|_{L^4}^2\\
&\lesssim \|\sigma^\delta\|_{L^2}\|\nabla^\perp\cdot b^\delta\|_{L^2}\|\nabla\nabla^\perp\cdot b^\delta\|_{L^2}\lesssim \eta\|\nabla\nabla^\perp\cdot b^\delta\|_{L^2}^2+C_\eta\|\sigma^\delta\|_{L^2}^2\|\nabla^\perp\cdot b^\delta\|_{L^2}^2,
\end{aligned} \tag{49}$$

Notice that

$$\begin{aligned}
&\int_\Omega \sigma^\delta\kappa(\theta_0)\partial_y\Big(\frac{\kappa(\theta_0+\sigma^\delta)-\kappa(\theta_0)-\sigma^\delta\kappa'(\theta_0)}{\kappa(\theta_0)}\Big)\,dxdy\\
&=-\int_\Omega[\kappa(\theta_0+\sigma^\delta)-\kappa(\theta_0)-\sigma^\delta\kappa'(\theta_0)]\partial_y\sigma^\delta\,dxdy\\
&-\int_\Omega\frac{\kappa(\theta_0+\sigma^\delta)-\kappa(\theta_0)-\sigma^\delta\kappa'(\theta_0)}{\kappa(\theta_0)}\sigma^\delta\frac{d}{dy}\kappa(\theta_0)\,dxdy.
\end{aligned} \tag{50}$$

Therefore, we have

$$\begin{aligned}
&\Big|\int_\Omega \sigma^\delta\kappa(\theta_0)\partial_y\Big(\frac{\kappa(\theta_0+\sigma^\delta)-\kappa(\theta_0)-\sigma^\delta\kappa'(\theta_0)}{\kappa(\theta_0)}\Big)\,dxdy\Big|\\
&\lesssim \|\kappa(\theta_0+\sigma^\delta)-\kappa(\theta_0)-\sigma^\delta\kappa'(\theta_0)\|_{L^2}(\|\nabla\sigma^\delta\|_{L^2}+\|\sigma^\delta\|_{L^2})\\
&\lesssim \|\sigma^\delta\|_{L^4}^2(\|\nabla\sigma^\delta\|_{L^2}+\|\sigma^\delta\|_{L^2})\lesssim \|\sigma^\delta\|_{L^2}\|\nabla\sigma^\delta\|_{L^2}^2+\|\sigma^\delta\|_{L^2}^2\|\nabla\sigma^\delta\|_{L^2}\\
&\lesssim \|\sigma^\delta\|_{L^2}^2\|\nabla\sigma^\delta\|_{L^2}^2.
\end{aligned} \tag{51}$$

On the other hand, by Poincare inequality for a strip, it follows from physical condition (2) and the assumption in (ii) of Theorem 1 that

$$\Big|\int_\Omega(\sigma^\delta)^2\,\kappa(\theta_0)\frac{d^2}{dy^2}(\kappa(\theta_0))\,dxdy\Big|\lesssim\beta_0\int_\Omega\kappa^2(\theta_0)|\nabla\sigma^\delta|^2\,dxdy.$$

Therefore, plugging (45)-(51) into (44), for β_0 small enough, we can get

$$\begin{aligned}
&\frac{d}{dt}\|(\sqrt{\kappa(\theta_0)}\sigma^\delta,u^\delta,b^\delta)\|_{L^2}^2+c_0\|(\nabla\sigma^\delta,\nabla u^\delta,\nabla b^\delta)\|_{L^2}^2\\
&\lesssim(\|u^\delta\|_{L^2}+\|\sigma^\delta\|_{L^2}^2+\|\sigma^\delta\|_{H^1})\|(\nabla\sigma^\delta,\nabla u^\delta,\nabla b^\delta)\|_{L^2}^2+\eta\|(\nabla^2u^\delta,\nabla^2\sigma^\delta,\nabla^2b^\delta)\|_{L^2}^2
\end{aligned} \tag{52}$$

for some positive constant c_0.

In order to get the H^1 estimate, one can apply the operator ∂_i (with $i = 1, 2$ and $\partial_1 = \partial_x,\ \ \partial_2 = \partial_y$) to the equations (4) to get

$$\begin{cases} \partial_t\partial_i\sigma^\delta + u^\delta\cdot\nabla\partial_i\sigma^\delta + \partial_i u^\delta\cdot\nabla\sigma^\delta - \mathrm{div}(\kappa(\theta_0)\nabla\partial_i\sigma^\delta) - \mathrm{div}(\partial_i\kappa(\theta_0)\nabla\sigma^\delta) \\ \qquad = \partial_y\partial_i\Big(\sigma^\delta\frac{d}{dy}\kappa(\theta_0)\Big) - \partial_i\Big(\frac{1}{\kappa(\theta_0)}u_2^\delta\Big) + \partial_i F^\delta, \\ \partial_t\partial_i u^\delta + u^\delta\cdot\nabla\partial_i u^\delta + \partial_i u^\delta\cdot\nabla u^\delta - \mathrm{div}(\mu(\theta_0)\nabla\partial_i u^\delta) - \mathrm{div}(\partial_i\mu(\theta_0)\nabla u^\delta) + \nabla\partial_i p^\delta \\ \qquad = b^\delta\cdot\nabla\partial_i b^\delta + \partial_i b^\delta\cdot\nabla b^\delta + \partial_i\sigma^\delta e_2 + \partial_i G^\delta, \\ \partial_t\partial_i b^\delta + u^\delta\cdot\nabla\partial_i b^\delta + \partial_i u^\delta\cdot\nabla b^\delta - \nabla^\perp(\gamma(\theta_0)\partial_i\nabla^\perp\cdot b^\delta) - \nabla^\perp(\partial_i\gamma(\theta_0)\nabla^\perp\cdot b^\delta) \\ \qquad = b^\delta\cdot\nabla\partial_i u^\delta + \partial_i b^\delta\cdot\nabla u^\delta + \partial_i H^\delta. \end{cases} \tag{53}$$

Multiplying the three equation in (53) by $\partial_i\sigma^\delta$, $\partial_i u^\delta$, and $\partial_i b^\delta$ respectively, then using integration by parts, we can obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_\Omega |(\nabla\sigma^\delta,\nabla u^\delta,\nabla b^\delta)|^2 dxdy + \int_\Omega \Big[\kappa(\theta_0)|\nabla^2\sigma^\delta|^2 + \mu(\theta_0)|\nabla^2 u^\delta|^2 + \gamma(\theta_0)|\nabla\nabla^\perp\cdot b^\delta|^2\Big]dxdy \\ &= -\int_\Omega \Big[\frac{d\kappa(\theta_0)}{dy}\nabla\sigma^\delta\cdot\nabla\partial_y\sigma^\delta + \frac{d\mu(\theta_0)}{dy}\nabla u^\delta\cdot\nabla\partial_y u^\delta + \frac{d\gamma(\theta_0)}{dy}(\nabla^\perp\cdot b^\delta)(\nabla^\perp\cdot\partial_y b^\delta)\Big]dxdy \\ &\quad + \sum_{i=1}^2\int_\Omega (\partial_i b^\delta\cdot\nabla b^\delta\cdot\partial_i u^\delta + \partial_i b^\delta\cdot\nabla u^\delta\cdot\partial_i b^\delta - \partial_i u^\delta\cdot\nabla b^\delta\cdot\partial_i b^\delta)dxdy \\ &\quad - \sum_{i=1}^2\int_\Omega \Big\{\Big[\partial_y\Big(\sigma^\delta\frac{d}{dy}\kappa(\theta_0)\Big) - \frac{u_2^\delta}{\kappa(\theta_0)} + F^\delta\Big]\partial_i^2\sigma^\delta + (\sigma^\delta e_2 + G^\delta)\cdot\partial_i^2 u^\delta\Big\}dxdy \\ &\quad - \sum_{i=1}^2\int_\Omega \partial_i u^\delta\cdot\nabla u^\delta\cdot\partial_i u^\delta dxdy - \sum_{i=1}^2\int_\Omega(\partial_i u^\delta\cdot\nabla\sigma^\delta)\partial_i\sigma^\delta dxdy. \end{aligned} \tag{54}$$

Thanks to the boundedness of $\kappa(\theta_0)$, $\mu(\theta_0)$, $\gamma(\theta_0)$ and their derivatives, it follows from Hölder inequality and Young inequality that

$$\begin{aligned} &\Big|\int_\Omega \Big[\frac{d\kappa(\theta_0)}{dy}\nabla\sigma^\delta\cdot\nabla\partial_y\sigma^\delta + \frac{d\mu(\theta_0)}{dy}\nabla u^\delta\cdot\nabla\partial_y u^\delta + \frac{d\gamma(\theta_0)}{dy}(\nabla^\perp\cdot b^\delta)\cdot(\nabla^\perp\cdot\partial_y b^\delta)\Big]dxdy\Big| \\ &\quad\lesssim \|(\nabla\sigma^\delta,\nabla u^\delta,\nabla b^\delta)\|_{L^2}\|(\nabla^2\sigma^\delta,\nabla^2 u^\delta,\nabla^2 b^\delta)\|_{L^2} \\ &\quad\lesssim \eta\|(\nabla^2\sigma^\delta,\nabla^2 u^\delta,\nabla^2 b^\delta)\|_{L^2}^2 + C_\eta\|(\nabla\sigma^\delta,\nabla u^\delta,\nabla b^\delta)\|_{L^2}^2. \end{aligned} \tag{55}$$

Similarly, we can prove

$$\begin{aligned} \Big|\sum_{i=1}^2\int_\Omega \partial_i u^\delta\cdot\nabla u^\delta\cdot\partial_i u^\delta\, dxdy\Big| &\lesssim \|\nabla u^\delta\|_{L^4}^2\|\nabla u^\delta\|_{L^2} \lesssim \|\nabla u^\delta\|_{L^2}^2\|\nabla^2 u^\delta\|_{L^2} \\ &\lesssim C_\eta\|\nabla u^\delta\|_{L^2}^4 + \eta\|\nabla^2 u^\delta\|_{L^2}^2, \end{aligned} \tag{56}$$

$$\Big|\sum_{i=1}^2\int_\Omega \partial_i u^\delta\cdot\nabla\sigma^\delta\cdot\partial_i\sigma^\delta dxdy\Big| \lesssim C_\eta\|\nabla\sigma^\delta\|_{L^2}^2\|\nabla u^\delta\|_{L^2}^2 + \eta\|\nabla^2\sigma^\delta\|_{L^2}^2, \tag{57}$$

$$\begin{aligned} &\Big|\sum_{i=1}^2\int_\Omega(\partial_i b^\delta\cdot\nabla b^\delta\cdot\partial_i u^\delta + \partial_i b^\delta\cdot\nabla u^\delta\cdot\partial_i b^\delta - \partial_i u^\delta\cdot\nabla b^\delta\cdot\partial_i b^\delta)dxdy\Big| \\ &\lesssim C_\eta\|\nabla b^\delta\|_{L^2}^2\|\nabla u^\delta\|_{L^2}^2 + \eta\|\nabla^2 b^\delta\|_{L^2}^2. \end{aligned} \tag{58}$$

On the other hand, direct estimates give that

$$\begin{aligned}
&\Big|\sum_{i=1}^{2}\int_{\Omega}\Big\{\Big[\partial_y\Big(\sigma^\delta\frac{d}{dy}\kappa(\theta_0)\Big)-\frac{u_2^\delta}{\kappa(\theta_0)}+F^\delta\Big]\partial_i^2\sigma^\delta+(\sigma^\delta e_2+G^\delta)\cdot\partial_i^2u^\delta+H^\delta\cdot\partial_i^2b^\delta\Big\}dxdy\Big|\\
&\lesssim\|(\nabla^2\sigma^\delta,\nabla^2u^\delta,\nabla^2b^\delta)\|_{L^2}(\|F^\delta\|_{L^2}+\|G^\delta\|_{L^2}+\|H^\delta\|_{L^2}+\|(\sigma^\delta,u^\delta)\|_{L^2}+\|\nabla\sigma^\delta\|_{L^2})\\
&\lesssim\eta\|(\nabla^2\sigma^\delta,\nabla^2u^\delta,\nabla^2b^\delta)\|_{L^2}^2+C_\eta(\|F^\delta\|_{L^2}^2+\|G^\delta\|_{L^2}^2+\|H^\delta\|_{L^2}^2+\|(\nabla\sigma^\delta,\nabla u^\delta)\|_{L^2}^2),
\end{aligned}\tag{59}$$

$$\begin{aligned}
&\|F^\delta\|_{L^2}+\|G^\delta\|_{L^2}+\|H^\delta\|_{L^2}\\
&\lesssim\|\nabla\sigma^\delta\|_{L^4}\|(\nabla\sigma^\delta,\nabla u^\delta,\nabla b^\delta)\|_{L^4}+\|\sigma^\delta\|_{L^\infty}\|(\nabla^2\sigma^\delta,\nabla^2u^\delta,\nabla^2b^\delta)\|_{L^2}\\
&\quad+\|\sigma^\delta\|_{L^4}^2+\|\nabla\sigma^\delta\|_{L^2},\\
&\lesssim(\|(\nabla\sigma^\delta,\nabla u^\delta,\nabla b^\delta)\|_{L^2}+\|\sigma^\delta\|_{L^2}^{\frac12}\|\nabla^2\sigma^\delta\|_{L^2}^{\frac12})\|(\nabla^2\sigma^\delta,\nabla^2u^\delta,\nabla^2b^\delta)\|_{L^2}\\
&\quad+(1+\|\sigma^\delta\|_{L^2})\|\nabla\sigma^\delta\|_{L^2}.
\end{aligned}\tag{60}$$

Therefore, substituting (55)-(60) to (54)), we can show that, for some positive constant c_1,

$$\begin{aligned}
&\frac{d}{dt}\|(\nabla\sigma^\delta,\nabla u^\delta,\nabla b^\delta)\|_{L^2}^2+c_1\|(\nabla^2\sigma^\delta,\nabla^2u^\delta,\nabla^2b^\delta)\|_{L^2}^2\\
&\lesssim(1+\|\sigma^\delta\|_{L^2}^2+\|\nabla u^\delta\|_{L^2}^2)\|(\nabla\sigma^\delta,\nabla u^\delta,\nabla b^\delta)\|_{L^2}^2\\
&\quad+(\|(\nabla\sigma^\delta,\nabla u^\delta,\nabla b^\delta)\|_{L^2}^2+\|\sigma^\delta\|_{L^2}\|\nabla^2\sigma^\delta\|_{L^2})\|(\nabla^2\sigma^\delta,\nabla^2u^\delta,\nabla^2b^\delta)\|_{L^2}^2.
\end{aligned}\tag{61}$$

Similarly, we can get for some positive constant c_2,

$$\begin{aligned}
&\frac{d}{dt}\|(\nabla^2\sigma^\delta,\nabla^2u^\delta,\nabla^2b^\delta)\|_{L^2}^2+c_2\|(\nabla^3\sigma^\delta,\nabla^3u^\delta,\nabla^3b^\delta)\|_{L^2}^2\\
&\lesssim(1+\|(\nabla\sigma^\delta,\nabla u^\delta,\nabla b^\delta)\|_{L^2}^2)\|(\nabla^2\sigma^\delta,\nabla^2u^\delta,\nabla^2b^\delta)\|_{L^2}^2\\
&\quad+\|(\nabla^2\sigma^\delta,\nabla^2u^\delta,\nabla^2b^\delta)\|_{L^2}^2\|(\nabla^3\sigma^\delta,\nabla^3u^\delta,\nabla^3b^\delta)\|_{L^2}^2.
\end{aligned}\tag{62}$$

Hence, it follows from (52), (61), and (62) that for some positive constant c_3,

$$\begin{aligned}
&\frac{d}{dt}\|(\sigma^\delta,u^\delta,b^\delta)\|_{H^2}^2+c_3\|(\nabla\sigma^\delta,\nabla u^\delta,\nabla b^\delta)\|_{H^2}^2\\
&\le C_4\|(\sigma^\delta,u^\delta,b^\delta)\|_{H^2}^2\|(\nabla\sigma^\delta,\nabla u^\delta,\nabla b^\delta)\|_{H^2}^2.
\end{aligned}\tag{63}$$

By a bootstrap argument, we can obtain that there is a positive constant $\delta_0\le\sqrt{\frac{c_3}{2C_4}}$, such that, for any $\delta\in(0,\delta_0)$, if $\|(\sigma_0^\delta,u_0^\delta,b_0^\delta)\|_{H^2}\le\delta$, then

$$\|(\sigma^\delta,u^\delta,b^\delta)\|_{L_t^\infty(H^2)}^2+\frac{c_3}{2}\|(\nabla\sigma^\delta,\nabla u^\delta,\nabla b^\delta)\|_{L_t^2(H^2)}^2\le\|(\sigma_0^\delta,u_0^\delta,b_0^\delta)\|_{H^2}^2\le\delta^2.$$

It follows from this and (63) that

$$\frac{d}{dt}\|(\sigma^\delta,u^\delta,b^\delta)\|_{H^2}^2+\frac{c_3}{2}\|(\nabla\sigma^\delta,\nabla u^\delta,\nabla b^\delta)\|_{H^2}^2\le0,$$

which, together with Poincare inequality, implies that

$$\frac{d}{dt}\|(\sigma^\delta,u^\delta,b^\delta)\|_{H^2}^2+c_4\|(\sigma^\delta,u^\delta,b^\delta)\|_{H^2}^2\le0\tag{64}$$

for some positive constant c_4. Therefore, (64) implies that, for any $t > 0$

$$\|(\sigma^\delta, u^\delta, b^\delta)(t)\|_{H^2} \leq \|(\sigma_0^\delta, u_0^\delta, b_0^\delta)\|_{H^2} e^{-c_4 t},$$

which gives (5). This finishes the proof. □

6. Conclusions

In this paper, we consider the 2D Boussinesq-MHD equations with the temperature-dependent fluid viscosity, thermal diffusivity and electrical conductivity in a channel. We get that if $a_+ \geq a_-$, and $\frac{d^2}{dy^2}\kappa(\theta_0(y)) \leq 0$ or $0 < \frac{d^2}{dy^2}\kappa(\theta_0(y)) \leq \beta_0$, with $\beta_0 > 0$ small enough constant, and then the equilibrium state ($\overline{u} = 0$, $\overline{B} = 0$, $\overline{\theta} = \theta_0(y)$) is nonlinearly asymptotically stable, and, if $a_+ < a_-$, then the equilibrium state ($\overline{u} = 0$, $\overline{B} = 0$, $\overline{\theta} = \theta_0(y)$) is nonlinearly unstable. There is one open interesting problem. How about the equilibirum state ($\overline{u} = (y, 0)$, $\overline{B} = (y, 0)$, $\overline{\theta} = \theta_0(y)$) or the equilibirum state ($\overline{u} = (y, 0)$, $\overline{B} = (y, 0)$, $\overline{\theta} = 0$)? We will consider this problem in another paper.

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