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Some Properties of Extended Euler's Function and Extended Dedekind's Function

Nicușor Minculete ^{1,*} and Diana Savin ²

¹ Faculty of Mathematics and Computer Science, Transilvania University of Brașov, Iuliu Maniu Street 50, 500091 Brașov, Romania

² Faculty of Mathematics and Computer Science, Ovidius University, Bd. Mamaia 124, 900527 Constanța, Romania; savin.diana@univ-ovidius.ro or dianet72@yahoo.com

* Correspondence: minculeten@yahoo.com

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Abstract: In this paper, we find some properties of Euler's function and Dedekind's function. We also generalize these results, from an algebraic point of view, for extended Euler's function and extended Dedekind's function, in algebraic number fields. Additionally, some known inequalities involving Euler's function and Dedekind's function, we generalize them for extended Euler's function and extended Dedekind's function, working in a ring of integers of algebraic number fields.

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1. Introduction and Preliminaries

Let Euler's function $\varphi : \mathbb{N}^* \rightarrow \mathbb{N}^*$, $\varphi(n) = |\{k \in \mathbb{N}^* | k \leq n, (k, n) = 1\}|$, (\forall) $n \in \mathbb{N}^*$. φ is sometimes called Euler's totient function. It is known that if $n, r \in \mathbb{N}, n \geq 2, r \geq 1, n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdot \dots \cdot p_r^{\alpha_r}$, where $p_1, p_2, \dots, p_r \in \mathbb{N}$ are distinct prime numbers and $\alpha_1, \alpha_2, \dots, \alpha_r \in \mathbb{N}^*$, then $\varphi(n) = n \cdot \left(1 - \frac{1}{p_1}\right) \cdot \left(1 - \frac{1}{p_2}\right) \cdot \dots \cdot \left(1 - \frac{1}{p_r}\right)$. Also, $\varphi(n)$ gives the number of invertible elements in $\mathbb{Z}/n\mathbb{Z}$. It is known that φ is multiplicative function, but it is not a completely multiplicative function.

We now give a few examples on the evaluation of $\varphi(n) : \varphi(2) = 1, \varphi(p) = p - 1$, for any prime integer $p, \varphi(36) = \varphi(2^2 \cdot 3^2) = 36 \cdot \left(1 - \frac{1}{2}\right) \cdot \left(1 - \frac{1}{3}\right) = 12$.

In [1], for an integer $n \not\equiv 2 \pmod{4}$, with ζ_n denote a primitive n -th root of unity, if we consider the n -th cyclotomic field $K = \mathbb{Q}(\zeta_n)$ we have that the fields extension $\mathbb{Q} \subset K$ is a Galois extension, of degree $[K : \mathbb{Q}] = \varphi(n)$.

The function $\psi : \mathbb{N}^* \rightarrow \mathbb{N}^*, \psi(1) = 1, \psi(n) = n \cdot \left(1 + \frac{1}{p_1}\right) \cdot \left(1 + \frac{1}{p_2}\right) \cdot \dots \cdot \left(1 + \frac{1}{p_r}\right)$, for $n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdot \dots \cdot p_r^{\alpha_r} \in \mathbb{N}, n \geq 2$ is called *Dedekind's function* (being discovered by Richard Dedekind).

In [2] Solé and Planat introduced function ψ_t as a generalization of the Dedekind ψ function, defined by

$$\psi_t(n) := n \prod_{p|n} \left(1 + 1/p + \dots + 1/p^{t-1}\right),$$

for any integer $t \geq 2$ ($\psi_2(n) = \psi(n)$).

Let K be an algebraic number field of degree $[K : \mathbb{Q}] = n$, where $n \in \mathbb{N}, n \geq 2$. We denote by \mathcal{O}_K the ring of integers of the field K . We denote by $\text{Spec}(\mathcal{O}_K)$ the set of the prime ideals of the ring \mathcal{O}_K . It is known that \mathcal{O}_K is a Dedekind domain. Let I be an ideal of \mathcal{O}_K . It is known that Euler's function and Dedekind's function were extended to the set of the ideals of the ring of integers \mathcal{O}_K . We denote this set by \mathbb{J} . Accordingly, the extended Euler's function and the extended Dedekind's function $\varphi_{ext}; \psi_{ext} :$

$\mathbb{J} \rightarrow \mathbb{N}^*$. These functions have been introduced, while taking into account that Dedekind domains have the factorization theorem for ideals analogous with the Fundamental theorem of arithmetic. Applying the fundamental theorem of Dedekind rings, there exist positive integers r and $\alpha_i, i = \overline{1, r}$ and the different prime ideals P_1, P_2, \dots, P_r in the ring \mathcal{O}_K such that

$$I = P_1^{\alpha_1} \cdot P_2^{\alpha_2} \cdot \dots \cdot P_r^{\alpha_r}.$$

This decomposition is unique, except the order of factors.

In [3], Miguel defined the extended Euler totient function type for a non-zero ideal of a Dedekind domain, because the factorization of ideals is unique. He extended Menon’s identity to residually finite Dedekind domains (rings of finite norm property).

The extended Euler’s function for an ideal I of the ring \mathcal{O}_K is defined, as follows:

$$\varphi_{ext}(I) = N(I) \cdot \left(1 - \frac{1}{N(P_1)}\right) \cdot \left(1 - \frac{1}{N(P_2)}\right) \cdot \dots \cdot \left(1 - \frac{1}{N(P_r)}\right),$$

where, by $N(I)$, we meant the norm of the ideal I . We recall that, by definition $N(I) = [\mathcal{O}_K : I]$.

Additionally, $\varphi_{ext}(I)$ gives the number of invertible elements of the factor ring \mathcal{O}_K/I .

If I and J are nonzero ideals of the ring \mathcal{O}_K such that $I + J = \mathcal{O}_K$, then

$$\varphi_{ext}(I \cdot J) = \varphi_{ext}(I) \cdot \varphi_{ext}(J).$$

The extended Dedekind’s function for an ideal I of the ring \mathcal{O}_K is defined, as follows:

$$\psi_{ext}(I) = N(I) \cdot \left(1 + \frac{1}{N(P_1)}\right) \cdot \left(1 + \frac{1}{N(P_2)}\right) \cdot \dots \cdot \left(1 + \frac{1}{N(P_r)}\right)$$

(see [1,4–7]).

Other extended arithmetic functions in algebraic number fields can be found in [8].

We now recall some properties of the norm of an ideal, properties that we will use in proving our results.

Proposition 1. ([1,7,9]). *Let K be an algebraic number field. Then:*

$$N(I \cdot J) = N(I) \cdot N(J),$$

for (\forall) nonzero ideals I, J of the ring \mathcal{O}_K .

Proposition 2. ([1,7,9]). *Let K be an algebraic number field. If I is an ideal of ring \mathcal{O}_K , such that $N(I)$ is a prime number, then $I \in \text{Spec}(\mathcal{O}_K)$.*

We now consider the Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} = \prod_{p \text{ prime}} \frac{1}{1 - \frac{1}{p^s}}, \text{ where } s \in \mathbb{C}, \text{ Re}(s) > 1.$$

The Dedekind zeta function of a number field K

$$\zeta_K(s) = \sum_I N(I)^{-s},$$

where the sum is over all ideals $I \neq (0)$ of the ring \mathcal{O}_K . The function is defined for all complex numbers s with $\text{Re}(s) > 1$.

This function can also be written as a Eulerian product, as follows:

$$\zeta_K(s) = \prod_{P \in \text{Spec}(\mathcal{O}_K)} \frac{1}{1 - \frac{1}{N(P)^s}}, \text{ where } s \in \mathbb{C}, \text{ Re}(s) > 1.$$

Proposition 3. ([10]). Let the quadratic field $K = \mathbb{Q}(\sqrt{5})$ and let ζ_K be the Dedekind zeta function of the quadratic field K . Subsequently: $\zeta_K(2) = \frac{2\pi^4}{75\sqrt{5}}$.

Now, we recall a result about the quadratic fields; we will use this result in proving our results.

Proposition 4. ([1,4,7]). Let a quadratic field $K = \mathbb{Q}(\sqrt{d})$, where $d \neq 0, 1$ is a square free integer and let \mathcal{O}_K be the ring of integers of the quadratic field K . Afterwards, we have:

(i) $\mathcal{O}_K = \mathbb{Z}[\sqrt{d}]$, if $d \equiv 2, 3 \pmod{4}$;

(ii) $\mathcal{O}_K = \mathbb{Z}[\frac{1+\sqrt{d}}{2}]$, if $d \equiv 1 \pmod{4}$.

2. Some Results about Euler’s Function, Dedekind’s Function and Generalized Dedekind’s Function

In [11], Sándor and Atanassov proved an inequality with the arithmetic functions φ and ψ given by

$$n^{\varphi(n)+\psi(n)} < \left(\frac{\varphi(n) + \psi(n)}{2}\right)^{\varphi(n)+\psi(n)} < \varphi(n)^{\varphi(n)}\psi(n)^{\psi(n)},$$

for any integer $n > 1$. The above inequality is an improvement of the inequality given by Kannan and Srikanth [12], thus:

$$n^{\varphi(n)+\psi(n)} < \varphi(n)^{\varphi(n)}\psi(n)^{\psi(n)}. \tag{1}$$

Alzer and Kwong proved [13] another improvement, thus, for all $n \geq 2$, we have

$$\frac{\varphi(n)^{\varphi(n)}\psi(n)^{\psi(n)}}{n^{\varphi(n)+\psi(n)}} \geq (1 - 1/n)^{n-1} (1 + 1/n)^{n+1} \geq 1.$$

Next, we give two refinements of inequality (1).

Proposition 5. For any integer $n > 1$, we have the following inequality:

$$\frac{\varphi(n)^{\varphi(n)}\psi(n)^{\psi(n)}}{n^{\varphi(n)+\psi(n)}} \geq e^{\varphi(n)+\psi(n)-2n} \geq 1.$$

Proof. It is easy to see that $x \log x \geq x - 1$, for every $x > 0$. Therefore, we deduce $a \log a + b \log b \geq a + b - 2$, for all $a, b > 0$. This implies the following inequality:

$$a^a \cdot b^b \geq e^{a+b-2}. \tag{2}$$

In (2), we take $a = \frac{\varphi(n)}{n}$ and $b = \frac{\psi(n)}{n}$ and deduce

$$\frac{\varphi(n)^{\varphi(n)}\psi(n)^{\psi(n)}}{n^{\varphi(n)+\psi(n)}} \geq e^{\varphi(n)+\psi(n)-2n}.$$

It is easy to prove by mathematical induction over $n \in \mathbb{N}, n \geq 2$, that:

$$\frac{\varphi(n)}{n} + \frac{\psi(n)}{n} = \prod_{p \text{ prime}, p|n} (1 - 1/p) + \prod_{p \text{ prime}, p|n} (1 + 1/p) \geq 2.$$

Consequently, we proved the statement. \square

Proposition 6. For any integer $n \geq 1$, we have the following inequality:

$$\frac{\varphi(n)^{\varphi(n)}\psi(n)^{\psi(n)}}{n^{\varphi(n)+\psi(n)}} \geq \left(\frac{\varphi(n)+\psi(n)}{2n}\right)^{\varphi(n)+\psi(n)} \cdot \left(\frac{\varphi(n)+\psi(n)}{2\sqrt{\varphi(n)\cdot\psi(n)}}\right)^{2\cdot\varphi(n)} \geq 1.$$

Proof. In [14], we have the inequality:

$$x^\lambda \cdot y^{1-\lambda} \cdot \left(\frac{x+y}{2\sqrt{xy}}\right)^{2\min\{\lambda,1-\lambda\}} \leq \lambda x + (1-\lambda) \cdot y,$$

for all $x, y > 0$ and $\lambda \in [0, 1]$.

In this inequality we make the substitution $x \rightarrow \frac{1}{x}, y \rightarrow \frac{1}{y}$ and obtain:

$$\left(\frac{x+y}{2\sqrt{xy}}\right)^{2\min\{\lambda,1-\lambda\}} \leq x^\lambda \cdot y^{1-\lambda} \cdot \left(\frac{\lambda}{x} + \frac{1-\lambda}{y}\right).$$

If we take $\lambda = \frac{x}{x+y}$, then, we have $1-\lambda = \frac{y}{x+y}$ and we deduce the following inequality:

$$\left(\frac{x+y}{2\sqrt{xy}}\right)^{2\frac{\min\{x,y\}}{x+y}} \leq (x^x \cdot y^y)^{\frac{1}{x+y}} \cdot \frac{2}{x+y},$$

which is equivalent with

$$\left(\frac{x+y}{2\sqrt{xy}}\right)^{2\min\{x,y\}} \cdot \left(\frac{x+y}{2}\right)^{x+y} \leq x^x \cdot y^y \tag{3}$$

For $x = \frac{\varphi(n)}{n}$ and $y = \frac{\psi(n)}{n}$, we deduce the first part of the inequality of the statement.

Since $\varphi(n) + \psi(n) \geq 2n$ and $\varphi(n) + \psi(n) \geq 2\sqrt{\varphi(n)\cdot\psi(n)}$, for all $n \in \mathbb{N}^*$, and then, we proved the inequality of the statement. \square

Proposition 7. For any integers $t \geq 2$ and $n \geq 2$, we have the following inequality:

$$\psi_t(n)\varphi(n)\zeta(t) \geq n^2.$$

Proof. We have

$$\psi_t(n) = n \prod_{p|n} \left(1 + 1/p + \dots + 1/p^{t-1}\right) = n \prod_{p|n} \frac{1-1/p^t}{1-1/p} = \frac{n^2}{\varphi(n)} \prod_{p|n} (1-1/p^t) \geq \frac{n^2}{\varphi(n)} \frac{1}{\zeta(t)}.$$

Therefore, we deduce the statement. \square

3. Some Results Involving Extended Euler’s Function and Extended Dedekind’s Function

In 1965, Kendall and Osborn ([15]) found the following property of Euler’s function:

$$\varphi(n) \geq \sqrt{n}, (\forall) n \in \mathbb{N}^* \setminus \{2, 6\}.$$

Here, we generalize this result, for extended Euler’s function:

Proposition 8. Let n be a positive integer, $n \geq 2$ and let K be an algebraic number field of degree $[K : \mathbb{Q}] = n$. Subsequently: $\varphi_{\text{ext}}(I) \geq \sqrt{N(I)}$, for (\forall) nonzero ideal I of the ring \mathcal{O}_K with $N(I) \neq 2, 6$.

Proof. Let $I \neq (0)$ be an ideal of \mathcal{O}_K . According to the fundamental theorem about Dedekind rings, $(\exists!) r \in \mathbb{N}^*$, the different ideals $P_1, P_2, \dots, P_r \in \text{Spec}(\mathcal{O}_K)$ and $\alpha_1, \alpha_2, \dots, \alpha_r \in \mathbb{N}^*$, such that

$$I = P_1^{\alpha_1} \cdot P_2^{\alpha_2} \cdot \dots \cdot P_r^{\alpha_r}.$$

Subsequently,

$$\varphi_{ext}(I) = N(I) \cdot \left(1 - \frac{1}{N(P_1)}\right) \cdot \left(1 - \frac{1}{N(P_2)}\right) \cdot \dots \cdot \left(1 - \frac{1}{N(P_r)}\right).$$

First, we prove that, for any nonzero ideal I of the domain \mathcal{O}_K with $N(I) \in \{2, 6\}$, the inequality from the statement is not true.

Case 1: if $N(I) = 2$, according to Proposition 2, it results that $I \in \text{Spec}(\mathcal{O}_K)$, so $\varphi_{ext}(I) = N(I) - 1 = 1 \not\geq \sqrt{N(I)} = \sqrt{2}$.

Case 2: if $N(I) = 6$, according to the fundamental theorem about Dedekind rings, Proposition 1 and Proposition 2, it results that $I = P_1 \cdot P_2$, where $P_1, P_2 \in \text{Spec}(\mathcal{O}_K)$, $P_1 \neq P_2$, $N(P_1) = 2$, $N(P_2) = 3$. Afterwards, we have:

$$\varphi_{ext}(I) = N(I) \cdot \left(1 - \frac{1}{N(P_1)}\right) \cdot \left(1 - \frac{1}{N(P_2)}\right) = 2 \not\geq \sqrt{N(I)} = \sqrt{6}.$$

Case 3: I is an ideal of the domain \mathcal{O}_K , with $N(I) \neq 2, 6$.

Subcase 3.(a): $I = P^m$, where $P \in \text{Spec}(\mathcal{O}_K)$, $m \in \mathbb{N}$, $m \geq 2$, $N(P) = 2$. It results that $N(I) \geq 4$ and this implies:

$$\varphi_{ext}(I) = N(I) \cdot \left(1 - \frac{1}{N(P)}\right) = \frac{1}{2} \cdot N(I) \geq \sqrt{N(I)}.$$

Subcase 3.(b): $I = P^m$, where $P \in \text{Spec}(\mathcal{O}_K)$, $m \in \mathbb{N}^*$, $N(P) \geq 3$. This implies $N(I) \geq 3$ and $1 - \frac{1}{N(P)} \geq \frac{2}{3}$. It results that

$$\varphi_{ext}(I) = N(I) \cdot \left(1 - \frac{1}{N(P)}\right) \geq \sqrt{N(I)}.$$

Subcase 3.(c) (the general subcase): $I = P_1^{\alpha_1} \cdot P_2^{\alpha_2} \cdot \dots \cdot P_r^{\alpha_r}$, where $r \in \mathbb{N}^*$ and P_1, P_2, \dots, P_r are distinct prime ideals of the Dedekind domain \mathcal{O}_K and $\alpha_1, \alpha_2, \dots, \alpha_r \in \mathbb{N}^*$. Applying the multiplicativity of function φ_{ext} and the results of the previous subcases, we obtain:

$$\begin{aligned} \varphi_{ext}(I) &= \varphi_{ext}(P_1) \cdot \varphi_{ext}(P_2) \cdot \dots \cdot \varphi_{ext}(P_r) \geq \\ &\geq \sqrt{N(P_1)} \cdot \sqrt{N(P_2)} \cdot \dots \cdot \sqrt{N(P_r)} = \sqrt{N(I)}. \end{aligned}$$

□

In 1988, Sierpinski and Schinzel ([16–18]) proved the following inequality involving Euler’s function:

$$\varphi(n) \leq n - \sqrt{n}, \quad (\forall) n \in \mathbb{N}^*,$$

where n is non prime. Now, we give a similar result, for extended Euler’s function:

Proposition 9. Let n be a positive integer, $n \geq 2$ and let K be an algebraic number field of degree $[K : \mathbb{Q}] = n$. Then:

$$\varphi_{ext}(I) \leq N(I) - \sqrt{N(I)}, \quad (\forall) I \notin \text{Spec}(\mathcal{O}_K).$$

Proof. Since I is not a prime ideal of the Dedekind \mathcal{O}_K , it results that $(\exists) P_i \in \text{Spec}(\mathcal{O}_K)$, such that $P_i^2 | I$ or $(\exists) P_i, P_j \in \text{Spec}(\mathcal{O}_K), P_i \neq P_j$, such that $P_i \cdot P_j | I$ and $N(P_i) \leq N(P_j)$. In both cases, it results that $N(P_i) \leq \sqrt{N(I)}$. Moreover, if $P \in \text{Spec}(\mathcal{O}_K)$ such that $P | I$, we remark that $(1 - \frac{1}{N(P)}) \in (0, 1)$. Using these, we obtain that:

$$\begin{aligned} \varphi_{\text{ext}}(I) &= N(I) \cdot \left(1 - \frac{1}{N(P_1)}\right) \cdot \left(1 - \frac{1}{N(P_2)}\right) \cdot \dots \cdot \left(1 - \frac{1}{N(P_r)}\right) \leq \\ &\leq N(I) \cdot \left(1 - \frac{1}{N(P_i)}\right) \leq N(I) \cdot \left(1 - \frac{1}{\sqrt{N(I)}}\right) \leq N(I) - \sqrt{N(I)}. \end{aligned}$$

□

Proposition 10. Let n be a positive integer, $n \geq 2$ and let K be an algebraic number field of degree $[K : \mathbb{Q}] = n$. Afterwards:

$$\psi_{\text{ext}}(I) \geq N(I) + \sqrt{N(I)}, (\forall) I \notin \text{Spec}(\mathcal{O}_K).$$

Proof. Because I is not a prime ideal of the Dedekind \mathcal{O}_K , it results that $(\exists) P_i \in \text{Spec}(\mathcal{O}_K)$ such that $P_i^2 | I$ or $(\exists) P_i, P_j \in \text{Spec}(\mathcal{O}_K), P_i \neq P_j$ such that $P_i \cdot P_j | I$ and $N(P_i) \leq N(P_j)$. So, we obtain that $N(P_i) \leq \sqrt{N(I)}$. Therefore, we have that:

$$\begin{aligned} \psi_{\text{ext}}(I) &= N(I) \cdot \left(1 + \frac{1}{N(P_1)}\right) \cdot \left(1 + \frac{1}{N(P_2)}\right) \cdot \dots \cdot \left(1 + \frac{1}{N(P_r)}\right) \geq \\ &\geq N(I) \cdot \left(1 + \frac{1}{\sqrt{N(I)}}\right) \geq N(I) + \sqrt{N(I)}. \end{aligned}$$

□

In 1940, T. Popovici ([19]) found the following inequality about Euler’s function:

$$\varphi^2(a \cdot b) \leq \varphi(a^2) \cdot \varphi(b^2) (\forall) a, b \in \mathbb{N}^*.$$

Now, we give a similar result, for extended Euler’s function:

Proposition 11. Let n be a positive integer, $n \geq 2$ and let K be an algebraic number field of degree $[K : \mathbb{Q}] = n$. Subsequently:

$$\varphi_{\text{ext}}^2(I \cdot J) \leq \varphi_{\text{ext}}(I^2) \cdot \varphi_{\text{ext}}(J^2), (\forall) \text{ ideals } I \text{ and } J \text{ of } \mathcal{O}_K.$$

Proof. Let I and J be two ideals of the domain \mathcal{O}_K . Since \mathcal{O}_K is a Dedekind ring, according to the fundamental theorem of Dedekind rings, $(\exists!) r, s, g \in \mathbb{N}^*$, the different ideals $P_1, P_2, \dots, P_r, P'_1, P'_2, \dots, P'_s, P''_1, P''_2, \dots, P''_g \in \text{Spec}(\mathcal{O}_K)$ and $\alpha_1, \alpha_2, \dots, \alpha_r, \beta_1, \beta_2, \dots, \beta_s, e_1, e_2, \dots, e_g, e_{g+1}, \dots, e_{2g} \in \mathbb{N}^*$ such that

$$I = P_1^{\alpha_1} \cdot P_2^{\alpha_2} \cdot \dots \cdot P_r^{\alpha_r} \cdot (P''_1)^{e_1} \cdot (P''_2)^{e_2} \cdot \dots \cdot (P''_g)^{e_g}$$

and

$$J = (P'_1)^{\beta_1} \cdot (P'_2)^{\beta_2} \cdot \dots \cdot (P'_s)^{\beta_s} \cdot (P''_1)^{e_{g+1}} \cdot (P''_2)^{e_{g+2}} \cdot \dots \cdot (P''_g)^{e_{2g}}.$$

Applying the definition of the extended Euler’s function and Proposition 1, we have:

$$\varphi_{\text{ext}}^2(I \cdot J) = N^2(I \cdot J) \cdot \left(1 - \frac{1}{N(P_1)}\right)^2 \cdot \left(1 - \frac{1}{N(P_2)}\right)^2 \cdot \dots \cdot \left(1 - \frac{1}{N(P_r)}\right)^2.$$

$$\begin{aligned} & \cdot \left(1 - \frac{1}{N(P'_1)}\right)^2 \cdot \left(1 - \frac{1}{N(P'_2)}\right)^2 \cdots \left(1 - \frac{1}{N(P'_s)}\right)^2 \cdot \\ & \cdot \left(1 - \frac{1}{N(P''_1)}\right)^2 \cdot \left(1 - \frac{1}{N(P''_2)}\right)^2 \cdots \left(1 - \frac{1}{N(P''_g)}\right)^2 \leq \\ & \leq N(I^2) \cdot N(J^2) \cdot \left(1 - \frac{1}{N(P_1)}\right) \cdot \left(1 - \frac{1}{N(P_2)}\right) \cdots \left(1 - \frac{1}{N(P_r)}\right) \cdot \\ & \cdot \left(1 - \frac{1}{N(P'_1)}\right) \cdot \left(1 - \frac{1}{N(P'_2)}\right) \cdots \left(1 - \frac{1}{N(P'_s)}\right) \cdot \\ & \cdot \left(1 - \frac{1}{N(P''_1)}\right)^2 \cdot \left(1 - \frac{1}{N(P''_2)}\right)^2 \cdots \left(1 - \frac{1}{N(P''_g)}\right)^2 = \varphi_{ext}(I^2) \cdot \varphi_{ext}(J^2). \end{aligned}$$

□

We return now to Proposition 7. If we take $t = 2$ in Proposition 7, we find a result from [20]:

$$\varphi(n) \cdot \psi(n) \geq \frac{6n^2}{\pi^2}, \quad (\forall) n \in \mathbb{N}^*.$$

Here, we generalize this result, for extended Euler’s function and extended Dedekind’s function and for the algebraic number field $K = \mathbb{Q}(\sqrt{5})$.

Proposition 12. *Let the quadratic field $K = \mathbb{Q}(\sqrt{5})$. Then:*

$$\varphi_{ext}(I) \cdot \psi_{ext}(I) \geq \frac{75\sqrt{5} \cdot (N(I))^2}{2\pi^4},$$

for (\forall) nonzero ideal I of the ring $\mathbb{Z}\left[\frac{1+\sqrt{5}}{2}\right]$.

Proof. According to Proposition 4, it results that $\mathcal{O}_K = \mathbb{Z}\left[\frac{1+\sqrt{5}}{2}\right]$.

Let I be a nonzero ideal of the ring $\mathbb{Z}\left[\frac{1+\sqrt{5}}{2}\right]$. According to the fundamental theorem of Dedekind rings, $(\exists!)$ positive integers r and $\alpha_i, i = \overline{1, r}$ and the different prime ideals P_1, P_2, \dots, P_r in the ring $\mathcal{O}_K = \mathbb{Z}\left[\frac{1+\sqrt{5}}{2}\right]$, such that

$$I = P_1^{\alpha_1} \cdot P_2^{\alpha_2} \cdot \dots \cdot P_r^{\alpha_r}.$$

Using the formulas of $\varphi_{ext}(I)$ and $\psi_{ext}(I)$, it results that:

$$\varphi_{ext}(I) \cdot \psi_{ext}(I) = (N(I))^2 \cdot \left(1 - \frac{1}{(N(P_1))^2}\right) \cdot \left(1 - \frac{1}{(N(P_2))^2}\right) \cdots \left(1 - \frac{1}{(N(P_r))^2}\right) \Leftrightarrow$$

$$\frac{\varphi_{ext}(I) \cdot \psi_{ext}(I)}{(N(I))^2} = \left(1 - \frac{1}{(N(P_1))^2}\right) \cdot \left(1 - \frac{1}{(N(P_2))^2}\right) \cdots \left(1 - \frac{1}{(N(P_r))^2}\right) \geq \frac{1}{\zeta_K(2)},$$

where ζ_K is the Dedekind zeta function of the quadratic field $K = \mathbb{Q}(\sqrt{5})$. From these, it results that:

$$\varphi_{ext}(I) \cdot \psi_{ext}(I) \geq \frac{(N(I))^2}{\zeta_K(2)}, \tag{4}$$

However, according to Proposition 3, we have

$$\zeta_K(2) = \frac{2\pi^4}{75\sqrt{5}}. \tag{5}$$

From (4) and (5), we obtain that:

$$\varphi_{ext}(I) \cdot \psi_{ext}(I) \geq \frac{75\sqrt{5} \cdot (N(I))^2}{2\pi^4},$$

for (\forall) nonzero ideal I of the ring $\mathbb{Z} \left[\frac{1+\sqrt{5}}{2} \right]$. \square

Now, we generalize Proposition 5 and Proposition 6 for extended Euler’s function and extended Dedekind’s function:

Proposition 13. *Let n be a positive integer, $n \geq 2$ and let K be an algebraic number field of degree $[K : \mathbb{Q}] = n$. Subsequently, we have the following inequality:*

$$\frac{\varphi_{ext}(I)^{\varphi_{ext}(I)} \cdot \psi_{ext}(I)^{\psi_{ext}(I)}}{N(I)^{\varphi_{ext}(I)+\psi_{ext}(I)}} \geq e^{\varphi_{ext}(I)+\psi_{ext}(I)-2N(I)} \geq 1,$$

for (\forall) nonzero ideal I of the ring \mathcal{O}_K , with $N(I) \geq 2$.

Proof. The idea of the proof is similar with the idea of the proof of Proposition 5.

In inequality (2), we take $a = \frac{\varphi_{ext}(I)}{N(I)}$ and $b = \frac{\psi_{ext}(I)}{N(I)}$ and obtain:

$$\frac{\varphi_{ext}(I)^{\varphi_{ext}(I)} \cdot \psi_{ext}(I)^{\psi_{ext}(I)}}{N(I)^{\varphi_{ext}(I)+\psi_{ext}(I)}} \geq e^{\varphi_{ext}(I)+\psi_{ext}(I)-2N(I)}.$$

Analogous to Proposition 5, it is easy to prove by mathematical induction over $N(I) \in \mathbb{N}$, $N(I) \geq 2$, that:

$$\frac{\varphi_{ext}(I)}{N(I)} + \frac{\psi_{ext}(I)}{N(I)} = \prod_{P \in \text{Spec}(\mathcal{O}_K), P|I} (1 - 1/N(P)) + \prod_{P \in \text{Spec}(\mathcal{O}_K), P|I} (1 + 1/N(P)) \geq 2.$$

So, we obtain that

$$e^{\varphi_{ext}(I)+\psi_{ext}(I)-2N(I)} \geq 1.$$

\square

Proposition 14. *Let n be a positive integer, $n \geq 2$ and let K be an algebraic number field of degree $[K : \mathbb{Q}] = n$. Afterwards, we have the following inequality:*

$$\begin{aligned} & \frac{\varphi_{ext}(I)^{\varphi_{ext}(I)} \cdot \psi_{ext}(I)^{\psi_{ext}(I)}}{N(I)^{\varphi_{ext}(I)+\psi_{ext}(I)}} \geq \\ & \geq \left(\frac{\varphi_{ext}(I) + \psi_{ext}(I)}{2N(I)} \right)^{\varphi_{ext}(I)+\psi_{ext}(I)} \cdot \left(\frac{\varphi_{ext}(I) + \psi_{ext}(I)}{2\sqrt{\varphi_{ext}(I) \cdot \psi_{ext}(I)}} \right)^{2 \cdot \varphi_{ext}(I)} \geq 1, \end{aligned}$$

for (\forall) nonzero ideal I of the ring \mathcal{O}_K , with $N(I) \geq 2$.

Proof. Let I be a nonzero ideal of the ring \mathcal{O}_K , with $N(I) \geq 2$. Applying inequality (3) for $x = \frac{\varphi_{ext}(I)}{N(I)}$ and $y = \frac{\psi_{ext}(I)}{N(I)}$, we obtain the first part of the inequality of the statement.

But, in the proof of Proposition 13, we showed that $\varphi_{ext}(I) + \psi_{ext}(I) \geq 2N(I)$ and since $\varphi_{ext}(I) + \psi_{ext}(I) \geq 2\sqrt{\varphi_{ext}(I) \cdot \psi_{ext}(I)}$, for all nonzero ideal I of the ring \mathcal{O}_K , with $N(I) \geq 2$, it results the inequality of the statement. \square

4. Conclusions

In this paper, starting from some inequalities satisfied by Euler's totient function φ and Dedekind's function ψ , we proved that the extended Euler's function φ_{ext} and the extended Dedekind's function ψ_{ext} satisfy similar inequalities.

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