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Resonant Anisotropic (p, q)-Equations

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Abstract: We consider an anisotropic Dirichlet problem which is driven by the $(p(z), q(z))$-Laplacian (that is, the sum of a $p(z)$-Laplacian and a $q(z)$-Laplacian), The reaction (source) term, is a Carathéodory function which asymptotically as $x \pm \infty$ can be resonant with respect to the principal eigenvalue of $(-\Delta_{p(z)}, W_0^{1,p(z)}(\Omega))$. First using truncation techniques and the direct method of the calculus of variations, we produce two smooth solutions of constant sign. In fact we show that there exist a smallest positive solution and a biggest negative solution. Then by combining variational tools, with suitable truncation techniques and the theory of critical groups, we show the existence of a nodal (sign changing) solution, located between the two extremal ones.

Keywords: anisotropic (p, q)-laplacian; resonance; principal eigenvalue; critical group; constant sign; nodal solutions

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1. Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. In this paper we study the following anisotropic (p, q)-equation

$$\begin{cases} -\Delta_{p(z)} u(z) - \Delta_{q(z)} u(z) = f(z, u(z)) & \text{in } \Omega, \\ u|_{\partial\Omega} = 0. \end{cases} \tag{1}$$

Given $r \in C(\overline{\Omega})$, we define

$$r_- = \min_{\overline{\Omega}} r \quad \text{and} \quad r_+ = \max_{\overline{\Omega}} r_+.$$

Let $E_1 = \{r \in C(\overline{\Omega}) : 1 < r_-\}$ and for $r \in E_1$ we introduce the anisotropic r-Laplacian differential operator defined by

$$\Delta_{r(z)} u = \operatorname{div}\left(|Du|^{r(z)-2} Du\right) \quad \forall u \in W_0^{1,r(z)}(\Omega).$$

In (1), the left hand side (the differential operator) is the sum of two such operators with different exponents. Equations driven by the sum of two differential operators of different nature, such as the anisotropic (p, q)-equations of the present work, arise in the mathematical models of many physical processes. We refer to the survey papers of Marano–Mosconi [1] and Rădulescu [2] and the references therein. In particular, anisotropic equations arise in elasticity (see Zhikov [3,4]) and in the study of electrorheological and magnetorheological fluids (see Růžička [5] and Versaci–Palumbo [6]). For other papers dealing with the sum of two differential operators of different nature (mostly (p, q)-Laplacian) we refer to Candito–Gasiński–Livrea [7], Gasiński–Klimczak–Papageorgiou [8], Gasiński–Papageorgiou [9–12], Gasiński–Winkert [13,14], and for anisotropic problems governed by the $p(z)$-Laplacian we refer to Gasiński–Papageorgiou [15,16]. Finally for the use of the eigenproblem to molecules we refer to Jäntsch [17], and Teng–Lu [18].

In the reaction (right hand side of (1)), the function $f(z,x)$ is a Carathéodory function (that is, for all $x \in \mathbb{R}$, $z \longmapsto f(z,x)$ is measurable and for almost all $z \in \Omega$, $x \longmapsto f(z,x)$ is continuous) and asymptotically as $x \to \pm\infty$, we can have resonance with respect to the principal eigenvalue of $(-\Delta_{p(z)}, W_0^{1,p(z)}(\Omega))$. Using variational tools from the critical point theory, together with suitable truncation techniques and Morse theory (critical groups), we show that problem (1) has at least three nontrivial smooth solutions, one positive, one negative and the third nodal (sign-changing). For isotropic problems, such three solutions theorem was proved for Dirichlet problems driven by the p-Laplacian by Liu [19] (Theorem 1.2). In that paper, the reaction $f(z,\cdot)$ asymptotically as $x \to \pm\infty$ is uniformly nonresonant with respect to the principal eigenvalue of $(-\Delta_p, W_0^{1,p}(\Omega))$ and no nodal solutions are obtained. For the same problem, the resonant case was examined by Liu–Su [20], who obtained two nontrivial solutions, but without providing sign information for them.

The study of anisotropic equations is lagging behind and there is only the work of Fan–Zhao [21], who produced nodal solutions for a class of radially symmetric equations driven by the anisotropic p-Laplacian. Our work here appears to be the first one producing three nontrivial smooth solutions with sign information for resonant anisotropic (p,q)-equations.

2. Mathematical Background—Hypotheses

The study of problem (1) requires the use of Lebesgue and Sobolev spaces with variable exponents. For a comprehensive presentation of such spaces, we refer to the book of Diening–Harjulehto–Hästö-Růžička [22].

By $M(\Omega)$ we denote the space of all functions $u\colon \Omega \to \mathbb{R}$ which are measurable. As usual, we identify two such functions which differ only on a Lebesgue-null set. Given $r \in E_1$, the variable exponent Lebesgue space $L^{r(z)}(\Omega)$ is defined by

$$L^{r(z)}(\Omega) = \left\{ u \in M(\Omega) : \int_\Omega |u|^{r(z)}\, dz < \infty \right\}.$$

We equip this space with the so-called "Luxemburg norm", defined by

$$\|u\|_{r(z)} = \inf \left\{ \lambda > 0 : \int_\Omega \left(\frac{|u(z)|}{\lambda} \right)^{r(z)} dz \leqslant 1 \right\}.$$

The space $(L^{r(z)}(\Omega), \|\cdot\|_{r(z)})$ is separable and uniformly convex (hence reflexive too, by the Milman–Pettis theorem; see Papageorgiou–Winkert [23] (Theorem 3.4.28, p. 225) or Gasiński–Papageorgiou [24] (Theorem 5.89, p. 853)). If

$$r'(z) = \frac{r(z)}{r(z)-1} \quad \forall z \in \overline{\Omega},$$

then $r' \in E_1$ and we have $L^{r(z)}(\Omega)^* = L^{r'(z)}(\Omega)$. In addition the following Hölder-type inequality holds

$$\int_\Omega |uv|\, dz \leqslant \left(\frac{1}{r_-} + \frac{1}{r'_-} \right) \|u\|_{r(z)} \|v\|_{r'(z)} \quad \forall u \in L^{r(z)}(\Omega),\ v \in L^{r'(z)}(\Omega).$$

These function spaces have many properties similar to the classical Lebesgue L^p-space. So, if $r_1, r_2 \in E_1$ and $r_1(z) \leqslant r_2(z)$ for all $z \in \overline{\Omega}$, then $L^{r_2(z)}(\Omega)$ embeds continuously into $L^{r_1(z)}(\Omega)$.

Using the variable exponent Lebesgue spaces, given $r \in E_1$, we can define the variable exponent Sobolev space $W^{1,r(z)}(\Omega)$ as follows

$$W^{1,r(z)}(\Omega) = \{u \in L^{r(z)}(\Omega) : \ |Du| \in L^{r(z)}(\Omega)\}.$$

In this definition, the gradient Du is understand in the weak sense. The space $W^{1,r(z)}(\Omega)$ is equipped with the following norm

$$\|u\|_{1,r(z)} = \|u\|_{r(z)} + \||Du|\|_{r(z)} \quad \forall u \in W^{1,r(z)}(\Omega).$$

For the sake of notational simplicity, we will write $\|Du\|_{r(z)} = \||Du|\|_{r(z)}$.

When $r \in E_1 \cap C^{0,1}(\overline{\Omega})$ (that is, $r \in E_1$ is Lipschitz continuous on $\overline{\Omega}$), then we can also define

$$W_0^{1,r(z)}(\Omega) = \overline{C_c^\infty(\Omega)}^{\|\cdot\|_{1,r(z)}}.$$

The spaces $W^{1,r(z)}(\Omega)$ and $W_0^{1,r(z)}(\Omega)$ are separable and uniformly convex (thus reflexive). For the space $W_0^{1,r(z)}(\Omega)$, the well-known Poincaré inequality is still valid, namely there exists $\widehat{c} > 0$ such that

$$\|u\|_{r(z)} \leqslant \widehat{c}\|Du\|_{r(z)} \quad \forall u \in W_0^{1,r(z)}(\Omega).$$

This inequality implies that on $W_0^{1,r(z)}(\Omega)$ we can consider the equivalent norm

$$\|u\|_{1,r(z)} = \|Du\|_{r(z)} \quad \forall u \in W_0^{1,r(z)}(\Omega).$$

The Sobolev embedding theorem can be extended to the present setting. More precisely, let $r \in E_1$ and set

$$r^*(z) = \begin{cases} \frac{Nr(z)}{N-r(z)} & \text{if} \quad r(z) < N, \\ +\infty & \text{if} \quad N \leqslant r(z), \end{cases}$$

for all $z \in \overline{\Omega}$ (the critical Sobolev exponent corresponding to $r \in E_1$). Suppose that $r \in E_1 \cap C^{0,1}(\overline{\Omega})$, $q \in E_1$, $q_+ < N$ and also

$$1 < q(z) \leqslant r^*(z) \quad (\text{resp. } 1 < q(z) < r^*(z)) \quad \forall z \in \overline{\Omega}.$$

Then the following embeddings are true

$$W_0^{1,r(z)}(\Omega) \subseteq L^{q(z)}(\Omega) \text{ continuously}$$
$$(\text{resp. } W_0^{1,r(z)}(\Omega) \subseteq L^{q(z)}(\Omega) \text{ compactly}).$$

The study of the anisotropic Lebesgue and Sobolev spaces uses the following modular function

$$\varrho_r(u) = \int_\Omega |u|^{r(z)}\, dz \quad \forall u \in L^{r(z)}(\Omega),$$

for $r \in E_1$. This modular function is closely related to the Luxemburg norm.

Proposition 1. *If $r \in E_1$, $u \in L^{r(z)}(\Omega)$ and $\{u_n\}_{n\in\mathbb{N}} \subseteq L^{r(z)}(\Omega)$, then*
(a) $\|u\|_{r(z)} = \vartheta \iff \varrho_r(\frac{u}{\vartheta}) = 1$.
(b) $\|u\|_{r(z)} < 1$ *(resp.* $= 1, > 1$*)* $\iff \varrho_r(u) < 1$ *(resp.* $= 1, > 1$*).*
(c) $\|u\|_{r(z)} < 1 \implies \|u\|_{r(z)}^{r_+} \leqslant \varrho_r(u) \leqslant \|u\|_{r(z)}^{r_-}$.
(d) $\|u\|_{r(z)} > 1 \implies \|u\|_{r(z)}^{r_-} \leqslant \varrho_r(u) \leqslant \|u\|_{r(z)}^{r_+}$.

(e) $\|u_n\|_{r(z)} \longrightarrow 0 \iff \varrho_r(u_n) \longrightarrow 0.$
(f) $\|u_n\|_{r(z)} \longrightarrow +\infty \iff \varrho_r(u_n) \longrightarrow +\infty.$

We know that for $r \in E_1 \cap C^{0,1}(\overline{\Omega})$, we have

$$W_0^{1,r(z)}(\Omega)^* = W^{-1,r'(z)}(\Omega).$$

So, to every $\xi \in W_0^{1,r(z)}(\Omega)^*$, correspond functions $\{f_k\}_{k=0}^N \subseteq L^{r(z)}(\Omega)$ such that

$$\langle \xi, u\rangle = \int_\Omega f_0 u\, dz + \int_\Omega (\widehat{f}, Du)_{\mathbb{R}^N}\, dz \quad \forall u \in W_0^{1,r(z)}(\Omega),$$

with $\widehat{f} = (f_k)_{k=1}^N$. Then we introduce the operator $A_{r(z)}\colon W^{1,r(z)}(\Omega) \longrightarrow W^{-1,r'(z)}(\Omega)$ defined by

$$\langle A_{r(z)}(u), h\rangle = \int_\Omega |Du|^{r(z)-2}(Du, Dh)_{\mathbb{R}^N}\, dz \quad \forall u, h \in W_0^{1,r(z)}(\Omega).$$

This operator has the following properties (see Gasiński–Papageorgiou [15] (Proposition 2.5)).

Proposition 2. *The operator $A_{r(z)}$ is bounded (that is, maps bounded sets into bounded sets), continuous, strictly monotone (hence maximal monotone too) and of type $(S)_+$, which means that*

"if $u_n \xrightarrow{w} u$ in $W_0^{1,r(z)}(\Omega)$ and $\limsup\limits_{n\to+\infty}\langle A_{r(z)}(u_n), u_n - u\rangle \leqslant 0$, than $u_n \longrightarrow u$ in $W_0^{1,r(z)}(\Omega)$."

In addition to the anisotropic spaces, we will also use the space

$$C_0^1(\Omega) = \{u \in C^1(\overline{\Omega}) : \ u|_{\partial\Omega} = 0\}.$$

This is an ordered Banach space with positive (order) cone

$$C_+ = \{u \in C_0^1(\Omega) : \ u(z) \geqslant 0 \text{ for all } z \in \overline{\Omega}\}.$$

This cone has a nonempty interior given by

$$\operatorname{int} C_+ = \{u \in C_+ : \ u(z) > 0 \text{ for all } z \in \Omega,\ \frac{\partial u}{\partial n}|_{\partial\Omega} < 0\},$$

with n being the outward unit normal on $\partial\Omega$.

We will need also some information about the spectrum of $(-\Delta_{r(z)}, W^{1,r(z)}(\Omega))$. So, with $r \in E_1 \cap C^{0,1}(\overline{\Omega})$, we consider the following eigenvalue problem

$$\begin{cases} -\Delta_{r(z)} u(z) = \widehat{\lambda}|u(z)|^{r(z)-2}u(z) & \text{in } \Omega, \\ u|_{\partial\Omega} = 0. \end{cases} \tag{2}$$

We say that $\widehat{\lambda} \in \mathbb{R}$ is an "eigenvalue", if problem (2) admits a nontrivial solution $\widehat{u} \in W_0^{1,r(z)}(\Omega)$, known as a corresponding "eigenfunction". We introduce the set

$$\mathcal{L} = \{\lambda : \ \lambda \text{ is an eigenvalue of } (2)\}.$$

In contrast to the isotropic case, in the anisotropic case it can happen that $\inf \mathcal{L} = 0$ (see Fan–Zhang–Zao [25] (Theorem 3.1)). However, if there exists $\eta \in \mathbb{R}^N$ $(N > 1)$ such that for all $z \in \Omega$, the function $\vartheta(t) = r(z + t\eta)$ is monotone on $T_z = \{t \in \mathbb{R} : \ z + t\eta \in \Omega\}$ and if $r \in C^1(\overline{\Omega})$, then there exists a principal eigenvalue $\widehat{\lambda}_1(r) > 0$ with corresponding eigenfunction $\widehat{u}_1(r) \in \operatorname{int} C_+$ (see Fan–Zhang–Zao [25] (Theorem 3.3)). Moreover, we have

$$\widehat{\lambda}_1(r) = \frac{\varrho_r(D\widehat{u}_1(r))}{\varrho_r(\widehat{u}_1(r))} \leqslant \frac{\varrho_r(Du)}{\varrho_r(u)} \quad \forall u \in W^{1,r(z)}(\Omega) \setminus \{0\}. \tag{3}$$

Let X be a Banach space, $\varphi \in C^1(X)$ and $c \in \mathbb{R}$. We introduce the following sets

$$\begin{aligned} K_\varphi &= \{u \in X : \varphi'(u) = 0\} \text{ (the critical set of } \varphi), \\ \varphi^c &= \{u \in X : \varphi(u) \leqslant c\}. \end{aligned}$$

Suppose that (Y_1, Y_2) is a topological pair such that $Y_2 \subseteq Y_1 \subseteq X$. For $k \in \mathbb{N}_0$, by $H_k(Y_1, Y_2)$ we denote the k-th relative singular homology group for the pair (Y_1, Y_2) with integer coefficients. Let $u \in K_\varphi$ be isolated and $\varphi(u) = c$. Then the critical groups of φ at u are defined by

$$C_k(\varphi, u) = H_k(\varphi^c \cap U, \varphi^c \cap U \setminus \{u\}) \quad \forall k \in \mathbb{N}_0,$$

with U being a neighbourhood of u such that $K_\varphi \cap \varphi^c \cap U = \{u\}$. The excision property of singular homology implies that this definition of critical groups is independent of the choice of the neighbourhood U.

Next let us fix our basic notation. If $x \in \mathbb{R}$, then $x^\pm = \max\{\pm x, 0\}$. So, given $u \in W_0^{1,r(z)}(\Omega)$, we set $u^\pm(\cdot) = u(\cdot)^\pm$. We have

$$u^\pm \in W_0^{1,r(z)}(\Omega), \quad u = u^+ - u^-, \quad |u| = u^+ + u^-.$$

Furthermore, for $r = p$, we write $\widehat{\lambda}_1 = \widehat{\lambda}_1(p)$ and $\|u\| = \|u\|_{1,p(z)}$. Recall that by the Poincaré inequality we have

$$\|u\| = \|Du\|_{p(z)} \quad \forall u \in W^{1,p(z)}(\Omega).$$

We say that a set $S \subseteq W_0^{1,p(z)}(\Omega)$ is "downward directed" if, for every pair $u_1, u_2 \in S$, we can find $u \in S$ such that $u \leqslant u_1$, $u \leqslant u_2$; similarly, we say that S is "upward directed", if for every pair $v_1, v_2 \in S$, we can find $v \in S$ such that $v_1 \leqslant v$, $v_2 \leqslant v$. Finally as for the Luxemburg norm, we write $\varrho_p(Du) = \varrho_p(|Du|)$ for all $u \in W_0^{1,p(z)}(\Omega)$.

Now we are ready to introduce the hypotheses on the data of problem (1).

Hypothesis 1. *$p \in C^1(\overline{\Omega})$ and there exists $\eta \in \mathbb{R}^N$ such that for all $z \in \Omega$, the function $t \longmapsto \vartheta(t) = p(z + t\eta)$ is monotone on $T_z = \{t \in \mathbb{R} : z + t\eta \in \Omega\}$, $q \in E_1$ and $q(z) < p(z)$ for all $z \in \overline{\Omega}$.*

Hypothesis 2. *$f \colon \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is a Carathéodory function such that $f(z, 0) = 0$ for a.a. $z \in \Omega$ and*

(i) $|f(z,x)| \leqslant a(z)(1 + |x|^{p(z)-1})$ *for a.a.* $z \in \Omega$, *all* $x \in \mathbb{R}$, *with* $a \in L^\infty(\Omega)$;

(ii) $\limsup\limits_{x \to \pm\infty} \dfrac{f(z,x)}{|x|^{p(z)-2}x} \leqslant \widehat{\lambda}_1$ *and* $\limsup\limits_{x \to \pm\infty} \dfrac{p_+ F(z,x)}{|x|^{p(z)}} \leqslant \widehat{\lambda}_1$ *uniformly for a.a.* $z \in \Omega$, *with* $F(z,x) = \int_0^x f(z,s)\,ds$;

(iii) there exist $M, \xi_0 > 0$ *such that*

$$\widehat{\lambda}_1 |x|^{p(z)} - p_+ F(z,x) \geqslant -\xi_0 \quad \text{for a.a. } z \in \Omega, \text{ all } |x| \geqslant M;$$

(iv) there exist $\delta_0 > 0$, $c_0 > 0$ *and* $\tau \in E_1$ *such that* $\tau_+ < q_-$ *and*

$$f(z,x)x \geqslant c_0 |x|^{\tau(z)} \text{ for a.a. } z \in \Omega, \text{ all } |x| \leqslant \delta_0.$$

Remark 1. Hypothesis 2(ii) incorporates in our framework problems which are resonant with respect to the principal eigenvalue $\widehat{\lambda}_1 > 0$. Hypothesis Hypothesis 2(iv) implies the presence of a local concave term near zero.

Example 1. *The following function satisfies hypotheses H_1:*

$$f(z,x) = \begin{cases} \widehat{\lambda}_1 |x|^{\tau(z)-2} x & \text{if} \quad |x| \leqslant 2\pi, \\ \widehat{c}_1(z)|x|^{p(z)-2} x \cos|x| & \text{if} \quad 2\pi < |x|, \end{cases}$$

where $\tau \in E_1$, $\tau < q_-$ and $\widehat{c}_1(z) = \widehat{\lambda}_1 (2\pi)^{\tau(z)-p(z)}$ (note that $\widehat{\lambda}_1 (2\pi)^{\tau_- - p_+} \leqslant \widehat{c}_1(z) \leqslant \widehat{\lambda}_1 (2\pi)^{\tau_+ - p_-}$); see Figure 1.

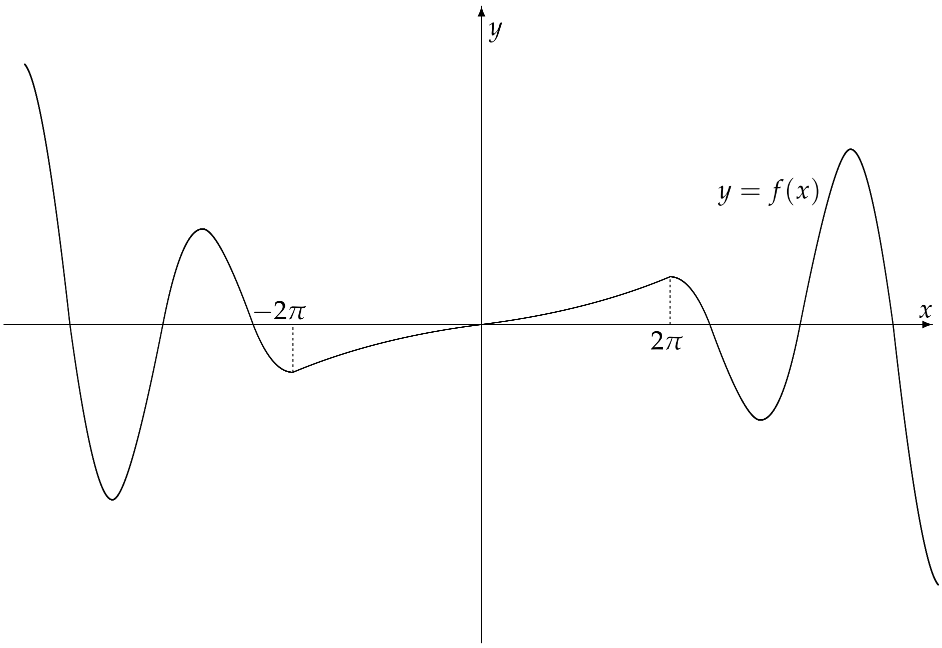

Figure 1. Function $f(x)$ from Example 1 (with $\tau(z) = 1.5$ and $p(z) = 2$).

We will need the following lemma.

Lemma 1. *If $\vartheta \in L^\infty(\Omega)$, $\vartheta(z) \leqslant \widehat{\lambda}_1$ for a.a. $z \in \Omega$ and $\vartheta \not\equiv \widehat{\lambda}_1$, then there exists $c_1 > 0$ such that*

$$c_1 \varrho_p(Du) \leqslant \varrho_p(Du) - \int_\Omega \vartheta(z)|u|^{p(z)}\,dz \quad \forall u \in W^{1,p(z)}(\Omega).$$

Proof. We proceed indirectly. So, suppose that the lemma is not true. Then we can find a sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,p(z)}(\Omega)$ such that

$$\varrho_p(Du_n) - \int_\Omega \vartheta(z)|u_n|^{p(z)}\,dz < \frac{1}{n}\varrho_p(Du_n).$$

Evidently we may assume that $u_n \geqslant 0$ for all $n \in \mathbb{N}$. We have

$$1 - \frac{1}{n} < \frac{1}{\varrho_p(Du_n)} \int_\Omega \vartheta(z) u_n^{p(z)}\,dz \quad \forall n \in \mathbb{N}. \tag{4}$$

Suppose that the sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,p(z)}(\Omega)$ is not bounded. Then by passing to a subsequence if necessary we may assume that

$$\|u_n\| \longrightarrow \infty,$$

so, by Proposition 1, also

$$\varrho_p(Du_n) \longrightarrow \infty. \tag{5}$$

We set $y_n = \frac{u_n}{\varrho_p(Du_n)^{\frac{1}{p(z)}}} \in W_0^{1,p(z)}(\Omega)$, $n \in \mathbb{N}$. Differentiating, we have

$$Dy_n = \frac{Du_n}{\varrho_p(Du_n)^{\frac{1}{p(z)}}} - \frac{1}{p(z)^2}\frac{\ln \varrho_p(Du_n)}{\varrho_p(Du_n)^{\frac{1+p(z)}{p(z)}}}Dp, \tag{6}$$

so

$$|Dy_n|^{p(z)} \leqslant \frac{|Du_n|^{p(z)}}{\varrho(Du_n)} + \chi_n,$$

with $\chi_n \to 0^+$ as $n \to +\infty$. Hence

$$\varrho_p(Dy_n) \leqslant 1 + \chi_n|\Omega|_N \quad \forall n \in \mathbb{N}, \tag{7}$$

where $|\cdot|_N$ denotes the Lebesgue measure on $\mathbb{R}^N$.

Then Proposition 1 and the Poincaré inequality, imply that the sequence $\{y_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,p(z)}(\Omega)$ is bounded.

Passing to a suitable subsequence if necessary, we may assume that

$$y_n \xrightarrow{w} y \quad \text{in } W_0^{1,p(z)}(\Omega) \quad \text{and} \quad y_n \longrightarrow y \quad \text{in } L^{p(z)}(\Omega). \tag{8}$$

The modular function ϱ_p is continuous, convex on $W_0^{1,p(z)}(\Omega)$. Therefore it is sequentially weakly lower semicontinuous. Hence we have

$$\varrho_p(Dy) \leqslant \liminf_{n\to\infty} \varrho_p(Dy_n)$$

See (8), so

$$\varrho_p(Dy) \leqslant 1 \tag{9}$$

See (7). Furthermore, (8) implies that

$$\int_\Omega \vartheta(z) y_n^{p(z)}\, dz \longrightarrow \int_\Omega \vartheta(z) y^{p(z)}\, dz.$$

Then from (4) we have

$$1 \leqslant \int_\Omega \vartheta(z) y^{p(z)}\, dz,$$

so

$$\varrho_p(Dy) \leqslant \int_\Omega \vartheta(z) y^{p(z)}\, dz \leqslant \widehat{\lambda}_1 \varrho_p(y) \tag{10}$$

See (9) and recall the hypothesis on ϑ, thus

$$\varrho_p(Dy) = \widehat{\lambda}_1 \varrho_p(y)$$

See (3) and hence

$$y = 0 \quad \text{or} \quad y(z) > 0 \quad \forall z \in \Omega.$$

Suppose that $y = 0$. then from (4), passing to the limit as $n \to \infty$, we have $1 \leqslant 0$, a contradiction. Next suppose that $y(z) > 0$ for all $z \in \Omega$. Using this in (10), from first inequality there, we obtain

$$\varrho_p(Dy) < \widehat{\lambda}_1 \varrho_p(y),$$

which contradicts (3). This proves the boundedness of the sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,p(z)}(\Omega)$. Then we may assume that

$$u_n \xrightarrow{w} u \quad \text{in } W_0^{1,p(z)}(\Omega) \quad \text{and} \quad u_n \longrightarrow u \quad \text{in } L^{p(z)}(\Omega).$$

From (10) and reasoning as above (replacing y_n with u_n), we reach again a contradiction. So, the assertion of the lemma is true. □

3. Solutions of Constant Sign

We introduce the energy (Euler) functional for problem (1) and the positive and negative truncations of it. So, we consider the following three functionals $\varphi, \varphi_\pm \colon W_0^{1,p(z)}(\Omega) \longrightarrow \mathbb{R}$:

$$\begin{aligned} \varphi(u) &= \int_\Omega \frac{1}{p(z)} |Du|^{p(z)}\, dz + \int_\Omega \frac{1}{q(z)} |Du|^{q(z)}\, dz - \int_\Omega F(z,u)\, dz, \\ \varphi_\pm(u) &= \int_\Omega \frac{1}{p(z)} |Du|^{p(z)}\, dz + \int_\Omega \frac{1}{q(z)} |Du|^{q(z)}\, dz - \int_\Omega F(z, \pm u^\pm)\, dz, \end{aligned}$$

for all $u \in W_0^{1,p(z)}(\Omega)$.

Proposition 3. *If hypotheses H_0 and H_1 hold, then the functionals φ, $\varphi_\pm$ are coercive.*

Proof. We do the proof for the functional φ_+, the proof for the functionals φ_- and φ being similar.
Hypotheses $H_1(i), (iii)$ imply that there exists $\xi_1 > 0$ such that

$$\widehat{\lambda}_1 |x|^{p(z)} - p_+ F(z,x) \geqslant -\xi_1 \quad \text{for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R}. \tag{11}$$

Suppose that φ_+ is not coercive. Then we can find a sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,p(z)}(\Omega)$ and $c_1 > 0$ such that

$$\|u_n\| \longrightarrow \infty \quad \text{and} \quad \varphi_+(u_n) \leqslant c_1 \quad \forall n \in \mathbb{N}. \tag{12}$$

From the inequality in (12) and since $q(z) < p(z)$ for all $z \in \overline{\Omega}$ (see hypotheses H_0), we have

$$\frac{1}{p_+}\left(\varrho_p(Du_n) + \varrho_q(Du_n) - \int_\Omega p_+ F(z, u_n^+)\, dz \right) \leqslant c_1 \quad \forall n \in \mathbb{N},$$

so

$$\frac{1}{p_+}\left(\widehat{\lambda}_1 \varrho_p(u_n^+) - \int_\Omega p_+ F(z, u_n^+)\, dz \right) + \frac{1}{p_+} \varrho_p(Du_n^-) \leqslant c_1 \quad \forall n \in \mathbb{N}$$

See (3), thus

$$\frac{1}{p_+} \int_\Omega (\widehat{\lambda}_1 (u_n^+)^{p(z)} - p_+ F(z, u_n^+))\, dz + \frac{1}{p_+} \varrho_p(Du_n^-) \leqslant c_1 \quad \forall n \in \mathbb{N},$$

hence

$$\varrho_p(Du_n^-) \leqslant c_2 \quad \forall n \in \mathbb{N},$$

with some $c_2 > 0$ (see (11)). Using also Proposition 1 and Poincaré's inequality, we conclude that

$$\text{the sequence } \{u_n^-\}_{n\in\mathbb{N}} \subseteq W_0^{1,p(z)}(\Omega) \text{ is bounded.} \tag{13}$$

From (12) and (13), it follows that

$$\|u_n^+\| \longrightarrow \infty,$$

so by Proposition 1, we have

$$\varrho_p(Du_n^+) \longrightarrow +\infty. \tag{14}$$

We set $v_n = \dfrac{u_n^+}{\varrho_p(Du_n^+)^{\frac{1}{p(z)}}} \in W_0^{1,p(z)}(\Omega)$, $n \in \mathbb{N}$. From the proof of Lemma 1 (see (6) and (7)), we have

$$0 < c_3 \leqslant \varrho_p(Dv_n) \leqslant 1 + \chi_n' \quad \forall n \in \mathbb{N},$$

with $\chi_n' \to 0^+$ and for some $c_3 > 0$, so the sequence $\{v_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,p(z)}(\Omega)$ is bounded.

So, we may assume that

$$v_n \xrightarrow{w} v \quad \text{in } W_0^{1,p(z)}(\Omega) \quad \text{and} \quad v_n \longrightarrow v \quad \text{in } L^{p(z)}(\Omega). \tag{15}$$

From (12) and (13), we have

$$\frac{1}{p_+}\varrho_p(Du_n^+) - \int_\Omega F(z,u_n^+)\,dz \leqslant c_4 \quad \forall n \in \mathbb{N},$$

for some $c_4 > 0$, so

$$\frac{1}{p_+}\varrho_p(Dv_n) - \int_\Omega \frac{F(z,u_n^+)}{\varrho_p(Du_n^+)}\,dz \leqslant \frac{c_4}{\varrho_p(Du_n^+)} \quad \forall n \in \mathbb{N},$$

thus

$$\varrho_p(Dv_n) \leqslant \frac{p_+c_4}{\varrho_p(Du_n^+)} + \int_\Omega \frac{p_+F(z,u_n^+)}{\varrho_p(Du_n^+)}\,dz \quad \forall n \in \mathbb{N}. \tag{16}$$

Using Hypothesis H2(ii), we obtain

$$\frac{p_+F(z,u_n^+)}{\varrho_p(Du_n^+)} \xrightarrow{w} \beta(\cdot)v^{p(z)} \quad \text{in } L^1(\Omega),$$

with $\beta \in L^\infty(\Omega)$, $\beta(z) \leqslant \widehat{\lambda}_1$ for a.a. $z \in \Omega$ (see Aizicovici–Papageorgiou–Staicu [26] (proof of Proposition 16)). So, if in (16) we pass to the limit as $n \to \infty$ and use (14) and (15) and the fact that the modular function ϱ_p is sequentially weakly lower semicontinuous (being continuous convex), we obtain

$$\varrho_p(Dv) \leqslant \int_\Omega \beta(z)v^{p(z)}\,dz. \tag{17}$$

If $\beta \not\equiv \widehat{\lambda}_1$, then from Lemma 1 and (17), we have

$$c_1\varrho_p(Dv) \leqslant 0,$$

so $v = 0$ (see Proposition 1). Then from (16) we see that

$$\varrho_p(Dv_n) \longrightarrow 0 \quad \text{as } n \to \infty,$$

a contradiction, since $0 < c_3 \leqslant \varrho_p(Dv_n)$ for all $n \in \mathbb{N}$.

If $\beta(z) = \widehat{\lambda}_1$ for a.a. $z \in \Omega$, then from (17) and (3) we have

$$\varrho_p(Dv) = \widehat{\lambda}_1\varrho_p(v),$$

so

$$v = 0 \quad \text{or} \quad v(z) > 0 \quad \forall z \in \Omega. \tag{18}$$

If $v = 0$, then as above we have a contradiction.
If $v(z) > 0$ for all $z \in \Omega$ (see (18)), we have

$$u_n^+(z) \longrightarrow +\infty \quad \text{for a.a. } z \in \Omega. \tag{19}$$

From (12) and (13), we have

$$\frac{1}{p_+}\left(\varrho_p(Du_n^+) - \int_\Omega p_+ F(z, u_n^+)\, dz\right) + \frac{1}{p_+}\varrho_q(Du_n^+) \leqslant c_5 \quad \forall n \in \mathbb{N},$$

for some $c_5 > 0$, so

$$\frac{1}{p_+}\left(\widehat{\lambda}_1 \varrho_p(u_n^+) - \int_\Omega p_+ F(z, u_n^+)\, dz\right) + \frac{1}{p_+}\varrho_q(Du_n^+) \leqslant c_5 \quad \forall n \in \mathbb{N}$$

See (3), thus

$$\frac{1}{p_+}\varrho_q(Du_n^+) \leqslant c_6 \quad \forall n \in \mathbb{N},$$

for some $c_6 > 0$ (see (11)) and hence

$$\frac{\widehat{\lambda}_1(q)}{p_+}\varrho_q(u_n^+) \leqslant c_6 \quad \forall n \in \mathbb{N}$$

See (3).
But from (19) and Fatou's lemma, we have that

$$\varrho_p(u_n^+) \longrightarrow +\infty \quad \text{as } n \to \infty,$$

a contradiction.

Therefore we have that the sequence $\{u_n^+\}_{n\in\mathbb{N}} \subseteq W_0^{1,p(z)}(\Omega)$ is bounded, and thus the sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,p(z)}(\Omega)$ is bounded (see (13)), a contradiction (see (12)). This proves the coercivity of φ_+.

In a similar fashion, we show that the functionals φ_- and φ are coercive too. □

Now that we have the coercivity of the functionals $\varphi_\pm$, we can use the direct method of the calculus of variations to produce two constant sign solutions.

Proposition 4. *If hypotheses H_0 and H_1 hold, then problem (1) has at least two constant sign solutions*

$$u_0 \in \operatorname{int} C_+ \quad \textit{and} \quad v_0 \in -\operatorname{int} C_+,$$

both local minimizers of the energy functional φ.

Proof. From Proposition 3 we know that φ_+ is coercive. Furthermore, using the Sobolev embedding theorem, we see that φ_+ is sequentially weakly lower semicontinuous. Hence by the Weierstrass–Tonelli theorem, we can find $u_0 \in W_0^{1,p(z)}(\Omega)$ such that

$$\varphi_+(u_0) = \min\{\varphi_+(u) : \ u \in W_0^{1,p(z)}(\Omega)\}, \tag{20}$$

so

$$\varphi'_+(u_0) = 0$$

and hence

$$\langle A_{p(z)}(u_0), h\rangle + \langle A_{q(z)}(u_0), h\rangle = \int_\Omega f(z, u_0^+) h\, dz \quad \forall h \in W_0^{1,p(z)}(\Omega). \tag{21}$$

We test (21) with $h = -u_0^- \in W_0^{1,p(z)}(\Omega)$ and obtain

$$\varrho_p(Du_0^-) + \varrho_q(Du_0^-) = 0,$$

so $u_0 \geqslant 0$.

Let $u \in \operatorname{int} C_+$ and choose $t \in (0,1)$ small such that

$$0 \leqslant tu(z) \leqslant \delta_0 \quad \forall z \in \overline{\Omega}.$$

Here δ_0 is as postulated in Hypothesis H2(iv). Using the fact that $t \in (0,1)$ and $\tau_+ < q_- < p_-$, we can write that

$$\varphi_+(tu) \leqslant \frac{t^{q_-}}{q_-}\left(\varrho_p(Du) + \varrho_q(Du)\right) - \frac{t^{\tau_+}}{\tau_+}\varrho_\tau(u).$$

So, choosing $t \in (0,1)$ even smaller, we see that

$$\varphi_+(tu) < 0,$$

so

$$\varphi_+(u_0) < 0 = \varphi_+(0)$$

See (20) and thus

$$u_0 \neq 0, \; u_0 \geqslant 0.$$

Then from (21) it follows that u_0 is a positive solution of (1) and we have

$$\begin{cases} -\Delta_{p(z)} u_0(z) - \Delta_{q(z)} u_0(z) = f(z, u_0(z)) & \text{in } \Omega, \\ u_0|_{\partial\Omega} = 0. \end{cases}$$

From Theorem 4.1 of Fan–Zhao [27], we know that $u_0 \in L^\infty(\Omega)$. Then Lemma 3.3 of Fukagai–Narukawa [28] (see also Tan–Fang [29] (Corollary 3.1)), we infer that $u_0 \in C_+ \setminus \{0\}$. Note that Hypothese H2(i),(iv) imply that

$$f(z,x)x \geqslant c_0|x|^{\tau(z)} - c_7|x|^{p(z)} \quad \text{for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R}.$$

So, from the anisotropic maximum principle of Papageorgiou–Qui–Rădulescu [30] (Proposition 4) (see also Zhang [31] (Theorem 1.2)), we have that $u_0 \in \operatorname{int} C_+$.

Note that $\varphi|_{C_+} = \varphi_+|_{C_+}$. Therefore we see that

$$u_0 \text{ is a local } C_0^1(\overline{\Omega})\text{-minimizer of } \varphi,$$

so by Proposition 3.3 of Gasiński–Papageorgiou [15] and Theorem 3.2 of Tan–Fang [29],

$$u_0 \text{ is a local } W_0^{1,p(z)}(\Omega)\text{-minimizer of } \varphi.$$

Similarly, working with the functional φ_-, we produce a negative solution $v_0 \in -\operatorname{int} C_+$ which is a local minimizer of φ. □

In fact we can show that there exist extremal constant sign solutions, that is, a smallest positive solution and a biggest negative solution. In Section 4, we will use these extremal constant sign solutions in order to generate a nodal (sign-changing) solution.

To obtain the extremal constant sign solutions, we need to do some preliminary work.

Let S_+ (resp. S_-) be the set of positive (resp. negative) solutions of Problem (1). From Proposition 4 and its proof, we know that

$$\emptyset \neq S_+ \subseteq \operatorname{int} C_+ \quad \text{and} \quad \emptyset \neq S_- \subseteq -\operatorname{int} C_+.$$

We will produce a lower bound for the set S_+ and an upper bound for the set S_-. To this end, note that on account of hypotheses $H_1(i), (iv)$, we have

$$f(z,x)x \geqslant c_0|x|^{\tau(z)} - c_8|x|^{p_+} \quad \text{for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R}, \tag{22}$$

for some $c_8 > 0$.

This unilateral growth condition on $f(z,\cdot)$ leads to the consideration of the following auxiliary anisotropic Dirichlet problem

$$\begin{cases} -\Delta_{p(z)}u(z) - \Delta_{q(z)}u(z) = c_0|u(z)|^{\tau(z)-2}u(z) - c_8|u(z)|^{p_+-2}u(z) & \text{in } \Omega, \\ u|_{\partial\Omega} = 0. \end{cases} \tag{23}$$

For this problem, we have the following existence and uniqueness result.

Proposition 5. *If hypotheses H_0 hold, then problem (23) has a unique positive solution $\overline{u} \in \operatorname{int} C_+$. Moreover, since problem is odd $\overline{v} = -\overline{u} \in -\operatorname{int} C_+$ is the unique negative solution of (23).*

Proof. First we prove the existence of a positive solution. For this purpose we introduce the C^1-functional $\psi_+ \colon W_0^{1,p(z)}(\Omega) \longrightarrow \mathbb{R}$ defined by

$$\begin{aligned} \psi_+(u) &= \int_\Omega \frac{1}{p(z)}|Du|^{p(z)}\,dz + \int_\Omega \frac{1}{q(z)}|Du|^{q(z)} + \frac{c_8}{p_+}\|u^+\|_{p^+}^{p^+} \\ &\quad -c_0\int_\Omega \frac{1}{\tau(z)}|Du^+|^{\tau(z)}\,dz \end{aligned}$$

for all $u \in W_0^{1,p(z)}(\Omega)$.

Since $\tau_+ < q_- < p_-$, we see that ψ_+ is coercive. Furthermore, it is sequentially weakly lower semicontinuous. So, we can find $\overline{u} \in W^{1,p(z)}(\Omega)$ such that

$$\psi_+(\overline{u}) = \min\{\psi_+(u) : \ u \in W_0^{1,p(z)}(\Omega)\}. \tag{24}$$

Given $u \in \operatorname{int} C_+$, since $\tau_+ < q_- < p_-$, as in the proof of Proposition 4, we see that for $t \in (0,1)$ small we have

$$\psi_+(tu) < 0,$$

so

$$\psi_+(\overline{u}) < 0 = \psi_+(0)$$

See (24), so $\overline{u} \neq 0$.

From (24), we have

$$\psi_+'(\overline{u}) = 0,$$

so

$$\langle A_{p(z)}(\overline{u}), h\rangle + \langle A_{q(z)}(\overline{u}), h\rangle = \int_\Omega c_0(\overline{u}^+)^{\tau(z)-1}h\,dz - \int_\Omega c_8(\overline{u}^+)^{p_+-1}h\,dz \tag{25}$$

for all $h \in W^{1,p(z)}(\Omega)$. Choosing $h = -\overline{u}^- \in W_0^{1,p(z)}(\Omega)$ in (25), we obtain

$$\varrho_p(D\overline{u}^-) + \varrho_q(D\overline{u}^-) = 0,$$

so $\overline{u} \geqslant 0$ and $\overline{u} \neq 0$.

Then from (25) we obtain

$$\begin{cases} -\Delta_{p(z)}\overline{u}(z) - \Delta_{q(z)}\overline{u}(z) = c_0\overline{u}(z)^{\tau(z)-1} - c_8\overline{u}(z)^{p_+-1} & \text{in } \Omega, \\ \overline{u}|_{\partial\Omega} = 0. \end{cases}$$

As before (see the proof of Proposition 4), using the anisotropic regularity theory and the anisotropic maximum principle, we obtain that $\overline{u} \in \operatorname{int} C_+$.

Next we show the uniqueness of this positive solution. For this purpose, we consider the integral functional $j\colon L^1(\Omega) \longrightarrow \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ defined by

$$j(u) = \begin{cases} \int_\Omega \frac{\tau_+}{p(z)}|Du^{\frac{1}{\tau_+}}|^{p(z)}\,dz + \int_\Omega \frac{\tau_+}{q(z)}|Du^{\frac{1}{\tau_+}}|^{q(z)}\,dz & \text{if } u \geqslant 0,\ u^{\frac{1}{\tau_+}} \in W_0^{1,p(z)}(\Omega), \\ +\infty & \text{otherwise.} \end{cases}$$

From Theorem 2.2 of Takáč-Giacomoni [32], we know that j is convex. Suppose that $\overline{y} \in W_0^{1,p(z)}(\Omega)$ is another positive solution of (23). Again we have that $\overline{y} \in \operatorname{int} C_+$. Then using Proposition 4.1.22 of Papageorgiou–Rădulescu–Repovš [33] (p. 274), we infer that

$$\frac{\overline{u}}{\overline{y}}, \frac{\overline{y}}{\overline{u}} \in L^\infty(\Omega).$$

So, if $\operatorname{dom} j = \{u \in L^1(\Omega) : j(u) < \infty\}$ (the effective domain of j) and $h = \overline{u}^{\tau_+} - \overline{y}^{\tau_+} \in W_0^{1,p(z)}(\Omega)$, then for $|t| < 1$ small we have

$$\overline{u}^{\tau_+} + th \in \operatorname{dom} j, \quad \overline{v}^{\tau_+} + th \in \operatorname{dom} j.$$

Hence the convexity of j implies the Gâteaux differentiability of j at $\overline{u}^{\tau_+}$ and at $\overline{y}^{\tau_+}$ in the direction h. Moreover, using the chain rule and Green's identity, we obtain

$$\begin{aligned} j'(\overline{u}^{\tau_+})(h) &= \int_\Omega \frac{-\Delta_{p(z)}\overline{u} - \Delta_{q(z)}\overline{u}}{\overline{u}^{\tau_+-1}} h\,dz \\ &= \int_\Omega \left(\frac{c_0}{\overline{u}^{\tau_+-\tau(z)}} - c_8\overline{u}^{p_+-\tau_+} \right) h\,dz, \\ j'(\overline{y}^{\tau_+})(h) &= \int_\Omega \frac{-\Delta_{p(z)}\overline{y} - \Delta_{q(z)}\overline{y}}{\overline{y}^{\tau_+-1}} h\,dz \\ &= \int_\Omega \left(\frac{c_0}{\overline{y}^{\tau_+-\tau(z)}} - c_8\overline{y}^{p_+-\tau_+} \right) h\,dz. \end{aligned}$$

The convexity of j implies the monotonicity of j'. Therefore

$$\begin{aligned} 0 \leqslant & \int_\Omega c_0 \left(\frac{1}{\overline{u}^{\tau_+-\tau(z)}} - \frac{1}{\overline{y}^{\tau_+-\tau(z)}} \right) (\overline{u}^{\tau_+} - \overline{y}^{\tau_+})\,dz \\ & - \int_\Omega c_8 (\overline{u}^{p_+-\tau_+} - \overline{y}^{p_+-\tau_+})(\overline{u}^{\tau_+} - \overline{y}^{\tau_+})\,dz \leqslant 0, \end{aligned}$$

so $\overline{u} = \overline{y}$. This proves the uniqueness of the positive solution $\overline{u} \in \operatorname{int} C_+$ of (23). Since problem (23) is odd, it follows that $\overline{v} = -\overline{u} \in -\operatorname{int} C_+$ is the unique negative solution of problem (23). □

These solutions will serve as bounds of S_+ and S_- respectively.

Proposition 6. *If hypotheses H_0 and H_1 hold, then $\overline{u} \leqslant u$ for all $u \in S_+$ and $v \leqslant \overline{v}$ for all $v \in S_-$.*

Proof. We do the proof for the set S_+, the proof for the set S_- being similar.

So, let $u \in S_+ \subseteq \operatorname{int} C_+$ and introduce the Carathéodory function k_+ defined by

$$k_+(z,x) = \begin{cases} c_0(x^+)^{\tau(z)-1} - c_8(x^+)^{p_+-1} & \text{if } \quad x \leqslant u(z), \\ c_0 u(z)^{\tau(z)-1} - c_8 u(z)^{p_+-1} & \text{if } \quad u(z) < x. \end{cases} \tag{26}$$

We set

$$K_+(z,x) = \int_0^x k_+(z,s)\,dz$$

and consider the C^1-functional $\sigma_+ : W_0^{1,p(z)}(\Omega) \longrightarrow \mathbb{R}$ defined by

$$\sigma_+(u) = \int_\Omega \frac{1}{p(z)} |Du|^{p(z)}\,dz + \int_\Omega \frac{1}{q(z)} |Du|^{q(z)}\,dz - \int_\Omega K_+(z,u)\,dz$$

for all $u \in W_0^{1,p(z)}(\Omega)$. Evidently σ_+ is coercive (see (26)) and sequentially weakly lower semicontinuous. Therefore we can find $\overline{u}_0 \in W_0^{1,p(z)}(\Omega)$ such that

$$\sigma_+(\overline{u}_0) = \min\{\sigma_+(u) : \, u \in W_0^{1,p(z)}(\Omega)\}. \tag{27}$$

If $w \in \operatorname{int} C_+$, then we can find $t \in (0,1)$ small such that $tw \leqslant u$ (recall that $u \in \operatorname{int} C_+$ and use Proposition 4.1.22 of Papageorgiou–Rădulescu–Repovš [33] (p. 274)). Using (26) and the fact that $\tau_+ < q_- < p_-$, we see that by taking $t \in (0,1)$ even smaller if necessary, we will have

$$\sigma_+(tw) < 0,$$

so

$$\sigma_+(\overline{u}_0) < 0 = \sigma_+(0)$$

See (27), thus $\overline{u}_0 \neq 0$.

From (27) we have

$$\sigma'_+(\overline{u}_0) = 0,$$

so

$$\langle A_{p(z)}(\overline{u}_0), h\rangle + \langle A_{q(z)}(\overline{u}_0), h\rangle = \int_\Omega k_+(z,\overline{u}_0) h\,dz \quad \forall h \in W_0^{1,p(z)}(\Omega). \tag{28}$$

In (28) first we use the test function $h = -\overline{u}_0^- \in W_0^{1,p(z)}(\Omega)$. We obtain

$$\varrho_p(D\overline{u}_0^-) + \varrho_q(D\overline{u}_0^-) = 0,$$

so $\overline{u}_0 \geqslant 0$, $\overline{u}_0 \neq 0$.

Next, in (28) we use $h = (\overline{u}_0 - u)^+ \in W_0^{1,p(z)}(\Omega)$. We have

$$\begin{aligned} & \langle A_{p(z)}(\overline{u}_0), (\overline{u}_0 - u)^+\rangle + \langle A_{q(z)}(\overline{u}_0), (\overline{u}_0 - u)^+\rangle \\ = & \int_\Omega (c_0 u^{\tau(z)-1} - c_8 u^{p_+-1})(\overline{u}_0 - u)^+\,dz \\ \leqslant & \int_\Omega f(z,u)(\overline{u}_0 - u)^+\,dz \\ = & \langle A_{p(z)}(u), (\overline{u}_0 - u)^+\rangle + \langle A_{q(z)}(u), (\overline{u}_0 - u)^+\rangle \end{aligned}$$

See (26), (22) and since $u \in S_+$, so $\overline{u}_0 \leqslant u$ (see Proposition 2).

Thus we have proved that

$$u_0 \in [0,u], \; u_0 \neq 0, \tag{29}$$

where $[0,u] = \{w \in W_0^{1,p(z)}(\Omega) : \ 0 \leqslant w(z) \leqslant u(z) \text{ for a.a. } z \in \Omega\}$.

From (29), (26), (28) and Proposition 5, it follows that

$$\overline{u}_0 = \overline{u} \in \operatorname{int} C_+,$$

so $\overline{u} \leqslant u$ for all $u \in S_+$ (see (29)).

In a similar fashion, we show that

$$v \leqslant \overline{v} \quad \text{for all } v \in S_-.$$

□

Next following some ideas of Filippakis–Papageorgiou [34], we show that S_+ is downward directed and S_- is upward directed.

Proposition 7. *If hypotheses H_0 and H_1 hold, then S_+ is downward directed and S_- is upward directed.*

Proof. We do the proof for S_+, the proof for S_- being similar.

For $\varepsilon > 0$, we consider the function

$$\gamma_\varepsilon(s) = \begin{cases} -\varepsilon & \text{if} \quad s < -\varepsilon, \\ s & \text{if} \quad -\varepsilon \leqslant s \leqslant \varepsilon, \\ \varepsilon & \text{if} \quad \varepsilon < s, \end{cases}$$

for $s \in \mathbb{R}$. The function γ_ε is Lipschitz continuous. Let $u \in W_0^{1,p(z)}(\Omega)$. Since $W_0^{1,p(z)}(\Omega) \subseteq W_0^{1,p_-}(\Omega)$, from the chain rule for isotropic Sobolev spaces (see Papageorgiou–Rădulescu–Repovš [33] (Proposition 1.4.2, p. 22)), we have

$$D(\gamma_\varepsilon(u)) = \gamma'_\varepsilon(u)Du$$

(recall that by Rademacher's theorem; see Gasiński–Papageorgiou [35] (Theorem 1.5.8, p. 56) γ_ε is differentiable almost everywhere). It follows that

$$\gamma_\varepsilon(u) \in W_0^{1,p(z)}(\Omega), \ D(\gamma_\varepsilon(u)) = \gamma'_\varepsilon(u)Du. \tag{30}$$

Let $u_1, u_2 \in S_+$ and consider $\vartheta \in C_c^1(\Omega)$, $\vartheta \geqslant 0$. We introduce the test functions

$$h_1 = \gamma_\varepsilon((u_1 - u_2)^-)\vartheta \quad \text{and} \quad h_2 = (\varepsilon - \gamma_\varepsilon((u_1 - u_2)^-))\vartheta.$$

Evidently $h_1, h_2 \in W_0^{1,p(z)}(\Omega) \cap L^\infty(\Omega)$ (see (30)). We have

$$\langle A_{p(z)}(u_1), h_1\rangle + \langle A_{q(z)}(u_1), h_1\rangle = \int_\Omega f(z,u_1)h_1\,dz,$$
$$\langle A_{p(z)}(u_2), h_2\rangle + \langle A_{q(z)}(u_2), h_2\rangle = \int_\Omega f(z,u_2)h_2\,dz.$$

We add these two equations, divide with $\varepsilon > 0$ and then let $\varepsilon \to 0^+$. Note that

$$\frac{1}{\varepsilon}\gamma_\varepsilon((u_1 - u_2)^-) \longrightarrow \chi_{\{u_1<u_2\}} \quad \text{for a.a. } z \in \Omega, \text{ as } \varepsilon \to 0^+$$

and

$$\chi_{\{u_1 \geqslant u_2\}} = 1 - \chi_{\{u_1<u_2\}}.$$

So, in the limit as $\varepsilon \to 0^+$, we obtain

$$\begin{aligned} &\langle A_{p(z)}(u_1), \chi_{\{u_1<u_2\}}\vartheta\rangle + \langle A_{q(z)}(u_1), \chi_{\{u_1<u_2\}}\vartheta\rangle \\ &+\langle A_{p(z)}(u_2), \chi_{\{u_1\geqslant u_2\}}\vartheta\rangle + \langle A_{q(z)}(u_2), \chi_{\{u_1\geqslant u_2\}}\vartheta\rangle \\ \geqslant\ & \int_{\{u_1<u_2\}} f(z,u_1)\vartheta\, dz + \int_{\{u_1\geqslant u_2\}} f(z,u_2)\vartheta\, dz \end{aligned} \tag{31}$$

(recall that $\vartheta \geqslant 0$).

Let $\widetilde{u} = \min\{u_1, u_2\} \in W_0^{1,p(z)}(\Omega)$. From (31) we infer that $\widetilde{u}$ is an upper solution for Problem (1). Then, by a standard truncation technique (see for example the proof of Proposition 6), we produce $u \in S_+$ satisfying $u \leqslant \widetilde{u}$. hence $u \leqslant u_1$, $u \leqslant u_2$ and this proves that S_+ is downward directed.

Similarly we show that S_- is upward directed. □

Now we are ready to produce the extremal constant sign solutions.

Proposition 8. *If hypotheses H_0 and H_1 hold, then S_+ has a smallest element $u_* \in \operatorname{int} C_+$ ($u_* \leqslant u$ for all $u \in S_+$), S_- has a biggest element $v_* \in -\operatorname{int} C_+$ ($v \leqslant v_*$ for all $v \in S_-$).*

Proof. From Proposition 7 we know that S_+ is downward directed. Using Lemma 3.10 of Hu–Papageorgiou [36] (p. 178), we can find a decreasing sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq S_+$ such that

$$\inf_{n\in\mathbb{N}} u_n = \inf S_+.$$

We have

$$\langle A_{p(z)}(u_n), h\rangle + \langle A_{q(z)}(u_n), h\rangle = \int_\Omega f(z,u_n)h\, dz \quad \forall h \in W_0^{1,p(z)}(\Omega),\ n \in \mathbb{N} \tag{32}$$

and

$$\overline{u} \leqslant u_n \leqslant u_1 \quad \forall n \in \mathbb{N} \tag{33}$$

(see Proposition 6).

Testing (32) with $h = u_n \in W_0^{1,p(z)}(\Omega)$ and using (33) and Hypothesis H2(i), we see that the sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,p(z)}(\Omega)$ is bounded. So, we may assume that

$$u_n \xrightarrow{w} u_* \quad \text{in } W_0^{1,p(z)}(\Omega) \quad \text{and} \quad u_n \longrightarrow u_* \quad \text{in } L^{p(z)}(\Omega). \tag{34}$$

In (32) we use $h = u_n - u_* \in W_0^{1,p(z)}(\Omega)$, pass to the limit as $n \to \infty$ and use (33) and Hypothesis H2(i). We obtain

$$\lim_{n\to\infty}\Big(\langle A_{p(z)}(u_n), u_n - u_*\rangle + \langle A_{q(z)}(u_n), u_n - u_*\rangle\Big) = 0,$$

so

$$\limsup_{n\to\infty}\Big(\langle A_{p(z)}(u_n), u_n - u_*\rangle + \langle A_{q(z)}(u_*), u_n - u_*\rangle\Big) \leqslant 0$$

since $A_{q(z)}$ is monotone, thus

$$\limsup_{n\to\infty}\langle A_{p(z)}(u_n), u_n - u_*\rangle \leqslant 0$$

See (34) and hence

$$u_n \longrightarrow u_* \quad \text{in } W_0^{1,p(z)}(\Omega) \tag{35}$$

(see Proposition 2).

So, if in (32) we pass to the limit as $n \to \infty$ and use (35), then

$$\langle A_{p(z)}(u_*), h\rangle + \langle A_{q(z)}(u_*), h\rangle = \int_\Omega f(z, u_*) h\, dz \quad \forall h \in W_0^{1,p(z)}(\Omega). \tag{36}$$

Furthermore, from (33), we have

$$\overline{u} \leqslant u_*. \tag{37}$$

Then (36) and (37) imply that $u_* \in S_+$, $u_* = \inf S_+$.

Similarly working with the set S_-, we produce $v_* \in S_-$ such that $v_* = \sup S_-$. Note that since S_- is upward directed, we can find an increasing sequence $\{v_n\}_{n\in\mathbb{N}}$ such that $\sup\limits_{n\in\mathbb{N}} v_n = \sup S_-$. □

4. Nodal Solution

In this section we produce a nodal solution for problem (1). The idea is to use truncations in order to focus on the order interval

$$[v_*, u_*] = \{h \in W_0^{1,p(z)}(\Omega) : \ v_*(z) \leqslant h(z) \leqslant u_*(z) \text{ for a.a. } z \in \Omega\}.$$

Then on account of the extremality of the solutions u_* and v_*, any nontrivial solution of (1) located in $[v_*, u_*]$ and distinct from u_* and v_* will be nodal. To produce such a solution, we will combine tools from critical point theory and from Morse theory (critical groups).

We start with a result which provides the critical groups of the energy functional φ at the origin. The result is a consequence of Hypothesis H2(iv) and follows from Proposition 6 of Leonardi–Papageorgiou [37].

Proposition 9. *If hypotheses H_0 and H_1 hold, then*

$$C_k(\varphi, 0) = 0 \quad \forall k \in \mathbb{N}_0.$$

As mentioned above, to concentrate on the order interval $[v_*, u_*]$, we will use truncations. For this purpose, we introduce the function g defined by

$$g(z,x) = \begin{cases} f(z, v_*(z)) & \text{if} \quad x < v_*(z), \\ f(z,x) & \text{if} \quad v_*(z) \leqslant x \leqslant u_*(z), \\ f(z, u_*(z)) & \text{if} \quad u_*(z) < x. \end{cases} \tag{38}$$

This is a Carathéodory function. We will also use the positive and negative truncations of $g(z,\cdot)$, namely the Carathéodory functions

$$g_\pm(z,x) = g(z, \pm x^\pm). \tag{39}$$

We set

$$G(z,x) = \int_0^x g(z,s)\, ds \quad \text{and} \quad G_\pm(z,x) = \int_0^x g_\pm(z,s)\, ds$$

and consider the C^1-functionals $\widehat{\xi}, \widehat{\xi}_\pm : W_0^{1,p(z)}(\Omega) \longrightarrow \mathbb{R}$ defined by

$$\begin{aligned} \widehat{\xi}(u) &= \int_\Omega \frac{1}{p(z)} |Du|^{p(z)}\, dz + \int_\Omega \frac{1}{q(z)} |Du|^{q(z)}\, dz - \int_\Omega G(z,u)\, dz, \\ \widehat{\xi}_\pm(u) &= \int_\Omega \frac{1}{p(z)} |Du|^{p(z)}\, dz + \int_\Omega \frac{1}{q(z)} |Du|^{q(z)}\, dz - \int_\Omega G(z, \pm u^\pm)\, dz \end{aligned}$$

for all $u \in W_0^{1,p(z)}(\Omega)$.

Since $u_* \in \operatorname{int} C_+$ and $v_* \in -\operatorname{int} C_+$, using Proposition 9 and a simple homotopy invariance argument as in the proof of Proposition 4.4 of Papageorgiou–Rădulescu–Repovš [38], we obtain the following result.

Proposition 10. *If hypotheses H_0 and H_1 hold, then*

$$C_k(\widehat{\xi}, 0) = C_k(\varphi, 0) = 0 \quad \forall k \in \mathbb{N}_0.$$

Now we are ready to produce nodal solutions.

Proposition 11. *If hypotheses H_0 and H_1 hold, then problem (1) has a nodal solution*

$$y_0 \in [v_*, u_*] \cap C_0^1(\overline{\Omega}).$$

Proof. Using (38) and (39), we can easily see that

$$K_{\widehat{\xi}} \subseteq [v_*, u_*] \cap C_0^1(\overline{\Omega}), \quad K_{\widehat{\xi}_+} \subseteq [0, u_*] \cap C_+, \quad K_{\widehat{\xi}_-} \subseteq [v_*, 0] \cap (-C_+).$$

On account of the extremality of u_* and v_*, we have

$$K_{\widehat{\xi}} \subseteq [v_*, u_*] \cap C_0^1(\overline{\Omega}), \quad K_{\widehat{\xi}_+} = \{0, u_*\}, \quad K_{\widehat{\xi}_-} = \{v_*, 0\}. \tag{40}$$

Claim. $u_* \in \operatorname{int} C_+$ and $v_* \in -\operatorname{int} C_+$ are local minimizers of $\widehat{\xi}$.

From (38) and (39) it is clear that the functionals $\widehat{\xi}_\pm$ are coercive. Furthermore, they are sequentially weakly lower semicontinuous. So, we can find $\widehat{u}_* \in W_0^{1,p(z)}(\Omega)$ such that

$$\widehat{\xi}_+(\widehat{u}_*) = \min \{ \widehat{\xi}_+(u) : \ u \in W_0^{1,p(z)}(\Omega) \}. \tag{41}$$

Let $u \in \operatorname{int} C_+$ and choose $t \in (0,1)$ small so that $tu \leqslant u_*$ (recall that $u_* \in \operatorname{int} C_+$ and use Proposition 4.1.22 of Papageorgiou–Rădulescu–Repovš [33] (p. 274)). Using Hypothesis H2(iv) and recalling that $\tau_+ < q_- < p_-$, by choosing $t \in (0,1)$ even smaller, we will have

$$\widehat{\xi}_+(tu) < 0,$$

so, by (41) also

$$\widehat{\xi}_+(\widehat{u}_*) < 0 = \widehat{\xi}_+(0),$$

thus

$$\widehat{u}_* \neq 0$$

and hence

$$\widehat{u}_* = u_*$$

(see (40) and (41)).

From (38) and (39) it is clear that

$$\widehat{\xi}|_{C_+} = \widehat{\xi}_+|_{C_+}.$$

Since $u_* \in \operatorname{int} C_+$, it follows that

$$u_* \text{ is a local } C_0^1(\overline{\Omega})\text{-minimizer of } \widehat{\xi},$$

thus by Proposition 3.3 of Gasiński–Papageorgiou [15] and Theorem 3.2 of Tan–Fang [29], also

$$u_* \text{ is a local } W_0^{1,p(z)}(\Omega)\text{-minimizer of } \widehat{\xi}.$$

Similarly for $v_* \in -\mathrm{int}\, C_+$ using this time the functional $\widehat{\xi}_-$.
This proves the Claim.

Without any loss of generality, we may assume that

$$\widehat{\xi}(v_*) \leqslant \widehat{\xi}(u_*).$$

The reasoning is similar if the opposite inequality holds.

From (40) we see that we may assume that $K_{\widehat{\xi}}$ is finite. Otherwise we already have an infinity of nodal solutions of (1) and so we are done. By Theorem 5.7.6 of Papageorgiou–Rădulescu–Repovš [33] (p. 449), we can find $\varrho \in (0,1)$ small such that

$$\widehat{\xi}(v_*) \leqslant \widehat{\xi}(u_*) < \inf\left\{\widehat{\xi}(u) : \|u - u_*\| = \varrho\right\} = \widehat{m}, \quad \|v_* - u_*\| > \varrho. \tag{42}$$

From (38) it is clear that $\widehat{\xi}$ is coercive. So, using Proposition 5.1.15 of Papageorgiou–Rădulescu–Repovš [33] (p. 369), we have that

$$\widehat{\xi} \text{ satisfies the Palais–Smale condition.} \tag{43}$$

Then (42) and (43) permit the use of the mountain pass theorem. So, we can find $y_0 \in W_0^{1,p(z)}(\Omega)$ such that

$$y_0 \in K_{\widehat{\xi}} \subseteq [v_*, u_*] \cap C_0^1(\overline{\Omega}), \quad \widehat{m} \leqslant \widehat{\xi}(y_0) \tag{44}$$

(see (40) and (42)). From (42) and (44), it follows that

$$y_0 \notin \{v_*, u_*\}.$$

Since y_0 is a critical point of $\widehat{\xi}$ of the mountain pass type, using Theorem 6.5.8 of Papageorgiou–Rădulescu–Repovš [33] (p. 527), we have

$$C_1(\widehat{\xi}, y_0) \neq 0. \tag{45}$$

From (45) and Proposition 10, we infer that $y_0 \neq 0$. Therefore

$$y_0 \in C_0^1(\overline{\Omega}) \text{ is a nodal solution of (1).}$$

□

Finally we can state the following multiplicity theorem for Problem (1).

Theorem 1. *If hypotheses H_0 and H_1 hold, then Problem* (1) *has at least three nontrivial solutions*

$$u_0 \in \mathrm{int}\, C_+, \quad v_0 \in -\mathrm{int}\, C_+ \quad \text{and} \quad y_0 \in [v_0, u_0] \cap C_0^1(\overline{\Omega}) \text{ nodal}.$$

Remark 2. *In this paper we examined resonant anisotropic problems in which the resonance occurs from the left of $\widehat{\lambda}_1$ (see Hypothesis H2(ii)). This made the relevant energy functionals coercive (see Proposition 3). It is an interesting open problem what can be said if the resonance if from the right of $\widehat{\lambda}_1$. In this case the functionals fail to be coercive.*

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