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Hypercompositional Algebra, Computer Science and Geometry

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Abstract: The various branches of Mathematics are not separated between themselves. On the contrary, they interact and extend into each other's sometimes seemingly different and unrelated areas and help them advance. In this sense, the Hypercompositional Algebra's path has crossed, among others, with the paths of the theory of Formal Languages, Automata and Geometry. This paper presents the course of development from the hypergroup, as it was initially defined in 1934 by F. Marty to the hypergroups which are endowed with more axioms and allow the proof of Theorems and Propositions that generalize Kleen's Theorem, determine the order and the grade of the states of an automaton, minimize it and describe its operation. The same hypergroups lie underneath Geometry and they produce results which give as Corollaries well known named Theorems in Geometry, like Helly's Theorem, Kakutani's Lemma, Stone's Theorem, Radon's Theorem, Caratheodory's Theorem and Steinitz's Theorem. This paper also highlights the close relationship between the hyperfields and the hypermodules to geometries, like projective geometries and spherical geometries.

Keywords: hypergroup; hyperfield; formal languages; automata; convex set; vector space; geometry

1. Introduction

This paper is written in the context of the special issue "Hypercompositional Algebra and Applications" in "*Mathematics*" and it aims to shed light on two areas where the Hypercompositional Algebra has expanded and has interacted with them: Computer Science and Geometry.

Hypercompositional Algebra is a branch of Abstract Algebra which appeared in the 1930s via the introduction of the hypergroup.

It is interesting that the group and the hypergroup are two algebraic structures which satisfy exactly the same axioms, i.e., the associativity and the reproductivity, but they differ in the law of synthesis. In the first one, the law of synthesis is a composition, while in the second one it is a hypercomposition. This difference makes the hypergroup a much more general algebraic structure than the group, and for this reason the hypergroup has been gradually enriched with further axioms, which are either more powerful or less powerful, leading thus to a significant number of special hypergroups. Among them, there exist hypergroups that were proved to be very useful for the study of Formal Languages and Automata, as well as convexity in Euclidian vector spaces. Furthermore, based on these hypergroups, there derived other hypercompositional structures, which are equally as useful in the study of Geometries (spherical, projective, tropical, etc.) and Computer Science.

A *binary operation* (or *composition*) " $\cdot$ " on a non-void set E is a rule which assigns a unique element of E to each element of $E \times E$. The notation $a_i \cdot a_j = a_k$, where a_i, a_j, a_k are elements of E, indicates that a_k is the result of the operation " $\cdot$ " performed on the operands a_i and a_j. When no confusion arises, the operation symbol " $\cdot$ " may be omitted, and we write $a_i a_j = a_k$. If the binary operation is *associative*, that is, if it satisfies the equality

$$a(bc) = (ab)c \text{ for all } a, b, c \in E$$

the pair $(E, \cdot)$ is called *semigroup*. An element $e \in E$ is an *identity* of $(E, \cdot)$ if for all $a \in E$, $ea = ae = a$. A triplet $(E, \cdot, e)$ is a *monoid* if $(E, \cdot)$ is a semigroup and e is its identity. If no ambiguity arises, we can denote a semigroup $(E, \cdot)$ or a monoid $(E, \cdot, e)$ simply by E. A binary operation on E is *reproductive* if it satisfies:

$$aE = Ea = E \text{ for all } a \in E$$

Definition 1. *(Definition of the group) The pair $(G, \cdot)$ is called group if G is a non-void set and $\cdot$ is an associative and reproductive binary operation on G.*

This definition does not appear in group theory books, but it is equivalent to the one mentioned in them. We introduce it here as we consider it to be the most appropriate in order to demonstrate the close relationship between the group and the hypergroup. Thus, the following two properties of the group derive from the above definition:

Property 1. *In a group G there is an element e, called the identity element, such that $ae = ea = a$ for all a in G.*

Property 2. *For each element a of a group G there exists an element a' of G, called the inverse of a, such that $a \cdot a' = a' \cdot a = e$.*

The proof of the above properties can be found in [1,2]. If no ambiguity arises, we may abbreviate $(G, \cdot)$ to G.

Example 1. *(The free monoid) Often, a finite non-empty set A is referred to as an alphabet. The elements of $A^{\{1,\dots,l\}}$, i.e., the functions from $\{1, \dots, l\}$ to A, are called strings (or words) of length l. When $l = 0$, then we have $A^{\varnothing}$ which is equal to $\{\varnothing\}$. The empty set $\varnothing$ is the only string of length 0 over A. This string is called the empty string and it is denoted by λ. If x is a string of length l over A and $x(i) = a_i$, $i = 1, \dots, l$, we write $x = (a_1, a_2, \dots, a_l)$. The set*

$$strings(A) = A^* = \bigcup_{n=0}^{\infty} A^{\{1,\dots,n\}}$$

becomes a monoid if we define

$$(a_1, a_2, \dots, a_l)(b_1, b_2, \dots b_k) = (a_1, a_2, \dots, a_l, b_1, b_2, \dots b_k)$$

The identity element of this binary operation, which is called string concatenation, is λ. The strings of length 1 generate the monoid, since every element $(a_1, \dots, a_l)$ is a finite product $(a_1) \dots (a_l)$ of strings of length 1. The function g from A to $A^{\{1\}}$ which is defined by

$$g(a) = (a),\ a \in A$$

is a bijection. Thus we may identify a string (a), of length 1 over A, with its only element a. This means that we can regard the sets $A^{\{1\}}$ and A as identical and consequently we may regard the elements of A^* as words $a_1 a_2 \dots a_l$ in the alphabet A. It is obvious that for all $x \in A^*$ there is exactly one natural number n and exactly one sequence of elements $a_1, a_2, \dots, a_n$ of A, such that $x = a_1 a_2 \dots a_n$. $A^*$is called the free monoid on the set (or alphabet) A.

A generalization of the binary operation is the *binary hyperoperation* or *hypercomposition* " $\cdot$ " on a non-void set E, which is a rule that assigns to each element of $E \times E$ a unique element of the power set $\mathcal{P}(E)$ of E. Therefore, if a_i, a_j are elements of E, then $a_i \cdot a_j \subseteq E$. When there is no likelihood of confusion, the symbol " $\cdot$ " can be omitted and we write $a_i a_j \subseteq E$. If $A,\ B$ are subsets of E, then $A \cdot B$ signifies the union $\bigcup\limits_{(a,b)\in A\times B} a \cdot b$. In particular if $A = \varnothing$ or $B = \varnothing$, then $AB = \varnothing$. In both cases, aA and Aa have the same meaning as $\{a\}A$ and $A\{a\}$ respectively. Generally, the singleton $\{a\}$ is identified with its member a.

Definition 2. *(Definition of the hypergroup) The pair $(H, \cdot)$ is called hypergroup, if H is a non-void set and $\cdot$ is an associative and reproductive binary hypercomposition on H.*

The following Proposition derives from the definition of the hypergroup:

Proposition 1. *In a hypergroup the result of the hypercomposition of any two elements is non-void.*

The proof of Proposition 1 can be found in [3,4]. If no ambiguity arises, we may abbreviate $(H, \cdot)$ to H. Two significant types of hypercompositions are the closed and the open ones. A hypercomposition is called *closed* [5] (or *containing* [6], or *extensive* [7]) if the two participating elements always belong to the result of the hypercomposition, while it is called *open* if the result of the hypercomposition of any two elements different from each other does not contain the two participating elements.

The notion of the hypergroup was introduced in 1934 by F. Marty, who used it in order to study problems in non-commutative algebra, such as cosets determined by non-invariant subgroups [8–10]. From Definitions 1 and 2 it is evident that both groups and hypergroups satisfy the same axioms and their only difference is that the law of synthesis of two elements is a composition in groups, while it is a hypercomposition in hypergroups. This difference makes the hypergroups much more general algebraic structures than the groups, to the extent that properties similar to the previous 1 and 2 generally cannot be proved for the hypergroups. Furthermore, in the hypergroups there exist different types of identities [11–13]. In general, an element $e \in H$ is an identity if $a \in ae \cap ea$ for all $a \in H$. An identity is called *scalar* if $a = ae = ea$ for all $a \in H$, while it is called *strong* if $a \in ae = ea \subseteq \{e, a\}$ for all $a \in H$. Obviously, if the hypergroup has an identity, then the hypercomposition cannot be open.

Besides, in groups, both equations $a = xb$ and $a = bx$ have a unique solution, while, in the hypergroups, the analogous relations $a \in xb$ and $a \in bx$ do not have unique solutions. Thus F. Marty in [8] defined the two induced hypercompositions (right and left division) that derive from the hypergroup's hypercomposition:

$$\frac{a}{|\,b} = \{x \in H \mid a \in xb\} \text{ and } \frac{a}{b\,|} = \{x \in H \mid a \in bx\}.$$

If H is a group, then $\frac{a}{|\,b} = ab^{-1}$ and $\frac{a}{b\,|} = b^{-1}a$. It is obvious that if "." is commutative, then the right and the left division coincide. For the sake of notational simplicity, $a\ /\ b$ or $a : b$ is used to denote the right division, or right hyperfraction, or just the division in the commutative hypergroups and $b \setminus a$ or $a..b$ is used to denote the left division, or left hyperfraction [14,15]. Using the induced hypercomposition we can create an axiom equivalent to the reproductive axiom, regarding which, the following Proposition is valid [3,4]:

Proposition 2. *In a hypergroup H, the non-empty result of the induced hypercompositions is equivalent to the reproductive axiom.*

W. Prenowitz enriched a commutative hypergroup of idempotent elements with one more axiom, in order to use it in the study of Geometry [16–21]. More precisely, in a commutative hypergroup H, all the elements of which satisfy the properties $aa = a$ and $a \,/\, a = a$, he introduced the *transposition axiom*:

$$a \,/\, b \cap c \,/\, d \neq \varnothing \text{ implies } ad \cap bc \neq \varnothing \text{ for all } a, b, c, d \in H.$$

He named this new hypergroup *join space*. For the sake of terminology unification, a commutative hypergroup which satisfies the transposition axiom is called *join hypergroup*. Prenowitz was followed by J. Jantosciak [20–23], V. W. Bryant, R. J. Webster [24], D. Freni, [25,26], J. Mittas, C. G. Massouros [2,27–29], A. Dramalidis [30,31], etc. In the course of his research, J. Jantosciak generalized the transposition axiom in an arbitrary hypergroup as follows:

$$b \setminus a \cap c \,/\, d \neq \varnothing \text{ implies } ad \cap bc \neq \varnothing \text{ for all } a, b, c, d \in H$$

and he named this hypergroup *transposition hypergroup* [15]. These algebraic structures attracted the interest of a big number of researchers, among whom there are, J. Jantosciak [15,22,23], I. Cristea [32–35], P. Corsini, [35–42], V. Leoreanu-Fortea [40–46], S. Hoskova-Mayerova, [47–51], J. Chvalina [48–52], P. Rackova [49,50], C. G. Massouros [3,6,12,13,53–62], G. G. Massouros [3,12,13,53–62], R. Ameri [63–65], M. M. Zahedi [63], I. Rosenberg [66], etc.

Furthermore, it has been proved that these hypergroups also comprise a useful tool in the study of Languages and Automata [67–71] and a constructive origin for the development of other, new, hypercompositional structures [53,57,58,72,73].

Definition 3. *A transposition hypergroup, which has a scalar identity e, is called quasicanonical hypergroup* [74,75] *or polygroup* [76–78].

In the quasicanonical hypergroups, there exist properties analogous to 1 and 2 which are valid in the groups:

Proposition 3. [15,53] *If $(Q, \cdot, e)$ is a quasicanonical hypergroup, then:*

(i) *for each $a \in Q$ there exists one and only one $a' \in Q$ such that $e \in aa' = a'a$*
(ii) $c \in ab \Rightarrow a \in cb' \Rightarrow b \in a'c$

The inverse is also true:

Proposition 4. [15,53] *If a hypergroup Q has a scalar identity e and*

(i) *for each $a \in Q$ there exists one and only one $a' \in Q$ such that $e \in aa' = a'a$*
(ii) $c \in ab \Rightarrow a \in cb' \Rightarrow b \in a'c$

then, the transposition axiom is valid in Q.

A commutative quasicanonical hypergroup is called *canonical hypergroup*. This hypergroup was first used by M. Krasner [79] but it owes its name to J. Mittas [80,81].

A non-empty subset K of a hypergroup H is called *semi-subhypergroup* when it is stable under the hypercomposition, i.e., $xy \subseteq K$ for all $x, y \in K$. K is a *subhypergroup* of H if it satisfies the reproductive axiom, i.e., if the equality $xK = Kx = K$ is valid for all $x \in K$. Since the structure of the hypergroup is much more complicated than that of the group, there are various kinds of subhypergroups. A subhypergroup K of H is called *closed from the right* (in H), (resp. *from the left*) if, for every element x in the complement K^c of K, it holds that $(xK) \cap K = \varnothing$ (resp.$(Kx) \cap K = \varnothing$). K is called *closed* if it is closed both, from the right and from the left [82–84]. It has been proved [1] that a subhypergroup is closed if and only if it is stable under the induced hypercompositions, i.e.,

$$a \mathbin{/} b \subseteq K \text{ and } b \setminus a \subseteq K \text{ for all } a, b \in K.$$

A subhypergroup K of a hypergroup is *invertible* if $a \mathbin{/} b \cap K \neq \varnothing$ implies $b \mathbin{/} a \cap K \neq \varnothing$, and $b \setminus a \cap K \neq \varnothing$ implies $a \setminus b \cap K \neq \varnothing$. From this definition it derives that every invertible subhypergroup is also closed, but the opposite is not valid.

Proposition 5. [1,14] *If a subset K of a hypergroup H is stable under the induced hypercompositions, then K is a subhypergroup of H.*

Proposition 6. [1,14] *If K is a closed subhypergroup of a hypergroup H and $a \in K$, then:*

$$a \mathbin{/} K = K \mathbin{/} a = aK = K = Ka = K \setminus a = a \setminus K.$$

It has been proved [14,60,62] that the set of the semi-subhypergroups (resp. the set of the closed subhypergroups) which contains a non-void subset E is a complete lattice. Hence, the minimum (in the sense of inclusion) semi-subhypergroup of a hypergroup H, which contains a given non-empty subset E of H, can be assigned to E. This semi-subhypergroup is denoted by $[E]$ and it is called the generated by E semi-subhypergroup of H. Similarly, $\langle E \rangle$ is the generated by E closed subhypergroup of H. For notational simplicity, if $E = \{a_1, \ldots, a_n\}$, then $[E] = [a_1, \ldots, a_n]$ and $\langle E \rangle = \langle a_1, \ldots, a_n \rangle$ are used instead.

Duality. Two statements of the theory of hypergroups are *dual statements* (see [15,53]), if each one of them results from the other by interchanging the order of the hypercomposition "·", that is, interchanging any hyperproduct ab with ba. Observe that the reproductive and the associative axioms are self-dual. Moreover, observe that the induced hypercompositions / and \ have dual definitions; hence, they must be interchanged during the construction of a dual statement. So, the transposition axiom is self-dual as well. Therefore, the following principle of duality holds for the theory of hypergroups:

Given a Theorem, the dual statement, which results from the interchange of the order of the hypercomposition (and the necessary interchange of / and \), is also a Theorem.

Special notation: In the following pages, apart from the typical algebraic notations, we are using Krasner's notation for the complement and difference. So, we denote with $A..B$ the set of elements that are in the set A, but not in the set B.

2. Formal Languages, Automata Theory and Hypercompositional Structures

Mathematically, a language whose words are written with letters from an alphabet Σ, is defined as a subset of the free monoid Σ^* generated by Σ. The above definition of the language is fairly general and it includes all the written natural languages as well as the artificial ones. In general, a language is defined in two ways: It is either presented as an exhaustive list of all its valid words, i.e., through a dictionary, or it is presented as a set of rules defining the acceptable words. Obviously the first method can only be used when the language is finite. All the natural languages, such as English or Greek are finite and they have their own dictionaries. Artificial languages, on the other hand, may be infinite, and they can only be defined by the second way.

In the artificial languages, precision and no guesswork are required, especially when computers are concerned. The regular expressions, which are very precise language-defining symbols, were created and developed as a language-defining symbolism. The languages that are associated with these expressions are called *regular languages.*

The regular expressions were introduced by Kleene [85] who also proved that they are equivalent in expressive power to finite automata. McNaughton and Yamada gave their own proof to this [86], while Brzozowski [87,88] and Brzozowski and McClusky [89] further developed the theory of regular expressions. In regular languages the expression $x + y$ where x and y are strings of characters from an alphabet Σ means "either x or y". Therefore $x + y = \{x, y\}$. In this way the monoid Σ^* is enriched with a hypercomposition. This hypercomposition is named *B-hypercomposition* [67–69].

Proposition 7. [67,68] *A non-void set equipped with the B-hypecomposition is a join hypergroup.*

A hypergroup equipped with the B-hypercomposition is called *B-hypergroup* [67–69]. Moreover, the empty set of words and its properties in the theory of the regular languages leads to the following extension: Let $0 \notin \Sigma^*$. In the set $\overline{\Sigma^*} = \Sigma^* \cup \{0\}$ a hypercomposition, called *dilated B-hypercomposition*, is defined as follows:

$$x + y = \{x, y\} \quad \text{if } x, y \in \overline{\Sigma^*} \text{ and } x \neq y$$
$$x + x = \{x,\ 0\} \quad \text{for all } x \in \overline{\Sigma^*}$$

The associativity and the commutativity of the dilated B-hypercomposition derive without difficulty. Moreover, the transposition axiom is verified, since

$$x \,/\, y = y \setminus x = \begin{cases} H, & if \;\; x = y \\ \{x, y\}, & if \;\; x \neq y \;\; and \;\; x = e \\ x, & if \;\; x \neq y \;\; and \;\; x \neq e \end{cases}$$

This join hypergoup is called *dilated B-hypergroup* and it has led to the definition of a new class of hypergroups, the class of the *fortified transposition hypergroups* and *fortified join hypergroups*.

An *automaton* $\mathcal{A}$ is a collection of five objects $(\Sigma, S, \delta, s_o, F)$, where Σ is the alphabet of input letters (a finite nonempty set of symbols), S is a finite nonvoid set of states, s_o is an element of S indicating the start (or initial) state, F is a (possibly empty) subset of S representing the set of the final (or accepting) states and δ is the state transition function with domain $S \times \Sigma$ and range S, in the case of a deterministic automaton (DFA), or $\mathcal{P}(S)$, the powerset of S, in the case of a nondeterministic automaton (NDFA). Σ^* denotes the set of words (or strings) formed by the letters of Σ –closure of Σ– and $\lambda \in \Sigma^*$ signifies the empty word. Given a DFA $\mathcal{A}$, the extended state transition function for $\mathcal{A}$, denoted δ^*, is a function with domain $S \times \Sigma^*$ and range S defined recursively as follows:

i. $\delta^*(s, a) = \delta(s, a)$ for all s in S and a in Σ
ii. $\delta^*(s, \lambda) = s$ for all s in S
iii. $\delta^*(s, ax) = \delta^*(\delta(s, a), x)$ for all s in S, x in Σ^* and a in Σ.

In [67–71] it is shown that the set of the states of an automaton, equipped with different hypercompositions, can be endowed with the structure of the hypergroup. The hypergroups that have derived in this way are named *attached hypergroups to the automaton*. To date, various types of attached hypergroups have been developed to represent the structure and operation of the automata with the use of the hypercompositional algebra tools. Between them are:

i. the attached hypergroups of the order, and
ii. the attached hypergroups of the grade.

Those two types of hypergroups were also used for the minimisation of the automata. In addition, in [69] another hypergroup, derived from a different definition of the hypercomposition, was attached to the set of the states of the automaton. Due to its definition, this hypergroup was named the *attached hypergroup of the paths* and it led to a new proof of Kleene' s Theorem. Furthermore, in [70], the *attached hypergroup of the operation* was defined in automata. One of its applications is that this hypergroup can indicate all the states on which an automaton can be found after the t-clock pulse. For the purpose of defining the attached hypergroup of the operation, the notions of the Prefix and the Suffix of a word needed to be introduced. Let x be a word in Σ^*, then:

$$Prefix(x) = \{y \in \Sigma^* \mid yz = x \text{ for some } z \in \Sigma^*\} \text{ and } Suffix(x) = \{z \in \Sigma^* \mid yz = x \text{ for some } y \in \Sigma^*\}$$

Let s be an element of S. Then:

$$I_s = \{x \in \Sigma^* \mid \delta^*(s_o, x) = s\} \text{ and } P_s = \left\{ \{s_i \in S \mid s_i = \delta^*(s_o, y),\ y \in Prefix(x),\ x \in I_s\} \right\}$$

Obviously, the states s_o and s are in P_s.

Lemma 1. *If $r \in P_s$, then $P_r \subseteq P_s$.*

Proof. $P_r = \{s_i \in S \mid s_i = \delta^*(s_o, y),\ y \in Prefix(v),\ v \in I_r\}$ and since $r \in P_s$, it holds that $\delta^*(r, z) = s$, for some z in $Suffix(x),\ x \in I_s$. Thus $\delta^*(s_i, y_i) = s,\ y_i \in Suffix$ and so the Lemma. □

With the use of the above notions, more hypercompositional structures can be attached on the set of the states of the automaton.

Proposition 8. *The set S of the states of an automaton equipped with the hypercomposition*

$$s + q = P_s \bigcup P_q \quad for\ all\ \ s,\ q \in S$$

becomes a join hypergroup.

Proof. Initially, notice that $s + S = \bigcup_{q \in S} (P_s \cup P_q) = S$. Hence, the reproductive axiom is valid. Next, the definition of the hypercomposition yields the equality:

$$s + (q + r) = s + (P_q \cup P_r) = P_s \cup (\bigcup_{u \in P_q \cup P_r} P_u)$$

Per Lemma 1, the right-hand side of the above equality is equal to $P_s \cup (P_q \cup P_r)$ which, however, is equal to $(P_s \cup P_q) \cup P_r$. Using again Lemma 1, we get the equality $(P_s \cup P_q) \cup P_r = (\bigcup_{v \in P_s \cup P_q} P_v) \cup P_r$

Thus:

$$(\bigcup_{v \in P_s \cup P_q} P_v) \cup P_r = (P_s \cup P_q) + r = (s + q) + r$$

and so, the associativity is valid. Next, observe that the hypercomposition is commutative and therefore:

$$s \ / \ q = q \setminus s = \begin{cases} S, & if\ \ s \in P_q \\ \{r \in S \mid P_s \subseteq P_r\}, & if\ \ s \notin P_q \end{cases}$$

Suppose that $s \ / \ q \cap p \ / \ r \neq \varnothing$. Then, $(s + r) \cap (q + p) = (P_s \cup P_r) \cap (P_q \cup P_p)$ which is non-empty, since it contains s_o. Hence the transposition axiom is valid and so the Proposition. □

Proposition 9. *The set S of the states of an automaton equipped with the hypercomposition*

$$s + q = P_s \cap P_q \quad for\ all\ \ s,\ q \in S$$

becomes a join semihypergroup.

Proof. Since $s_o \in P_r$, for all $r \in S$, the result of the hypercomposition is always non-void. On the other hand

$$s \;/\; q = q \setminus s = \begin{cases} S, \; if \;\; s \in P_q \\ \varnothing, \; if \;\; s \notin P_q \end{cases}$$

hence, since $s \;/\; q$, with s, q in S, is not always nonvoid, the reproductive axiom is not valid. The associativity can be verified in the same way as in the previous Proposition. Finally if $s \,/\, q \cap p \,/\, r \neq \varnothing$, then $s \;/\; q \cap p \;/\; r = S$ and so the intersection $(s+r) \cap (q+p)$ which is equal to $(P_s \cap P_r) \cap (P_q \cap P_p)$ is non-empty, since it contains s_o. □

G. G. Massouros [68–73], G. G. Massouros and J. D. Mittas [67] and after them J. Chvalina [90–92], L. Chvalinova [90], M. Novak [91–94], S. Křehlík [91–93], M. M. Zahedi [95], M. Ghorani [95,96] etc, studied automata using algebraic hypercompositional structures.

Formal Languages and Automata theory are very close to Graph theory. P. Corsini [97,98], M. Gionfriddo [99], Nieminen [100,101], I. Rosenberg [66], M. De Salvo and G. Lo Faro [102–104], I. Cristea et al. [105–108], C. Massouros and G. Massouros [109,110], C. Massouros and C. Tsitouras [111,112] and others studied hypergroups associated with graphs. In the following we will present how to attach a join hypergroup to a graph. In general, a *graph* is a set of points called *vertices* connected by lines, which are called *edges*. A *path* in a graph is a sequence of no repeated vertices $v_1, \; v_2, \; \ldots, \; v_n$, such that $\overline{v_1 v_2}, \; \overline{v_2 v_3}, \; \ldots, \; \overline{v_{n-1} v_n}$ are edges in the graph. A graph is said to be *connected* if every pair of its vertices is connected by a path. A *tree* is a connected graph with no cycles. Let $\mathcal{T}$ be a tree. In the set V of its vertices a hypercompostion "·" can be introduced as follows: for each two vertices $x, \; y$ in V, $x \cdot y$ is the set of all vertices which belong to the path that connects vertex x with vertex y. Obviously this hypercomposition is a closed hypercomposition, i.e., $x, \; y$ are contained in $x \cdot y$ for every $x, \; y$ in V.

Proposition 10. *If V is the set of the vertices of a tree $\mathcal{T}$, then $(V, \cdot)$ is a join hypergroup.*

Proof. Since $\{x, y\} \subseteq xy$, it derives that $xV = V$ for each x in V and therefore the reproductive axiom is valid. Moreover, since $\mathcal{T}$ is an undirected graph, the hypecomposition is commutative. Next, let $x, \; y, \; z$ be three arbitrary vertices of $\mathcal{T}$. If any of these three vertices, e.g., z, belongs to the path that the other two define, then $(xy)z = x(yz) = xy$. If $x, \; y, \; z$ do not belong to the same path, then there exists only one vertex v in xy such that $vz \cap xy = \{v\}$. Indeed if there existed a second vertex w such that $wz \cap xy = \{w\}$, then the tree $\mathcal{T}$ would have a cycle, which is absurd. So $(xy)z = xy \cup vz$ and $x(yz) = xv \cup yz$. Since $xy \cup vz = xv \cup yz$, it derives that $(xy)z = x(yz)$. Now, for the transposition axiom, suppose that $x, \; y, \; z, \; w$ are vertices of $\mathcal{T}$ such that $x \;/\; y \cap z \;/\; w \neq \varnothing$. If $x, \; y, \; z, \; w$ are in the same path, then considering all their possible arrangements in their path, it derives that $xw \cap yz \neq \varnothing$. Next, suppose that the four vertices do not belong to the same path. Thus, suppose that z does not belong to the path defined by $y, \; w$. Then, $z \notin yw$. Consider zy and zw. As indicated above, since there are no cycles in $\mathcal{T}$, there exists only one vertex v in xy such that $zy = yv \cup vz$ and $zw = wv \cup vz$. Now, we distinguish between the cases:

(i) if x, y, w do not belong to the same path, then for the same reasons as above there exists only one s in xy such that $xy = ys \cup sx$ and $sw = ws \cup sx$. Since $x \;/\; y \cap z \;/\; w \neq \varnothing$, there exists r in V such that $x \in ry$ and $z \in rw$. Thus, since $\mathcal{T}$ contains no cycles, and in order for srv not to form a cycle, s and v must coincide. Hence, $v \in xw \cap yz$ and therefore $xw \cap yz \neq \varnothing$.

(ii) if x belongs to the same path with y and w, then:

- (ii_a) if $x \in yw$, then $yw = yx \cup xw$ and $xw \subseteq x \;/\; y$. Hence, $v = x$, $x \in xw \cap yz$ and therefore $xw \cap yz \neq \varnothing$.
- (ii_b) if $x \notin yw$, then $x \;/\; y \cap z \;/\; w = \varnothing$. □

A *spanning tree* of a connected graph is a tree whose vertex set is the same as the vertex set of the graph, and whose edge set is a subset of the edge set of the graph. Any connected graph has at least one spanning tree and there exist algorithms, which find such trees. Hence, any connected graph can be endowed with the join hypergroup structure through its spanning trees. Moreover, since a connected graph may have more than one spanning trees, more than one join hypergroups can be associated to a graph. On the other hand, in any connected or not connected graph, a hypergroup can be attached according to the following Proposition:

Proposition 11. *The set V of the vertices of a graph, is equipped with the structure of the hypergroup, if the result of the hypercomposition of two vertices v_i and v_j is the set of the vertices which appear in all the possible paths that connect v_i to v_j, or the set $\{v_i, v_j\}$, if there do not exist any connecting paths from vertex v_i to vertex v_j.*

2.1. Fortified Transposition Hypergroups

Definition 4. *A fortified transposition hypergroup (FTH) is a transposition hypergroup H with a unique strong identity e, which satisfies the axiom:*

for every $x \in H..\{e\}$ there exists one and only one element $y \in H..\{e\}$, such that $e \in xy$ and $e \in yx$.
y is denoted by x^{-1} and it is called inverse or symmetric of x. When the hypercomposition is written additively, the strong identity is denoted by 0, the unique element y is called opposite or negative instead of inverse and the notation $-x$ is used. If the hypercomposition is commutative, the hypergroup is called fortified join hypergroup (FJH).

It has been proved that every FTH consists of two types of elements, the *canonical* (*c-elements*) and the *attractive* (*a-elements*) [53,57]. An element x is called canonical if $ex = xe$ is the singleton $\{x\}$, while it is called attractive if $ex = xe = \{e, x\}$. We denote with A the set of the a-elements and with C the set of the c-elements. By convention $e \in A$.

Proposition 12. [53,57]

(i) *if x is a non-identity attractive element, then* $e / x = e \cdot x^{-1} = \left\{x^{-1}, e\right\} = x^{-1} \cdot e = x \setminus e$
(ii) *if x is a canonical element, then* $e / x = x^{-1} = x \setminus e$
(iii) *if x, y are attractive elements and* $x \neq y$*, then* $x \cdot y^{-1} = x / y \cup \left\{y^{-1}\right\}$ *and* $y^{-1} \cdot x = y \setminus x \cup \left\{y^{-1}\right\}$
(iv) *if x, y are canonical elements, then* $x \cdot y^{-1} = x / y$ *and* $y^{-1} \cdot x = y \setminus x$.

Theorem 1. [57]

(i) *the result of the hypercomposition of two a-elements is a subset of A and it always contains these two elements.*
(ii) *the result of the hypercomposition of two non-symmetric c-elements consists of c-elements,*
(iii) *the result of the hypercomposition of two symmetric c-elements contains all the a-elements.*
(iv) *the result of the hypercomposition of an a-element with a c-element is the c-element.*

Theorem 2. [53,57] *If H is a FTH, then the set A of the attractive elements is the minimum (in the sense of inclusion) closed subhypergroup of H.*

The proof of the above Theorems as well as other properties of the theory of the FTHs and FJHs can be found in [53,55–57,61]. The next two Propositions refer to the reversibility in FTHs.

Lemma 2. *If* $w \in x \cdot y$*, then* $w \cdot x^{-1} \cap e \cdot y \neq \varnothing$ *and* $w \cdot y^{-1} \cap e \cdot x \neq \varnothing$

Proof. $w \in x \cdot y$ implies $x \in w \ / \ y$ and $y \in x \setminus w$. Moreover $x \in x^{-1} \setminus e$ and $y \in e \ / y^{-1}$. Consequently, $w \ / \ y \cap x^{-1} \setminus e \neq \varnothing$ and $x \setminus w \cap y \in e \ / \ y^{-1} \neq \varnothing$. Next, the transposition axiom gives the Lemma. □

Proposition 13. *If $w \in x \cdot y$, and if any one of x, y is a canonical element, then*

$$x \in w \cdot y^{-1} \ \text{ and } \ y \in x^{-1} \cdot w$$

Proof. We distinguish between two cases:

(i) If $x, y \in C$, then $e \cdot x = x$ and $e \cdot y = y$. Next Lemma 2 applies and yields the Proposition.
(ii) Suppose that $x \in A$ and $y \in C$. Then, according to Theorem 1(iv), $x \cdot y = y$; thus, $w = y$. Via Theorem 1(iii), $A \subseteq y \cdot y^{-1}$; thus, $x \in y \cdot y^{-1}$. Per Theorem 2, $x^{-1} \in A$, consequently $y = x^{-1}y$.

Hence the Proposition is proved. □

Proposition 14. *Suppose that x, y are attractive elements and $w \in x \cdot y$.*

(i) if $w = y = e$, then $e \in xe$ implies that $e \in ex$, while $x \notin ee$
(ii) if $w = x \neq y$, then $x \in x \cdot y$ implies that $x \in xy^{-1}$, while, generally, $y \notin xx^{-1}$
(iii) in any other case $w \in x \cdot y$ implies $x \in w \cdot y^{-1}$ and $y \in x^{-1} \cdot w$

Sketch of Proof. Cases (i) and (ii) are direct consequences of the Theorem 1, while case (iii) derives from the application of Lemma 2. □

The property which is described in Proposition 13 is called *reversibility* and because of Proposition 14, this property holds partially in the case of TPH.

Another distinction between the elements of the FTH stems from the fact that the equality $\left(xx^{-1}\right)^{-1} = xx^{-1}$ (or $-(x - x) = x - x$ in the additive case) is not always valid. The elements that satisfy the above equality are called *normal*, while the others are called *abnormal* [53,57].

Example 2. *Let H be a totally ordered set, dense and symmetric around a center denoted by $0 \in H$. With regards to this center the partition $H = H^- \cup \{0\} \cup H^+$ can be defined, according to which, for every $x \in H^-$ and $y \in H^+$ it is $x < 0 < y$ and $x \le y \Rightarrow -y \le -x$ for every x, $y \in H$, where $-x$ is the symmetric of x with regards to 0. Then H, equipped with the hypercomposition:*

$$x + y = \{x, \ y\}, \qquad if \ \ y \neq -x$$

and

$$x + (-x) = [\,0, \ |x|\,] \ \cup \{\,-|x|\,\}$$

becomes a FJH in which $x - x \neq -(x - x)$, for every $x \neq 0$.

Proposition 15. *The canonical elements of a FTH are normal.*

Proof. Let x be a canonical element. Because of Theorem 1, $A \subseteq xx^{-1}$ while, according to Theorem 2, $A^{-1} = A$. Thus $A^{-1} \subseteq xx^{-1}$ and therefore $A \subseteq \left(xx^{-1}\right)^{-1}$. Suppose that z is a canonical element in xx^{-1}. Per Proposition 13, $x \in zx$. So $xx^{-1} \subseteq z\left(xx^{-1}\right)$. Hence, we have the sequence of implications:

$$e \in z\left(xx^{-1}\right); \ z^{-1} \in xx^{-1}; \ z \in \left(xx^{-1}\right)^{-1}$$

So, $xx^{-1} \subseteq \left(xx^{-1}\right)^{-1}$. Furthermore $xx^{-1} \subseteq \left(xx^{-1}\right)^{-1}$ implies that $\left(xx^{-1}\right)^{-1} \subseteq \left[\left(xx^{-1}\right)^{-1}\right]^{-1} = xx^{-1}$ and therefore the Proposition holds. □

An important Theorem that is valid for TFH [53] is the following structure Theorem:

Theorem 3. *A transposition hypergroup H containing a strong identity e is isomorphic to the expansion of the quasicanonical hypergroup $C \cup \{e\}$ by the transposition hypergroup A of all attractive elements through the identity e.*

The special properties of the FTH give different types of subhypergroups. There exist subhypergroups of a FTH that do not contain the symmetric of each one of their elements, while there exist others that do. This leads to the definition of the symmetric subhypergroups. A subhypergroup K of a FTH is *symmetric*, if $x \in K$ implies $x^{-1} \in K$. It is known that the intersection of two subhypergroups is not always a subhypergroup. In the case of the symmetric subhypergroups though, the intersection of two such subhypergroups is always a symmetric subhypergoup [57,62]. Therefore, the set of the symmetric subhypergroups of a FTH consist a complete lattice. It is proved that the lattice of the closed subhypergroups of a FTH is a sublattice of the lattice of the symmetric subhypergroups of the FTH [57,62]. An analytic and detailed study of these subhypergroups is provided in the papers [1,60,62]. Here, we will present the study of the *monogene symmetric subhypergroups*, i.e., symmetric subhypergroups generated by a single element. So, let H be a FJH, let x be an arbitrary element of H and let $M(x)$ be the monogene symmetric subhypergroup which is generated by this element. Then:

$$x^n = \begin{cases} x \cdot x \ldots \cdot x & (n \ times) & if \ n > 0 \\ e & & if \ n = 0 \\ x^{-1} \cdot x^{-1} \ldots \cdot x^{-1} & (-n \ times) & if \ n < 0 \end{cases} \tag{1}$$

and:

$$x^m \cdot x^n = \begin{cases} x^{m+n} & if \ mn > 0 \\ x^{m+n} \cdot \left(x \cdot x^{-1}\right)^{\min\{|m|,|n|\}} & if \ mn < 0 \end{cases} \tag{2}$$

From the above, it derives that:

$$x^{m+n} \subseteq x^m \cdot x^n$$

Theorem 4. *If x is an arbitrary element of a FJH, then the monogene symmetric subhypergroup which is generated by this element is:*

$$M(x) = \bigcup_{(m,n) \in \mathbb{Z} \times \mathbb{N}_0} x^m \cdot \left(x \cdot x^{-1}\right)^n$$

Proof. The symmetric subhypergroup of a normal FTH which is generated from a non-empty set X consists of the unions of all the finite products of the elements that are contained in the union $X^{-1} \cup X$ [62]; thus, from (1) we have:

$$M(x) = \bigcup_{(k,l) \in \mathbb{N}^2} x^k \cdot \left(x^{-1}\right)^l = \bigcup_{(k,l) \in \mathbb{N}^2} x^k \cdot x^{-l}$$

According to (2), it is $x^k \cdot x^{-l} = x^{k-l} \cdot \left(x \cdot x^{-1}\right)^{\min\{k,l\}}$. But $k - l \in \mathbb{Z}$; therefore, the Theorem is established. □

From the above Theorem, Proposition 13 and Theorem 1, we have the following Corollary:

Corollary 1. *Every monogene symmetric subhypergroup M(x) with generator a canonical element x is closed, it contains all the attractive elements and also*

$$M(x) = \bigcup_{(m,n)\in\mathbb{Z}^2} x^m \cdot \left(x \cdot x^{-1}\right)^n$$

Remark 1.

(i) *Since* $e \in x \cdot x^{-1}$ *the inclusion* $x^m \cdot \left(x \cdot x^{-1}\right)^n \subseteq x^m \cdot \left(x \cdot x^{-1}\right)^q$ *is valid for* $n < q$.
(ii) *for* $x = e$, *it is* $M(e) = \{e\}$.

Let us define now a symbol $\omega(x)$ (which can even be the $+\infty$), and name it *order of* x and simultaneously *order of the monogene subhypergroup* $M(x)$. Two cases can appear such that one revokes the other:

I. For any $(m, n) \in \mathbb{Z} \times \mathbb{N}_0$, with $m \neq 0$, we have:

$$e \notin x^m \cdot \left(x \cdot x^{-1}\right)^n$$

Then we define the order of x and of $M(x)$ to be the infinity and we write $\omega(x) = +\infty$.

Proposition 16. *If* $\omega(x) = +\infty$, *then* x *is a canonical element.*

Proof. Suppose that x belongs to the set A of the attractive elements. Then, per Theorem 1.(i), $x^m \subseteq A$ and $x \in x^m$. Consequently:

$$e \in x \cdot e \subseteq x^m \cdot e \subseteq x^m \cdot \left(x \cdot x^{-1}\right)^n$$

This contradicts our assumption and therefore x is a canonical element. □

The previous Proposition and Theorem 1 result to the following Corollary:

Corollary 2. *If* $\omega(x) = +\infty$, *then* x^m *does not contain attractive elements for every* $m \in \mathbb{Z}^*$.

Proposition 17. *If* $\omega(x) = +\infty$, *then*

$$x^{m+n} \cap x^n = \varnothing,\ if\ m > 0\ \ and\ \ x^{n-m} \cap x^n = \varnothing,\ if\ m < 0$$

for any $(m, n) \in \mathbb{Z} \times \mathbb{N}_0$, *with* $m \neq 0$.

Proof. From $e \notin x^m \cdot \left(x \cdot x^{-1}\right)^n$ it derives that $x^m \cap \left(x \cdot x^{-1}\right)^{-n} = \varnothing$. According to Proposition 16, x is a canonical element and therefore, because of Proposition 15, x is normal; thus, $\left(x \cdot x^{-1}\right)^{-n} = \left(x \cdot x^{-1}\right)^n$. Therefore $x^m \cap \left(x \cdot x^{-1}\right)^n = \varnothing$ or $x^m \cap \left(x^n \cdot x^{-n}\right) = \varnothing$. So, the Proposition follows from the reversibility. □

Proposition 18. $\omega(x) = +\infty$, *if and only if*

(i) $x^m \subseteq C$, *for every* $m \in \mathbb{Z}^*$
(ii) $x^{m_1} \cap x^{m_2} = \varnothing$, *for every* $m_1, m_2 \in \mathbb{Z}$ *with* $m_1 \neq m_2$.

Proof. If $\omega(x) = +\infty$, per Corollary 2, x^m does not contain attractive elements for every $m \in \mathbb{Z}^*$. Moreover, if $n \neq 0$, then (ii) derives from Proposition 17. If $n = 0$, then $e \notin x^m$ and assuming that $m = m_1 + m_2$ with $m_1 m_2 > 0$, we successively have:

$$e \notin x^m;\ e \notin x^{m_1+m_2};\ e \notin x^{m_1}x^{m_2};\ x^{-m_1} \cap x^{m_2} = \varnothing$$

Conversely now. If for every $m \in \mathbb{Z}^*$, the intersection $x^m \cap A$ is void and if for every $m_1, m_2 \in \mathbb{Z}$ with $m_1 \neq m_2$, the intersection $x^{m_1} \cap x^{m_2}$ is also void, then $e \notin x^{m_1}x^{-m_2}$ and therefore:

$$e \notin x^{m_1-m_2} \ \ if \ \ m_1 m_2 < 0$$

and

$$e \notin x^{m_1-m_2}\left(xx^{-1}\right)^{\min\{|m_1|, |m_2|\}} \ \ if \ \ m_1 m_2 > 0$$

Thus,

$$e \notin x^m \cdot \left(x \cdot x^{-1}\right)^n \text{ for every } (m,n) \in \mathbb{Z} \times \mathbb{N}_0$$

So, the Proposition holds. □

II. There exist $(m,n) \in \mathbb{Z} \times \mathbb{N}_0$ with $m \neq 0$ such that:

$$e \in x^m \cdot \left(x \cdot x^{-1}\right)^n$$

Proposition 19. *Let p be the minimum positive integer for which there exists $s \in \mathbb{N}_0$ such that $e \in x^p \cdot \left(x \cdot x^{-1}\right)^s$. Then for a given $m \in \mathbb{Z}^*$ there exist $n \in \mathbb{N}$ such that $e \in x^m \cdot \left(x \cdot x^{-1}\right)^n$ if and only if m is divided by p.*

Proof. Let $m = kp$, $k \in \mathbb{Z}$. From $e \in x^p \cdot \left(x \cdot x^{-1}\right)^r$ it derives that

$$e \in x^{kp} \cdot \left(x \cdot x^{-1}\right)^{kr} = x^m \cdot \left(x \cdot x^{-1}\right)^n$$

Therefore, the Proposition.

Conversely now. If x is an a-element, then $e \in x \cdot \left(x \cdot x^{-1}\right)^n$ for every $n \in \mathbb{N}$, so $p = 1$, and thus the Proposition. Next, if x is a c-element, and $e \in x^m \cdot \left(x \cdot x^{-1}\right)^n$ with $m = kp + r$, $k \in \mathbb{Z}$, $0 < r < p$. Then:

$$e \in x^m \cdot \left(x \cdot x^{-1}\right)^n = x^{kp+r} \cdot \left(x \cdot x^{-1}\right)^n \subseteq x^{kp} \cdot x^r \cdot \left(x \cdot x^{-1}\right)^n$$

According to our hypothesis $e \in x^p \cdot \left(x \cdot x^{-1}\right)^s$. Moreover, per Theorem 1, the sum of two non-opposite c-elements does not contain any a-elements. Consequently, there do not exist a-elements in x^p, and so

$$x^{-p} \subseteq x^{-p} \cdot x^p \cdot \left(x \cdot x^{-1}\right)^n = \left(x^{-1} \cdot x\right)^p \cdot \left(x \cdot x^{-1}\right)^n = \left(x \cdot x^{-1}\right)^{p+n}$$

Thus

$$x^{kp} \subseteq \left(x \cdot x^{-1}\right)^{-k(p+n)} = \left(x \cdot x^{-1}\right)^{|k|(p+n)}$$

and therefore

$$e \in \left(x \cdot x^{-1}\right)^{|k|(p+n)} \cdot x^r \cdot \left(x \cdot x^{-1}\right)^n = x^r \cdot \left(x \cdot x^{-1}\right)^{|k|(p+n)+n}$$

This contradicts the supposition, according to which p is the minimum element with the property $e \in x^p \cdot \left(x \cdot x^{-1}\right)^s$. Thus $r = 0$, and so $m = kp$. □

For $m = kp$, $k \in \mathbb{Z}$, let q_k be the minimum non-negative integer for which $e \in x^{kp} \cdot \left(x \cdot x^{-1}\right)^{q_k}$. Thus a function $q : \mathbb{Z} \to \mathbb{N}_0$ is defined which corresponds each k in $\mathbb{Z}$ to the non-negative integer q_k.

Definition 5. *The pair $\omega(x) = (p, q)$ is called order of x and of $M(x)$. The number p is called principal order of x and of $M(x)$, while the function q is called associated order of x and of $M(x)$.*

Consequently, according to the above definition, if x is an attractive element, then $e \in x \cdot \left(x \cdot x^{-1}\right)$ and therefore $\omega(x) = (1, q)$ with $q(k) = 1$ for every $k \in \mathbb{Z}^*$. Moreover, if x is a self-inverse canonical element, then $e \in x^2 \cdot \left(x \cdot x^{-1}\right)^0$, if $x \notin x \cdot x^{-1}$ and $e \in x \cdot \left(x \cdot x^{-1}\right)$, if $x \in x \cdot x^{-1}$ and thus $\omega(x) = (2, q)$ with $q(k) = 0$ in the first case and $\omega(x) = (1, q)$ with $q(k) = 1$ in the second case (for every $k \in \mathbb{Z}$).

Moreover, we remark that the order of e is $\omega(e) = (1, q)$, with $q(k) = 0$, for every $k \in \mathbb{Z}$, and e is the only element which has this property. Yet, it is possible that there exist non-identity elements $x \in H$ with prime order 1, and this happens if and only if there exists an integer n such that $x^{-1} \in \left(x \cdot x^{-1}\right)^n$, as for example when x is a self-inverse canonical element.

2.2. The Hyperringoid

Let Σ^* be the set of strings over an alphabet Σ. Then:

Proposition 20. *String concatenation is distributive over the B-hypecomposition.*

Proof. Let $a, b, c \in \Sigma^*$. Then, $a(b + c) = a\{b, c\} = \{ab, ac\} = ab + ac$. □

Via the thorough verification of the distributive axiom in all the different cases and taking into consideration that 0 is a bilaterally absorbing element with respect to the string concatenation on the set $\overline{\Sigma^*}$, it can also be proved that:

Proposition 21. *String concatenation is distributive over the dilated B-hypecomposition.*

Consequently, Σ^* and $\overline{\Sigma^*}$ are algebraic structures equipped with a composition and a hypercomposition which are related with the distributive law.

Definition 6. *A hyperringoid is a non-empty set Y equipped with an operation "·" and a hyperoperation "+" such that:*

i. *(Y,+) is a hypergroup*
ii. *(Y, ·) is a semigroup*
iii. *the operation "·" distributes on both sides over the hyperoperation "+ ".*

If the hypergroup $(Y, +)$ has extra properties, which make it a special hypergroup, it gives birth to corresponding special hyperringoids. So, if $(Y, +)$ is a join hypergroup, then the hyperringoid is called *join*. A distinct join hyperringoid is the *B-hyperringoid*, in which the hypergroup is a B-hypergroup. A *fortified join hyperringoid* or *join hyperring* is a hyperringoid whose additive part is a fortified join hypergroup and whose zero element is bilaterally absorbing with respect to the multiplication. A special join hyperring is the *join B-hyperring*, in which the hypergroup is a dilated B-hypergroup. If the additive part of a fortified join hyperringoid becomes a canonical hypergroup, then it is called *hyperring*. The hyperrigoid was introduced in 1990 [67] as the trigger for the study of languages and automata with the use of tools from hypercompositional algebra. An extensive study of the fundamental properties of hyperringoids can be found in [61,113–116].

Example 3. *Let $(R, +, \cdot)$ be a ring. If in R* we define the hypercomposition:

$$a \oplus b = \{a, b, a + b\}, \quad for\ all\ \ a, b \in R$$

then $(R, \oplus, \cdot)$ *is a join hyperring.*

Example 4. *Let* $\leq$ *be a linear order (also called a total order or chain) on* Y, *i.e., a binary reflexive and transitive relation such that for all* $y, y' \in Y$, $y \neq y'$ *either* $y \leq y'$ *or* $y' \leq y$ *is valid. For* $y, y' \in Y$, $y < y'$, *the set* $\{z \in Y \mid y \leq z \leq y'\}$ *is denoted by* $[y, y']$ *and the set* $\{z \in Y \mid y < z < y'\}$ *is denoted by* $]y, y'[$. *The order is dense if no* $]y, y'[$ *is void. Suppose that* $(Y, \cdot, \leq)$ *is a totally ordered group, i.e.,* $(Y, \cdot)$ *is a group such that for all* $y \leq y'$ *and* $x \in Y$, *it holds that* $x \cdot y \leq x \cdot y'$ *and* $y \cdot x \leq y' \cdot x$. *If the order is dense, then the set* Y *can be equipped with the hypercomposition:*

$$x + y = \begin{cases} x & if\ x = y \\ ]min\{x, y\}, max\{x, y\}[& if\ x \neq y \end{cases}$$

and the triplet $(Y, +, \cdot)$ *becomes a join hyperringoid. Indeed, since the equalities*

$$x + y =]\, min\{x, y\},\ max\{x, y\}\, [= y + x$$

and

$$(x + y) + z =]\, min\{x, y, z\},\ max\{x, y, z\}\, [= \ x + (y + z)$$

are valid for every $x, y, z \in Y$, *the hypercomposition is commutative and associative. Moreover,*

$$x \ / \ y = y \setminus x = \begin{cases} x & if\ x = y \\ \{t \in Y \ : \ x < t\} & if\ y < x \\ \{t \in Y \ : \ t < x\} & if\ x < y \end{cases}$$

Thus, when the intersection $(x \ / \ y) \cap (z \ / \ w)$ *is non-void, the intersection* $(x + w) \cap (z + y)$ *is also non-void. So, the transposition axion is valid. Therefore* $(Y, +)$ *is a join hypergroup. Moreover,*

$$x \cdot (y + z) = \begin{cases} x \cdot y = x \cdot y + x \cdot z & if\ y = z \\ x \cdot]y, z[= x \cdot \bigcup\limits_{y<t<z} \{t\} = \bigcup\limits_{y<t<z} \{x \cdot t\} = x \cdot y + x \cdot z, & if\ y \neq z \end{cases}$$

It is worth mentioning that the hypercomposition:

$$x + y = [min\{x, y\},\ max\{x, y\}], \ \ for\ all\ \ x, y \in Y$$

endows $(Y, \cdot)$ *with the join hyperringoid structure as well.*

As per Proposition 20, the set of the words Σ^* over an alphabet Σ can be equipped with the structure of the B-hyperringoid. This hyperringoid has the property that each one of its elements, which are the words of the language, has a unique factorization into irreducible elements, which are the letters of the alphabet. So, this hyperringoid has a finite prime subset, that is a finite set of initial and irreducible elements, such that each one of its elements has a unique factorization with factors from this set. In this sense, this hyperringoid has a property similar to the one of the Gauss' rings. Moreover, because of Proposition 21, $\overline{\Sigma^*}$ can be equipped with the structure of the join B-hyperring which has the same property.

Definition 7. *A linguistic hyperringoid (resp. linguistic join hyperring) is a unitary B-hyperringoid (resp. join B-hyperring) which has a finite prime subset* P *and which is non-commutative for* $|P| > 1$.

It is obvious that every B-hyperringoid or join B-hyperring is not a linguistic one.

Proposition 22. *From every non-commutative free monoid with finite base, there derives a linguistic hyperringoid.*

Example 5. *Let* $\{0,1\}^{2\times 2}$ *express the set of* 2×2 *matrices, which consist of the elements 0,1, that is the following 16 matrices:*

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, A_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, A_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$
$$B_1 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, B_2 = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, B_3 = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, B_4 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, B_5 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, B_6 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$
$$C_1 = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, C_2 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, C_3 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, C_4 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$
$$D_1 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$
$$E_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Consider the set T of all 2×2 *matrices deriving from products of the above matrices, except the zero matrix. T becomes a B-hyperringoid under B-hypercomposition and matrix multiplication. Observe that none of the matrices* A_1, B_6, C_4 *can be written as the product of any two matrices from the set T while all the matrices in T result from products of these three matrices. Therefore, T is a linguistic hyperringoid, whose prime subset is* $\{A_1, B_6, C_4\}$. *Furthermore, if T is enriched with the zero matrix, then it becomes a linguistic join hyperring.*

M. Krasner was the first one who introduced and studied hypercompositional structures with an operation and a hyperoperation. The first structure of this kind was the hyperfield, an additive-multiplicative hypercompositional structure whose additive part is a canonical hypergroup and the multiplicative part a commutative group. The hyperfield was introduced by M. Krasner in [79] as the proper algebraic tool in order to define a certain approximation of complete valued fields by sequences of such fields. Later on, Krasner introduced the hyperring which is related to the hyperfield in the same way as the ring is related to the field [117]. Afterwards, J. Mittas introduced the *superring* and the *superfield*, in which both the addition and the multiplication are hypercompositions and more precisely, the additive part is a canonical hypergroup and the multiplicative part is a semi-hypergroup [118–120]. In the recent bibliography, a structure whose additive part is a hypergroup and the multiplicative part is a semi-group is also referred to with the term additive hyperring and similarly, the term multiplicative hyperring is used when the multiplicative part is a hypergroup.

Rings and Krasner's hyperrings have many common elementary algebraic properties, e.g., in both structures the following are true:

(i) $x(-y) = (-x)y = -xy$
(ii) $(-x)(-y) = xy$
(iii) $w(x-y) = wx - wy,\ (x-y)w = xw - yw$

In the hyperringoids though, these properties are not generally valid, as it can be seen in the following example:

Example 6. *Let S be a multiplicative semigroup having a bilaterally absorbing element 0. Consider the set:*

$$P = (\{0\}\times S)\ \cup\ (S\times\{0\})$$

With the use of the hypercomposition "+":

$$\begin{aligned}
&(x,0)+(y,0) = \{\,(x,0),\ (y,0)\,\}\\
&(0,x)+(0,y) = \{\,(0,x),\ (0,y)\,\}\\
&(x,0)+(0,y) = (0,y)+(x,0) = \{\,(x,0),\ (0,y)\,\}\ \ for\ \ x\neq y\\
&(x,0)+(0,x) = (0,x)+(x,0) = \{\,(x,0),\ (0,x),\ (0,0)\,\}
\end{aligned}$$

P becomes a fortified join hypergroup with neutral element $(0, 0)$. *If* $(0, x)$ *is denoted by* $\overline{x}$ *and* $(0, 0)$ *by* $\overline{0}$, *then the opposite of* $\overline{x}$ *is* $-\overline{x} = (x, 0)$. *Obviously this hypergroup has not c-elements. Now let us introduce in P a multiplication defined as follows:*

$$(x_1, y_1) \cdot (x_2, y_2) = (x_1 x_2, y_1 y_2)$$

This multiplication makes $(P, +, \cdot)$ *a join hyperring, in which*

$$-(\overline{x}\,\overline{y}) = -[(0, x)(0, y)] = -(0, xy) = (xy, 0) \neq \overline{0} \text{ while } \overline{x}(-\overline{y}) = (0, x)(y, 0) = (0, 0) = \overline{0}$$

Similarly, $(-\overline{x})\overline{y} = \overline{0} \neq -(\overline{x}\,\overline{y})$. *Furthermore*

$$(-\overline{x})(-\overline{y}) = (x, 0)(y, 0) = (xy, 0) = -\overline{xy}$$

More examples of hyperringoids can be found in [61,113–116].

3. Hypercompositional Algebra and Geometry

It is very well known that there exists a close relation between Algebra and Geometry. So, as it should be expected, this relation also appears between Hypercompositional Algebra and Geometry. It is really of exceptional interest that the axioms of the hypergroup are directly related to certain Euclid's postulates [121]. Indeed, according to the first postulate of Euclid:

> "Ηιτήσθω ἀπό παντός σημείου ἐπί πᾶν σημεῖον εὐθεῖαν γραμμήν ἀγαγεῖν" [121]
>
> (Let the following be postulated: to draw a straight line from any point to any point [122])

So, to any pair of points (a, b), the segment of the straight line *ab* can be mapped. This segment always exists and it is a nonempty set of points. In fact, it is a multivalued result of the composition of two elements. Thus, a hypercomposition has been defined in the set of the points. Next, according to the second postulate:

> "Καὶ πεπερασμένην εὐθεῖαν κατά τό συνεχές ἐπ' εὐθείας ἐκβαλεῖν" [121]
>
> (To produce a finite straight line continuously in a straight line [122])

The sets $a\ /\ b$ and $b\ /\ a$ are nonempty. Therefore, as per Proposition 2, the reproductive axiom is valid. Besides, it is easy to prove that the associativity holds in the set of the points. It is only necessary to keep in mind the definition of the equal figures given by Euclid in the "Common Notions":

> "Τὰ τῷ αὐτῷ ἲσα καὶ ἀλλήλοις ἐστίν ἲσα" [121]
>
> (Things which are equal to the same thing are also equal to one another [122])

So, the set of the points is a hypergroup. Moreover, through similar reasoning, it can be proved that any Euclidean space of dimension *n* can become a hypergroup. Indeed:

Proposition 23. *Let* $(V, +)$ *be α linear space over an ordered field* $(F, +, \cdot)$. *Then V, with the hypercomposition:*

$$xy = \left\{ \kappa x + \lambda y \,\middle|\, \kappa, \lambda \in F_{+}^{*},\ \kappa + \lambda = 1 \right\}$$

becomes α join hypergroup.

This hypergroup is called attached hypergroup. Properties of vector spaces can be found via the attached hypergroup. Thus, for example:

Proposition 24. *In a vector space V over an ordered field F, the elements a_i, $i = 1, \dots, k$ are affinely dependent if and only if there exist distinct integers $s_1, \dots, s_n, t_1, \dots, t_m$ that belong to $\{1, \dots, k\}$ such that:*

$$[a_{s_1}, \dots, a_{s_n}] \cap [a_{t_1}, \dots, a_{t_m}] \neq \varnothing$$

In fact, several hypergroups can be attached to a vector space [28]. The connection of hypercompositional structures with Geometry was initiated by W. Prenowitz [16–19]. The classical geometries, descriptive geometries, spherical geometries and projective geometries can be treated as certain kinds of hypergoups, all satisfying the transposition axiom. The hypercomposition plays the central role in this approach. It assigns the appropriate connection between any two distinct points. Thus, in Euclidian geometry, it gives the points of the segment; in spherical geometry, it gives the points of the minor arc of the great circle; in projective geometry it gives the point of the line. This development is dimension free and it is applicable to spaces of arbitrary dimension, finite or infinite.

3.1. Hypergroups and Convexity

Several geometric notions can be described with the use of the hypercomposition. One such notion is the convexity. It is known that a figure is called *convex*, if the segment joining any pair of its points lies entirely in it. As mentioned above, the set of the points of the plane, as well as the set of the points of any vector space V over an ordered field, becomes a hypergroup under the hypercomposition defined in Proposition 23. From this point of view, that is, with the use of the hypercomposition, a subset E of V is convex if $ab \subseteq E$, for all $a, b \in E$. However, a subset E of a hypergroup which has this property is a semi-subhypergroup [14]. Thus:

Proposition 25. *The convex subsets of a vector space V are the semi-subhypergroups of its attached hypergroup.*

Consequently, the properties of the convex sets of a vector space are simple applications of the properties of the semi-subhypergroups, or the subhypergroups of a hypergroup, and more precisely, the attached hypergroup. So, this approach, except from the fact that it leads to remarkable results, it also gives the opportunity to generalize the already known Theorems of the vector spaces in sets with fewer axioms than the ones of the vector spaces. Next, we will present some well-known named Theorems that arise as corollaries of more general Theorems which are valid in hypercompositional algebra.

In hypergroups the following Theorem holds [2,14]:

Theorem 5. *Let H be a hypergroup in which every set with cardinality greater than n has two disjoint subsets A, B such that $[A] \cap [B] \neq \varnothing$. If $(Y_i)_{i \in I}$ with card $I \geq n$ is a finite family of semi-subhypergroups of H, in which the intersection of every n elements is non-void, then all the sets Y_i have a non-void intersection.*

The combination of Propositions 24, 25 and Theorem 5 gives the corollary:

Corollary 3. *(Helly's Theorem). Let $(C_i)_{i \in I}$ be a finite family of convex sets in $\mathbb{R}^d$, with $d + 1 <$ card I. Then, if any $d + 1$ of the sets C_i have a non-empty intersection, all the sets C_i have a non-empty intersection.*

Next, the following Theorem stands for a join hypergroup:

Theorem 6. *Let A, B be two disjoint semi-subhypergroups in a join hypergroup and let x be an idempotent element not in the union $A \cup B$. Then $[A \cup \{x\}] \cap B = \varnothing$ or $[B \cup \{x\}] \cap A = \varnothing$*

The proof of Theorem 6 is found in [2,14] and it is repeated here for the purpose of demonstrating the techniques which are used for it.

Proof. Suppose that $[A \cup \{x\}] \cap B \neq \varnothing$ and $[B \cup \{x\}] \cap A \neq \varnothing$. Since x is idempotent the equalities $[A \cup \{x\}] = Ax$ and $[B \cup \{x\}] = Bx$ are valid. Thus, there exist $a \in A$ and $b \in B$, such that $ax \cap B \neq \varnothing$ and $bx \cap A \neq \varnothing$. Hence, $x \in B / a$ and $x \in A / b$. Thus, $B / a \cap A / b \neq \varnothing$. Next, the application of the transposition axiom, gives $Bb \cap Aa \neq \varnothing$. However, $Bb \subseteq B$ and $Aa \subseteq A$, since A, B are semi-subhypergroups. Therefore, $A \cap B \neq \varnothing$, which contradicts the Theorem's assumption. □

Corollary 4. *Let H be a join hypergroup endowed with an open hypercomposition. If A, B are two disjoint semi-subhypergroups of H and x is an element not in the union* $A \cup B$, then:

$$[A \cup \{x\}] \cap B = \varnothing \ \text{ or } \ [B \cup \{x\}] \cap A = \varnothing.$$

The attached hypergroup of a vector space, which is defined in Proposition 23, is a join hypergroup whose hypercomposition is open, so Corollary 4 applies to it and we get the Kakutani's Lemma:

Corollary 5. *(Kakutani's Lemma). If A, B are disjoint convex sets in a vector space and x is a point not in their union, then either the convex envelope of* $A \cup \{x\}$ *and B or the convex envelope of* $B \cup \{x\}$ *and A are disjoint.*

Next in [2] it is proved that the following Theorem is valid:

Theorem 7. *Let H be a join hypergroup consisting of idempotent elements and suppose that A, B are two disjoint semi-subhypergroups in H. Then, there exist disjoint semi-subhypergroups M, N such that* $A \subseteq M$, $B \subseteq N$ *and* $H = M \cup N$.

A direct consequence of Theorem 7 is Stone's Theorem:

Corollary 6. *(Stone's Theorem). If A, B are disjoint convex sets in a vector space V, there exist disjoint convex sets M and N, such that* $A \subseteq M$, $B \subseteq N$ *and* $V = M \cup N$.

During his study of Geometry with hypercompositional structures, W. Prenowitz introduced the *exchange spaces* which are join spaces satisfying the axiom:

$$\text{if } \ c \in \langle a, b\rangle \ \text{ and } \ c \neq a, \ \text{ then } \ \langle a, b\rangle = \langle a, c\rangle$$

The above axiom enabled Prenowitz to develop a theory of linear independence and dimension of a type familiar to classical geometry. Moreover, a generalization of this theory has been achieved by Freni, who developed the notions of independence, dimension, etc., in a hypergroup H that satisfies only the axiom:

$$x \in \langle A \cup \{y\}\rangle,\ x \notin \langle A\rangle \ \Rightarrow \ y \in \langle A \cup \{x\}\rangle,\ \text{for every } \ x, y \in H \ \text{ and } \ A \subseteq H.$$

Freni called these hypergroups *cambiste* [25,26]. A subset B of a hypergroup H is called *free* or *independent* if either $B = \varnothing$, or $x \notin \langle B - \{x\}\rangle$ for all $x \in B$, otherwise it is called *non-free* or *dependent*. B generates H, if $\langle B\rangle = H$, in which case B is a set of generators of H. A free set of generators is a *basis* of H. Freni proved that all the bases of a cambiste hypergroup have the same cardinality. The *dimension* of a cambiste hypergroup H is the cardinality of any basis of H. The dimension theory gives very interesting results in convexity hypergroups. A *convexity hypergroup* is a join hypergroup which satisfies the axioms:

i. the hypercomposition is open,
ii. $ab \cap ac \neq \varnothing$ implies $b = c$ or $b \in ac$ or $c \in ab$.

Prenowitz, defined this hypercompositional structure with equivalent axioms to the above, named it *convexity space* and used it, as did Bryant and Webster [24], for generalizing some of the theory of linear spaces. In [2] it is proved that every convexity hypergroup is a cambiste hypergroup. Moreover in [2] it is proved that the following Theorem stands for convexity hypergroups:

Theorem 8. *Every n+1 elements of a n-dimensional convexity hypergroup H are correlated.*

One can easily see that the attached hypergroup of a vector space is a convexity hypergroup and, moreover, if the dimension of the attached hypergroup H_V of a vector space V is n, then the dimension of V is $n-1$. Thus, we have the following corollary of Theorem 8:

Corollary 7. *(Radon's Theorem). Any set of d+2 points in $\mathbb{R}^d$ can be partitioned into two disjoint subsets, whose convex hulls intersect.*

Furthermore, the following Theorem is proved in [2]:

Theorem 9. *If x is an element of a n-dimensional convexity hypergroup H and $a_1, \ldots, a_n, a_{n+1}$ are n+1 elements of H such that $x \in a_1 \cdots a_n a_{n+1}$, then there exists a proper subset of these elements which contains x in their hyperproduct.*

A direct consequence of this Theorem is Caratheodory's Theorem:

Corollary 8. *(Caratheodory's Theorem). Any convex combination of points in $\mathbb{R}^d$ is a convex combination of at most d+1 of them.*

In addition, Theorems of the hypercompositional algebra are proved in [2], which give as corollaries generalizations and extensions of Caratheodory's Theorem.

An element a of a semi-subhypergroup S is called *interior* element of S if for each $x \in S$, $x \neq a$, there exists $y \in S$, $y \neq a$, such that $a \in xy$. In [2] it is proved that any interior element of a semi-subhypergroup S of a n-dimensional convexity hypergroup, is interior to a finitely generated semi-subhypergroup of S. More precisely, the following Theorem is valid [2]:

Theorem 10. *Let a be an interior element of a semi-subhypergroup S of a n-dimensional convexity hypergroup H. Then a is interior element of a semi-subhypergroup of S, which is generated by at most* $2n$ *elements.*

A corollary of this Theorem, when H is $\mathbb{R}^d$, is Steinitz's Theorem:

Corollary 9. *(Steinitz's Theorem). Any point interior to the convex hull of a set E in $\mathbb{R}^d$ is interior to the convex hull of a subset of E, containing* $2d$ *points at the most.*

D. Freni in [123] extended the use of the hypergroup in more general geometric structures, called geometric spaces. A *geometric space* is a pair (S, B) such that S is a non-empty set, whose elements are called points, and B is a non-empty family of subsets of S, whose elements are called blocks. Freni was followed by S. Mirvakili, S.M. Anvariyeh and B. Davvaz [124,125].

3.2. Hyperfields and Geometry

As it is mentioned in the previous Section 2.2, the hyperfield was introduced by M. Krasner in order to define a certain approximation of a complete valued field by a sequence of such fields [79]. The construction of this hyperfield, which was named by Krasner himself *residual hyperfield*, is also described in his paper [117].

Definition 8. *A hyperring is a hypercompositional structure $(H, +, \cdot)$, where H is a non-empty set, "$\cdot$" is an internal composition on H, and "$+$" is a hypercomposition on H. This structure satisfies the axioms:*

i. $(H, +)$ *is a canonical hypergroup,*

ii. $(H, \cdot)$ *is a multiplicative semigroup in which the zero element 0 of the canonical hypergroup is a bilaterally absorbing element,*

iii. *the multiplication is distributive over the hypercomposition (hyperaddition), i.e.,*

$$z(x+y) = zx + zy \text{ and } (x+y)z = xz + yz$$

for all $x, y, z \in H$.

If $H..\{0\}$ *is a multiplicative group then* $(H, +, \cdot)$ *is called hyperfield.*

J. Mittas studied these hypercompositional structures in a series of papers [126–133]. Among the plenitude of examples which are found in these papers, we will mention the one which is presented in the first paragraph of [130].

Example 7. *Let (E,·) be a totally ordered semigroup, having a minimum element 0, which is bilaterally absorbing with regards to the multiplication. The following hypercomposition is defined on E:*

$$x + y = \begin{cases} \max\{x, y\} & if\ \ x \neq y \\ \{z \in E \mid z \leq x\} & if\ \ x = y \end{cases}$$

Then (E,+,·) is a hyperring. If E..{0} is a multiplicative group, then (E,+,·) is a hyperfield.

We referred to Mittas' example, because nowadays, this particular hyperfield is called *tropical hyperfield* (see, e.g., [134–138]) and it is proved to be an appropriate and effective algebraic tool for the study of tropical geometry.

M. Krasner worked on the occurrence frequency of such structures as the hyperrings and hyperfields and he generalized his previous construction of the residual hyperfields. He observed that, if R is a ring and G is a normal subgroup of R's multiplicative semigroup, then the multiplicative classes $\overline{x} = xG$, $x \in R$, form a partition of R and that the product of two such classes, as subsets of R, is a class *modG* as well, while their sum is a union of such classes. Next, he proved that the set $\overline{R} = R \ / \ G$ of these classes becomes a hyperring, if the product of $\overline{R}$'s two elements is defined to be their set-wise product and their sum to be the set of the classes contained in their set-wise sum [117]:

$$\overline{x} \cdot \overline{y} = xyG$$

and

$$\overline{x} + \overline{y} = \{zG \mid z \in xG + yG\}$$

He also proved that if R is a field, then R/G is a hyperfield. Krasner named these hypercompositional structures *quotient hyperring* and *quotient hyperfield*, respectively.

In the recent bibliography, there appear hyperfields with different and not always successful names, all of which belong to the class of the quotient hyperfields. For instance:

(a) starting from the papers [139,140] by A. Connes and C. Consani, there appeared many papers (e.g., [135–138]) which gave the name «*Krasners' hyperfield*» to the hyperfield which is constructed over the set {0, 1} using the hypercomposition:

$$0 + 0 = 0,\ 0 + 1 = 1 + 0 = 1,\ 1 + 1 = \{0, 1\}$$

Oleg Viro, in his paper [135] is reasonably noticing that «*To the best of my knowledge, K did not appear in Krasner's papers*». Actually, this is a quotient hyperfield. Indeed, let F be a field and let F^* be its multiplicative subgroup. Then the quotient hyperfield $F \ / \ F^* = \{0, F^*\}$ is isomorphic to the hyperfield considered by A. Connes and C. Consani. Hence the two-element non-trivial hyperfield is isomorphic to a quotient hyperfield.

(b) Papers [139,140] by A. Connes and C. Consani, show the construction of the hyperfield, which is now called «*sign hyperfield*» in the recent bibliography, over the set {−1, 0, 1} with the following hypercomposition:

$$0+0=0,\ 0+1=1+0=1,\ 1+1=1,\ -1-1=-1,\ 1-1=-1+1=\{-1,0,1\}$$

However, this hyperfield is a quotient hyperfield as well. Indeed, let F be an ordered field and let F^+ be its positive cone. Then the quotient hyperfield $F\ /\ F^+ = \{-F^+, 0, F^+\}$ is isomorphic to the hyperfield which is called sign hyperfield.

(c) The so called «*phase hyperfield*» (see e.g., [135,136]) in the recent bibliography, is just the quotient hyperfield $\mathbb{C}\ /\ \mathbb{R}^+$, where $\mathbb{C}$ is the field of complex numbers and $\mathbb{R}^+$ the set of the positive real numbers. The elements of this hyperfield are the rays of the complex field with origin the point (0,0). The sum of two elements $z\mathbb{R}^+$, $w\mathbb{R}^+$ of $\mathbb{C}\ /\ \mathbb{R}^+$ with $z\mathbb{R}^+ \neq w\mathbb{R}^+$ is the set $\{(zp+wq)\mathbb{R}^+ \mid p,q \in \mathbb{R}^+\}$, which consists of all the interior rays $x\mathbb{R}^+$ of the convex angle which is created from these two elements, while the sum of two opposite elements gives the participating elements and the zero element. This hyperfield is presented in detail in [141].

Krasner, immediately realized that if all hyperrings could be isomorphically embedded into quotient hyperrings, then several conclusions of their theory could be deduced in a very straightforward manner, through the use of the ring theory. So, he raised the question whether all the hyperrings are isomorphic to subhyperrings of quotient hyperrings or not. He also raised a similar question regarding the hyperfields [117]. These questions were answered by C. Massouros [142–144] and then by A. Nakassis [145], via the following Theorems:

Theorem 11. [142,143] *Let $(\Theta,\cdot)$ be a multiplicative group. Let $H=\Theta\cup\{0\}$, where 0 is a multiplicatively absorbing element. If H is equipped with the hypercomposition:*

$$\begin{array}{ll} x+0=0+x=x & \text{for all } x\in H,\\ x+x=H..\{x\} & \text{for all } x\in\Theta,\\ x+y=y+x=\{x,y\} & \text{for all } x,y\in\Theta \text{ with } x\neq y, \end{array}$$

then, $(H,+,\cdot)$ is a hyperfield, which does not belong to the class of quotient hyperfields when Θ is a periodic group.

Theorem 12. [144] *Let $\overline{\Theta}=\Theta\otimes\{1,-1\}$ be the direct product of the multiplicative groups Θ and $\{-1,1\}$, where $\operatorname{card}\Theta>2$. Moreover, let $K=\overline{\Theta}\cup\{0\}$ be the union of $\overline{\Theta}$ with the multiplicatively absorbing element 0. If K is equipped with the hypercomposition:*

$$\begin{array}{ll} w+0=0+w=w & \text{for all } w\in \mathrm{K},\\ (x,i)+(x,i)=\mathrm{K}..\{(x,i),(x,-i),0\} & \text{for all } (x,i)\in\overline{\Theta},\\ (x,i)+(x,-i)=\mathrm{K}..\{(x,i),(x,-i)\} & \text{for all } (x,i)\in\overline{\Theta},\\ (x,i)+(y,j)=\{(x,i),\ (x,-i),\ (y,j),\ (y,-j)\} & \text{for all } (x,i),(y,j)\in\overline{\Theta} \text{ with } (y,j)\neq(x,i),\ (x,-i), \end{array}$$

then, $(K,+,\cdot)$ is a hyperfield which does not belong to the class of quotient hyperfields when Θ is a periodic group.

Proposition 26. [145] *Let $(T^*,\cdot)$ be a multiplicative group of m, $m>3$ elements. Let $T=T^*\cup\{0\}$, where 0 is a multiplicatively absorbing element. If T is equipped with the hypercomposition:*

$$\begin{array}{ll} a+0=0+a=a & \text{for all } a\in T,\\ a+a=\{0,a\} & \text{for all } a\in T^*,\\ a+b=b+a=T..\{0,a,b\} & \text{for all } a,b\in T^* \text{ with } a\neq b, \end{array}$$

then, $(T, +, \cdot)$ is a hyperfield.

Theorem 13. [145] *If T^* is a finite multiplicative group of m, $m > 3$ elements and if the hyperfield T is isomorphic to a quotient hyperfield F / Q, then $Q \cup \{0\}$ is a field of m-1 elements while F is a field of $(m-1)^2$ elements.*

Clearly, we can choose the cardinality of T^* in such a way that T cannot be isomorphic to a quotient hyperfield. In [144,145] one can find non-quotient hyperrings as well.

Therefore, we know 4 different classes of hyperfields, so far: the class of the quotient hyperfields and the three ones which are constructed via the Theorems 11, 12 and 13.

The open and closed hypercompositions [5] in the hyperfields are of special interest. Regarding these, we have the following:

Proposition 27. *In a hyperfield K the sum $x + y$ of any two non-opposite elements $x, y \neq 0$ does not contain the participating elements if and only if, the difference $x - x$ equals to $\{-x,\ 0,\ x\}$, for every $x \neq 0$.*

Proposition 28. *In a hyperfield K the sum $x + y$ of any two non-opposite elements $x,\ y \neq 0$ contains these two elements if and only if, the difference $x - x$ equals to H, for every $x \neq 0$.*

For the proofs of the above Propositions 27 and 28, see [141]. With regard to Proposition 28, it is worth mentioning that there exist hyperfields in which, the sum $x + y$ contains only the two addends x, y, i.e. $x + y = \{x, y\}$, when $y \neq -x$ and $x, y \neq 0$ [141].

Theorem 14. [141] *Let $(K, +, \cdot)$ be a hyperfield. Let $\dagger$ be a hypercomposition on K, defined as follows:*

$$\begin{array}{ll} x \dagger y = (x + y) \cup \{x, y\} & if\ \ y \neq -x\ \ and\ \ x, y \neq 0 \\ x \dagger (-x) = K & for\ all\ \ x \in K..\{0\} \\ x \dagger 0 = 0 \dagger x = x & for\ all\ \ x \in K \end{array}$$

Then, $(K, \dagger, \cdot)$ is a hyperfield and moreover, if $(K, +, \cdot)$ is a quotient hyperfield, then $(K, \dagger, \cdot)$ is a quotient hyperfield as well.

Corollary 10. *If $(K, +, \cdot)$ is a field, then $(K, \dagger, \cdot)$ is a quotient hyperfield.*

The following problem in field theory is raised from the study of the isomorphism of the quotient hyperfields to the hyperfields which are constructed with the process given in Theorem 14:

> *when does a subgroup G of the multiplicative group of a field F have the ability to generate F via the subtraction of G from itself?* [141,143]

A partial answer to this problem, which is available so far, regarding the finite fields is given with the following theorem:

Theorem 15. [146] *Let F be a finite field and G be a subgroup of its multiplicative group of indexn and order m. Then, G-G = F, if and only if:*

$n = 2\ \ and\ \ m > 2,$
$n = 3\ \ and\ \ m > 5,$
$n = 4, -1 \in G\ \ and\ \ m > 11,$
$n = 4, -1 \notin G\ \ and\ \ m > 3,$
$n = 5, charF = 2\ \ and\ \ m > 8,$
$n = 5, charF = 3\ \ and\ \ m > 9,$
$n = 5, charF \neq 2, 3\ \ and\ \ m > 23.$

Closely related to the hyperfield is the hypermodule and the vector space.

Definition 9. *A left hypermodule over a unitary hyperring P is a canonical hypergroup M with an external composition* $(a, m) \to am$*, from* $P \times M$ *to M satisfying the conditions:*

i. $a(m+n) = am + an,$
ii. $(a+b)m = am + bm,$
iii. $(ab)m = a(bm),$
iv. $1m = m$ *and* $0m = 0$

for all $a, b \in P$ *and all* $m, n \in M$.

The *right hypermodule* is defined in a similar way. A hypermodule over a hyperfield is called *vector hyperspace*.

Suppose V and W are hypermodules over the hyperring P. The cartesian product $V \times W$ can become a hypercompositional structure over P, when the operation and the hyperoperation, for $v,\ v_1,\ v_2 \in V,\ w,\ w_1,\ w_2 \in W$, and $\alpha \in P$, are defined componentwise, as follows:

$$(v_1,\ w_1) + (v_2,\ w_2) = \bigcup \{\, (v, w) \,\big|\, v \in v_1 + v_2,\ w \in w_1 + w_2\}$$

$$a(v, w) \ = \ (av, aw)$$

The resulting hypercompositional structure is called the *direct sum* of V and W.

Theorem 16. *The direct sum of the hypermodules is not a hypermodule.*

Proof. Let V and W be two hypermodules over a hyperring P. Then:

$$(a+b)(v,w) \ = \ \bigcup \{c(v,w) \,\big|\, c \in a+b\} \ = \ \bigcup \{(cv, cw) \,\big|\, c \in a+b\}$$

On the other hand:

$$\begin{aligned} a(v,w) + b(v,w) \ &= \ (av, aw) + (bv, bw) \ = \ \cup\{(x,y) \,\big|\, x \in av + bv,\ y \in aw + bw\} \ = \\ &= \ \cup\{(x,y) \,\big|\, x \in (a+b)v,\ y \in (a+b)w\} \ = \ \cup\{(sv, rw) \,\big|\, s, r \in a+b\} \end{aligned}$$

Therefore:

$$(a+b)(v,w) \subseteq a(v,w) + b(v,w)$$

Consequently axiom (ii) is not valid. □

Remark 2. ***Errors in Published Papers****. Unfortunately, there exist plenty of papers which incorrectly consider that the direct sum of hypermodules is a hypermodule. For instance, they mistakenly consider that if P is a hyperring or a hyperfield, then* P^n *is a hypermodule or a vector hyperspace over P respectively. Due to this error, a lot of, if not all the conclusions of certain papers are incorrect. We are not going to specifically refer to such papers, as we do not wish to add negative citations in our paper, but we refer positively to the paper by P. Ameri, M Eyvazi and S. Hoskova-Mayerova [120], where the authors have presented a counterexample which shows that the polynomials over a hyperring give a superring in the sense of Mittas [118,119] and not a hyperring, as it is mistakenly mentioned in a previously published paper which is referred there. This error can also be highlighted with the same method as the one in Theorem 16, since the polynomials over a hyperring P can be considered as the ordered sets* $(a_0, a_1, \ldots)$ *where* a_i*, i=0, 1,* $\ldots$ *are their coefficients.*

Following the above remark, we can naturally introduce the definition:

Definition 10. *A left weak hypermodule over a unitary hyperring P is a canonical hypergroup M with an external composition $(a, m) \to am$, from $P \times M$ to M satisfying the conditions (i), (iii), (iv) of the Definition 9 and, in place of (ii), the condition:*

ii′. $(a+b)m \subseteq am + bm$, for all $a, b \in P$ and all $m \in M$.

The quotient hypermodule over a quotient hyperring is constructed in [147], as follows:

Let M be a P–module, where P is a unitary ring, and let G be a subgroup of the multiplicative semigroup of P, which satisfies the condition $aG\,bG = abG$, for all $a, b \in P$. Note that this condition is equivalent to the normality of G only when $P..\{0\}$ is a group, which appears only in the case of division rings (see [144]). Next, we introduce in M the following equivalence relation:

$$x \sim y \Leftrightarrow x = ty,\ t \in G$$

After that, we equip $\overline{M}$ with the following hypercomposition, where $\overline{M}$ is the set of equivalence classes of M modulo ~:

$$\overline{x} \dotplus \overline{y} = \left\{ \overline{w} \in \overline{M} \;\middle|\; w = xp + yq,\ p, q \in G \right\}$$

i.e., $\overline{x} \dotplus \overline{y}$ consists of all the classes $\overline{w} \in \overline{M}$ which are contained in the set-wise sum of $\overline{x}, \overline{y}$. Then $\left(\overline{M}, \dotplus\right)$ becomes a canonical hypergroup. Let $\overline{P}$ be the quotient hyperring of P over G. We consider the external composition from $\overline{P} \times \overline{M}$ to $\overline{M}$ defined as follows:

$$\overline{a}\,\overline{x} = \overline{ax} \text{ for each } \overline{a} \in \overline{P},\ \overline{x} \in M.$$

This composition satisfies the axioms of the hypermodule and so $\overline{M}$ becomes a $\overline{P}$- hypermodule.

If M is a module over a division ring D, then, using the multiplicative group D^* of D we can construct the quotient hyperring $\overline{D} = D \,/\, D^* = \{0, D^*\}$ and the relevant quotient hypermodule $\overline{M}$. For any $\overline{a} \in \overline{M}$ it holds that $\overline{a} + \overline{a} = \{0, \overline{a}\}$. In [147] it is shown that this hypermodule is strongly related to the projective geometries. A. Connes and C. Consani, in [139,140] also prove that the projective geometries, in which the lines have at least four points, are exactly vector hyperspaces over the quotient hyperfield with two elements. Moreover, if V is a vector space over an ordered field F, then, using the positive cone F^+ of F we can construct the vector hyperspace $\overline{V}$ over the quotient hyperfield $\overline{F} = F \,/\, F^+ = \{F^-, 0, F^+\}$. In [147] it is shown that every Euclidean spherical geometry can be considered as a quotient vector hyperspace over the quotient hyperfield with three elements.

Modern algebraic geometry is based on abstract algebra which offers its techniques for the study of geometrical problems. In this sense, the hyperfields, were connected to the conic sections via a number of papers [148–150], where the definition of an elliptic curve over a field F was naturally extended to the definition of an elliptic hypercurve over a quotient Krasner hyperfield. The conclusions obtained in [148–150] were extended to cryptography as well.

4. Conclusions and Open Problems

In this paper we have initially presented the relationship between the groups and the hypergroups. It is interesting that the groups and the hypergroups are two algebraic structures which satisfy exactly the same axioms, i.e., the associativity and the reproductivity, but they differ in the law of synthesis. In the first ones, the law of synthesis is a composition, while in the second ones, it is a hypercomposition. This difference makes the hypergroups much more general algebraic structures than the groups, and for this reason, the hypergroups have been gradually enriched with further axioms, which are either more powerful or less powerful and they lead to a significant number of special hypergroups.

We have also presented the connection of the Hypercompositional Algebra to the Formal Languages and Automata theory as well as its close relationship to Geometry. It is very interesting that the transposition hypergroup, which appears in the Formal Languages, is the proper algebraic tool for the study of the convexity in Geometry. The study of this hypergroup has led to general Theorems which have as corollaries well known named Theorems in the vector spaces. Different types of transposition hypergroups, as for example the fortified transposition hypergroup, give birth to hypercompositional structures like the hyperringoid, the linguistic hyperringoid the join hyperring the algebraic structure of which is an area with a plentitude of hitherto open problems. Moreover, the hyperfield and the hypermodule describe fully and accurately the projective and the spherical geometries, while they are directly connected to other geometries as well. Moreover, the classification problem of hyperfields gives birth to the question:

> *when does a subgroup G of the multiplicative group of a field F have the ability to generate F via the subtraction of G from itself?*

This question is answered for certain finite fields only and still remains to be answered in its entirety.

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References

1. Massouros, C.G. Some properties of certain subhypergroups. *Ratio Math.* **2013**, *25*, 67–76.
2. Massouros, C.G. On connections between vector spaces and hypercompositional structures. *Ital. J. Pure Appl. Math.* **2015**, *34*, 133–150.
3. Massouros, C.G.; Massouros, G.G. The transposition axiom in hypercompositional structures. *Ratio Math.* **2011**, *21*, 75–90.
4. Massouros, C.G.; Massouros, G.G. On certain fundamental properties of hypergroups and fuzzy hypergroups—Mimic fuzzy hypergroups. *Internat. J. Risk Theory* **2012**, *2*, 71–82.
5. Massouros, C.G.; Massouros, G.G. On open and closed hypercompositions. *AIP Conf. Proc.* **2017**, *1978*, 340002. [CrossRef]
6. Massouros, C.G.; Dramalidis, A. Transposition Hv-groups. *Ars Comb.* **2012**, *106*, 143–160.
7. Novák, M.; Křehlík, Š. *EL*-hyperstructures revisited. *Soft Comput.* **2018**, *22*, 7269–7280. [CrossRef]
8. Marty, F. Sur une généralisation de la notion de groupe. Huitième Congrès des mathématiciens Scand. *Stockholm* **1934**, 45–49.
9. Marty, F. Rôle de la notion de hypergroupe dans l'étude de groupes non abéliens. *C.R. Acad. Sci.* **1935**, *201*, 636–638.
10. Marty, F. Sur les groupes et hypergroupes attachés à une fraction rationelle. *Ann. L'école Norm.* **1936**, *53*, 83–123. [CrossRef]
11. Massouros, C.G.; Massouros, G.G. Identities in multivalued algebraic structures. *AIP Conf. Proc.* **2010**, *1281*, 2065–2068. [CrossRef]
12. Massouros, C.G.; Massouros, G.G. On the algebraic structure of transposition hypergroups with idempotent identity. *Iran. J. Math. Sci. Inform.* **2013**, *8*, 57–74. [CrossRef]
13. Massouros, C.G.; Massouros, G.G. Transposition hypergroups with idempotent identity. *Int. J. Algebr. Hyperstruct. Appl.* **2014**, *1*, 15–27.
14. Massouros, C.G. On the semi-subhypergroups of a hypergroup. *Int. J. Math. Math. Sci.* **1985**, *8*, 725–728. [CrossRef]
15. Jantosciak, J. Transposition hypergroups, Noncommutative Join Spaces. *J. Algebra* **1997**, *187*, 97–119. [CrossRef]
16. Prenowitz, W. Projective Geometries as multigroups. *Am. J. Math.* **1943**, *65*, 235–256. [CrossRef]
17. Prenowitz, W. Descriptive Geometries as multigroups. *Trans. Am. Math. Soc.* **1946**, *59*, 333–380. [CrossRef]
18. Prenowitz, W. Spherical Geometries and mutigroups. *Can. J. Math.* **1950**, *2*, 100–119. [CrossRef]
19. Prenowitz, W. A Contemporary Approach to Classical Geometry. *Am. Math. Month* **1961**, *68*, 1–67. [CrossRef]
20. Prenowitz, W.; Jantosciak, J. Geometries and Join Spaces. *J. Reine Angew. Math.* **1972**, *257*, 100–128.
21. Prenowitz, W.; Jantosciak, J. *Join. Geometries. A Theory of Convex Sets and Linear Geometry*; Springer: Berlin/Heidelberg, Germany, 1979.

22. Jantosciak, J. Classical geometries as hypergroups. In Proceedings of the Atti Convegno Ipergruppi Altre Structure Multivoche e Loro Applicazioni, Udine, Italia, 15–18 October 1985; pp. 93–104.
23. Jantosciak, J. A brif survey of the theory of join spaces. In Proceedings of the 5th International Congress on AHA, Iasi, Rommania, 4–10 July 1993; Hadronic Press: Palm Harbor, FA, USA, 1994; pp. 109–122.
24. Bryant, V.W.; Webster, R.J. Generalizations of the Theorems of Radon, Helly and Caratheodory. *Monatsh. Math.* **1969**, *73*, 309–315. [CrossRef]
25. Freni, D. Sur les hypergroupes cambistes. *Rend. Ist. Lomb.* **1985**, *119*, 175–186.
26. Freni, D. Sur la théorie de la dimension dans les hypergroupes. *Acta Univ. Carol. Math. Phys.* **1986**, *27*, 67–80.
27. Massouros, C.G. Hypergroups and convexity. *Riv. Mat. Pura Appl.* **1989**, *4*, 7–26.
28. Mittas, J.; Massouros, C.G. Hypergroups defined from linear spaces. *Bull. Greek Math. Soc.* **1989**, *30*, 63–78.
29. Massouros, C.G. Hypergroups and Geometry, Mem. *Acad. Romana Math. Spec. Issue* **1996**, *19*, 185–191.
30. Dramalidis, A. On geometrical hyperstructures of finite order. *Ratio Math.* **2011**, *21*, 43–58.
31. Dramalidis, A. Some geometrical P-H_V-hyperstructures. In *New Frontiers in Hyperstructures*; Hadronic Press: Palm Harbor, FL, USA, 1996; pp. 93–102.
32. Cristea, I. Complete hypergroups, 1-hypergroups and fuzzy sets. *An. Stalele Univ. Ovidius Constanta* **2000**, *10*, 25–38.
33. Cristea, I. A property of the connection between fuzzy sets and hypergroupoids. *Ital. J. Pure Appl. Math.* **2007**, *21*, 73–82.
34. Stefanescu, M.; Cristea, I. On the fuzzy grade of hypergroups. *Fuzzy Sets Syst.* **2008**, *159*, 1097–1106. [CrossRef]
35. Corsini, P.; Cristea, I. Fuzzy sets and non complete 1-hypergroups. *An. Stalele Univ. Ovidius Constanta* **2005**, *13*, 27–54.
36. Corsini, P. Graphs and Join Spaces. *J. Comb. Inf. Syst. Sci.* **1991**, *16*, 313–318.
37. Corsini, P. *Prolegomena of Hypergroup Theory*; Aviani Editore: Udine, Italy, 1993.
38. Corsini, P. Join Spaces, Power Sets, Fuzzy Sets. In Proceedings of the 5th International Congress on Algebraic Hyperstructures and Applications, Iasi, Romania, 4–10 July 1993; Hadronic Press: Palm Harbor, FL, USA, 1994; pp. 45–52.
39. Corsini, P. Binary relations, interval structures and join spaces. *J. Appl. Math. Comput.* **2002**, *10*, 209–216. [CrossRef]
40. Corsini, P.; Leoreanu, V. Fuzzy sets and join spaces associated with rough sets. *Rend. Circ. Mat. Palermo* **2002**, *51*, 527–536. [CrossRef]
41. Corsini, P.; Leoreanu, V. *Applications of Hyperstructures Theory*; Kluwer Academic Publishers: Berlin, Germany, 2003.
42. Corsini, P.; Leoreanu-Fotea, V. On the grade of a sequence of fuzzy sets and join spaces determined by a hypergraph. *Southeast Asian Bull. Math.* **2010**, *34*, 231–242.
43. Leoreanu, V. On the heart of join spaces and of regular hypergroups. *Riv. Mat. Pura Appl.* **1995**, *17*, 133–142.
44. Leoreanu, V. Direct limit and inverse limit of join spaces associated with fuzzy sets. *PUMA* **2000**, *11*, 509–516.
45. Leoreanu-Fotea, V. Fuzzy join n-ary spaces and fuzzy canonical n-ary hypergroups. *Fuzzy Sets Syst.* **2010**, *161*, 3166–3173. [CrossRef]
46. Leoreanu-Fotea, V.; Rosenberg, I. Join Spaces Determined by Lattices. *J. Mult.-Valued Logic. Soft Comput.* **2010**, *16*, 7–16.
47. Hoskova, S. Abelization of join spaces of affine transformations of ordered field with proximity. *Appl. Gen. Topol.* **2005**, *6*, 57–65. [CrossRef]
48. Chvalina, J.; Hoskova, S. Modelling of join spaces with proximities by first-order linear partial differential operators. *Ital. J. Pure Appl. Math.* **2007**, *21*, 177–190.
49. Hoskova, S.; Chvalina, J.; Rackova, P. Transposition hypergroups of Fredholm integral operators and related hyperstructures (Part I). *J. Basic Sci.* **2006**, *3*, 11–17.
50. Hoskova, S.; Chvalina, J.; Rackova, P. Transposition hypergroups of Fredholm integral operators and related hyperstructures (Part II). *J. Basic Sci.* **2008**, *4*, 55–60.
51. Chvalina, J.; Hoskova-Mayerova, S. On Certain Proximities and Preorderings on the Transposition Hypergroups of Linear First-Order Partial Differential Operators. *An. Stalele Univ. Ovidius Constanta Ser. Mat.* **2014**, 22, 85–103. [CrossRef]

52. Chvalina, J.; Chvalinova, L. Transposition hypergroups formed by transformation operators on rings of differentiable functions. *Ital. J. Pure Appl. Math.* **2004**, *15*, 93–106.
53. Jantosciak, J.; Massouros, C.G. Strong Identities and fortification in Transposition hypergroups. *J. Discret. Math. Sci. Cryptogr.* **2003**, *6*, 169–193. [CrossRef]
54. Massouros, C.G. Canonical and Join Hypergroups. *An. Sti. Univ. Al. I. Cuza Iasi* **1996**, *1*, 175–186.
55. Massouros, C.G. Normal homomorphisms of fortified join hypergroups, Algebraic Hyperstructures and Applications. In Proceedings of the 5th International Congress, Iasi, Romania, 4–10 July 1993; Hadronic Press: Palm Harbor, FL, USA, 1994; pp. 133–142.
56. Massouros, C.G. Isomorphism Theorems in fortified transposition hypergroups. *AIP Conf. Proc.* **2013**, *1558*, 2059–2062. [CrossRef]
57. Massouros, G.G.; Massouros, C.G.; Mittas, I.D. Fortified join hypergroups. *Ann. Math. Blaise Pascal* **1996**, *3*, 155–169. [CrossRef]
58. Massouros, G.G.; Zafiropoulos, F.A.; Massouros, C.G. Transposition polysymmetrical hypergroups. Algebraic Hyperstructures and Applications. In Proceedings of the 8th International Congress, Samothraki, Greece, 1–9 September 2002; Spanidis Press: Xanthi, Greece, 2003; pp. 91–202.
59. Massouros, C.G.; Massouros, G.G. Transposition polysymmetrical hypergroups with strong identity. *J. Basic Sci.* **2008**, *4*, 85–93.
60. Massouros, C.G.; Massouros, G.G. On subhypergroups of fortified transposition hypergroups. *AIP Conf. Proc.* **2013**, *1558*, 2055–2058. [CrossRef]
61. Massouros, G.G. Fortified join hypergroups and join hyperrings. *An. Sti. Univ. Al. I. Cuza, Iasi Sect. I Mat.* **1995**, *1*, 37–44.
62. Massouros, G.G. The subhypergroups of the fortified join hypergroup. *Italian J. Pure Appl. Math.* **1997**, *2*, 51–63.
63. Ameri, R.; Zahedi, M.M. Hypergroup and join space induced by a fuzzy subset. *Pure Math. Appl.* **1997**, *8*, 155–168.
64. Ameri, R. Topological transposition hypergroups. *Italian J. Pure Appl. Math.* **2003**, *13*, 181–186.
65. Ameri, R. Fuzzy transposition hypergroups. *Italian J. Pure Appl. Math.* **2005**, *18*, 147–154.
66. Rosenberg, I.G. Hypergroups and join spaces determined by relations. *Italian J. Pure Appl. Math.* **1998**, *4*, 93–101.
67. Massouros, G.G.; Mittas, J.D. Languages—Automata and hypercompositional structures, Algebraic Hyperstructures and Applications. In Proceedings of the 4th International Congress, Xanthi, Greece, 27–30 June 1990; World Scientific: Singapore, 1991; pp. 137–147.
68. Massouros, G.G. Automata, Languages and Hypercompositional Structures. Ph.D. Thesis, National Technical University of Athens, Athens, Greece, 1993.
69. Massouros, G.G. Automata and hypermoduloids, Algebraic Hyperstructures and Applications. In Proceedings of the 5th International Congress, Iasi, Romania, 4–10 July 1993; Hadronic Press: Palm Harbor, FL, USA, 1994; pp. 251–265.
70. Massouros, G.G. An automaton during its operation, Algebraic Hyperstructures and Applications. In Proceedings of the 5th International Congress, Iasi, Romania, 4–10 July 1993; Hadronic Press: Palm Harbor, FA, USA, 1994; pp. 267–276.
71. Massouros, G.G. On the attached hypergroups of the order of an automaton. *J. Discret. Math. Sci. Cryptogr.* **2003**, *6*, 207–215. [CrossRef]
72. Massouros, G.G. Hypercompositional structures in the theory of languages and automata. *An. Sti. Univ. Al. I. Cuza Iasi Sect. Inform.* **1994**, *3*, 65–73.
73. Massouros, G.G. Hypercompositional structures from the computer theory. *Ratio Math.* **1999**, *13*, 37–42.
74. Bonansinga, P. Sugli ipergruppi quasicanonici. *Atti Soc. Pelor. Sc. Fis. Mat. Nat.* **1981**, *27*, 9–17.
75. Massouros, C.G. Quasicanonical hypergroups. In Proceedings of the 4th Internat. Cong. on Algebraic Hyperstructures and Applications, Xanthi, Greece, 27–30 June 1990; World Scientific: Singapore, 1991; pp. 129–136.
76. Ioulidis, S. Polygroups et certaines de leurs propriétiés. *Bull. Greek Math. Soc.* **1981**, *22*, 95–104.
77. Comer, S. Polygroups derived from cogroups. *J. Algebra* **1984**, *89*, 397–405. [CrossRef]
78. Davvaz, B. *Polygroup Theory and Related Systems*; Publisher World Scientific: Singapore, 2013.

79. Krasner, M. *Approximation des corps valués complets de caractéristique p≠0 par ceux de caractéristique 0, Colloque d' Algèbre Supérieure (Bruxelles, Decembre 1956)*; Centre Belge de Recherches Mathématiques, Établissements Ceuterick, Louvain, Librairie Gauthier-Villars: Paris, France, 1957; pp. 129–206.
80. Mittas, J. Hypergroupes canoniques hypervalues, C.R. *Acad. Sci. Paris* **1970**, *271*, 4–7.
81. Mittas, J. Hypergroupes canoniques. *Math. Balk.* **1972**, *2*, 165–179.
82. Krasner, M. La loi de Jordan-Holder dans les hypergroupes et les suites generatrices des corps de nombres P-adiqes. *Duke Math. J.* **1940**, *6*, 120–140. [CrossRef]
83. Krasner, M. Cours d' Algebre superieure, Theories des valuation et de Galois. *Cours Fac. Sci. L' Univ. Paris* **1967**, *68*, 1–305.
84. Sureau, Y. Sous-hypergroupe engendre par deux sous-hypergroupes et sous-hypergroupe ultra-clos d'un hypergroupe, C.R. *Acad. Sc. Paris* **1977**, *284*, 983–984.
85. Kleene, S.C. Representation of Events in Nerve Nets and Finite Automata. In *Automata Studies*; Shannon, C.E., McCarthy, J., Eds.; Princeton University: Princeton, NJ, USA, 1956; pp. 3–42.
86. McNaughton, R.; Yamada, H. Regular Expressions and State Graphs for Automata. *IEEE Trans. Electron. Comput.* **1960**, *9*, 39–47. [CrossRef]
87. Brzozowski, J.A. A Survey of Regular Expressions and their Applications. *IEEE Trans. Electron. Comput.* **1962**, *11*, 324–335. [CrossRef]
88. Brzozowski, J.A. Derivatives of Regular Expressions. *J. ACM* **1964**, *11*, 481–494. [CrossRef]
89. Brzozowski, J.A.; McCluskey, E.J. Signal Flow Graph Techniques for Sequential Circuit State Diagrams. *IEEE Trans. Electron. Comput.* **1963**, *EC-12*, 67–76. [CrossRef]
90. Chvalina, J.; Chvalinova, L. State hypergroups of Automata. *Acta Math. Inform. Univ. Ostrav.* **1996**, *4*, 105–120.
91. Chvalina, J.; Křehlík, S.; Novák, M. Cartesian composition and the problem of generalizing the MAC condition to quasi-multiautomata. *An. Stalele Univ. Ovidius Constanta* **2016**, *24*, 79–100. [CrossRef]
92. Chvalina, J.; Novák, M.; Křehlík, S. Hyperstructure generalizations of quasi-automata induced by modelling functions and signal processing. *AIP Conf. Proc.* **2019**, *2116*, 310006. [CrossRef]
93. Novák, M.; Křehlík, S.; Stanek, D. n-ary Cartesian composition of automata. *Soft Comput.* **2019**, *24*, 1837–1849. [CrossRef]
94. Novák, M. Some remarks on constructions of strongly connected multiautomata with the input semihypergroup being a centralizer of certain transformation operators. *J. Appl. Math.* **2008**, *I*, 65–72.
95. Chorani, M.; Zahedi, M.M. Some hypergroups induced by tree automata. *Aust. J. Basic Appl. Sci.* **2012**, *6*, 680–692.
96. Chorani, M. State hyperstructures of tree automata based on lattice-valued logic. *RAIRO-Theor. Inf. Appl.* **2018**, *52*, 23–42.
97. Corsini, P. Binary relations and hypergroupoids. *Italian J. Pure Appl. Math.* **2000**, *7*, 11–18.
98. Corsini, P. Hypergraphs and hypergroups. *Algebra Universalis* **1996**, *35*, 548–555. [CrossRef]
99. Gionfriddo, M. *Hypergroups Associated with Multihomomorphisms between Generalized Graphs*; Corsini, P., Ed.; Convegno su Sistemi Binary e Loro Applicazioni: Taormina, Italy, 1978; pp. 161–174.
100. Nieminen, J. Join Space Graph. *J. Geom.* **1988**, *33*, 99–103. [CrossRef]
101. Nieminen, J. Chordal Graphs and Join Spaces. *J. Geom.* **1989**, *34*, 146–151. [CrossRef]
102. De Salvo, M.; Lo Faro, G. Hypergroups and binary relations. *Mult. Valued Log.* **2002**, *8*, 645–657.
103. De Salvo, M.; Lo Faro, G. A new class of hypergroupoids associated to binary relations. *J. Mult. Valued Logic. Soft Comput.* **2003**, *9*, 361–375.
104. De Salvo, M.; Freni, D.; Lo Faro, G. Fully simple semihypergroups. *J. Algebra* **2014**, *399*, 358–377. [CrossRef]
105. Cristea, I.; Jafarpour, M.; Mousavi, S.S.; Soleymani, A. Enumeration of Rosenberg hypergroups. *Comput. Math. Appl.* **2011**, *32*, 72–81. [CrossRef]
106. Cristea, I.; Stefanescu, M. Binary relations and reduced hypergroups. *Discrete Math.* **2008**, *308*, 3537–3544. [CrossRef]
107. Cristea, I.; Stefanescu, M.; Angheluta, C. About the fundamental relations defined on the hypergroupoids associated with binary relations. *Electron. J. Combin.* **2011**, *32*, 72–81. [CrossRef]
108. Aghabozorgi, H.; Jafarpour, M.; Dolatabadi, M.; Cristea, I. An algorithm to compute the number of Rosenberg hypergroups of order less than 7. *Ital. J. Pure Appl. Math.* **2019**, *42*, 262–270.

109. Massouros, C.G.; Massouros, G.G. Hypergroups associated with graphs and automata. *AIP Conf. Proc.* **2009**, *1168*, 164–167. [CrossRef]
110. Massouros, C.G. On path hypercompositions in graphs and automata. *MATEC Web Conf.* **2016**, *41*, 5003. [CrossRef]
111. Massouros, C.G.; Tsitouras, C.G. Enumeration of hypercompositional structures defined by binary relations. *Ital. J. Pure Appl. Math.* **2011**, *28*, 43–54.
112. Tsitouras, C.G.; Massouros, C.G. Enumeration of Rosenberg type hypercompositional structures defined by binary relations. *Eur. J. Comb.* **2012**, *33*, 1777–1786. [CrossRef]
113. Massouros, G.G. The hyperringoid. *Mult. Valued Logic.* **1998**, *3*, 217–234.
114. Massouros, G.G.; Massouros, C.G. Homomorphic relations on Hyperringoids and Join Hyperrings. *Ratio Mat.* **1999**, *13*, 61–70.
115. Massouros, G.G. Solving equations and systems in the environment of a hyperringoid, Algebraic Hyperstructures and Applications. In Proceedings of the 6th International Congress, Prague, Czech Republic, 1–9 September 1996; Democritus Univ. of Thrace Press: Komotini, Greece, 1997; pp. 103–113.
116. Massouros, C.G.; Massouros, G.G. On join hyperrings, Algebraic Hyperstructures and Applications. In Proceedings of the 10th International Congress, Brno, Czech Republic, 3–9 September 1996; pp. 203–215.
117. Krasner, M. A class of hyperrings and hyperfields. *Int. J. Math. Math. Sci.* **1983**, *6*, 307–312. [CrossRef]
118. Mittas, J. Sur certaines classes de structures hypercompositionnelles. *Proc. Acad. Athens* **1973**, *48*, 298–318.
119. Mittas, J. Sur les structures hypercompositionnelles, Algebraic Hyperstructures and Applications. In Proceedings of the 4th International Congress, Xanthi, Greece, 27–30 June 1990; World Scientific: Singapore, 1991; pp. 9–31.
120. Ameri, R.; Eyvazi, M.; Hoskova-Mayerova, S. Superring of Polynomials over a Hyperring. *Mathematics* **2019**, *7*, 902. [CrossRef]
121. Ευκλείδης (Euclid). *Στοιχεία (Elements) c. 300 BC*; Heiberg, I., Stamatis, E., Eds.; O.E.Δ.B: Athens, Greece, 1975.
122. Heath, T.L. *The Thirteen Books of Euclid's Elements*, 2nd ed.; Dover Publications: New York, NY, USA, 1956.
123. Freni, D. Strongly Transitive Geometric Spaces: Applications to Hypergroups and Semigroups Theory. *Commun. Algebra* **2004**, *32*, 969–988. [CrossRef]
124. Mirvakili, S.; Davvaz, B. Strongly transitive geometric spaces: Applications to hyperrings. *Revista UMA* **2012**, *53*, 43–53.
125. Anvariyeh, S.M.; Davvaz, B. Strongly transitive geometric spaces associated with (m, n)-ary hypermodules. *An. Sti. Univ. Al. I. Cuza Iasi* **2013**, *59*, 85–101. [CrossRef]
126. Mittas, J. Hyperanneaux et certaines de leurs propriétés, C.R. *Acad. Sci. Paris* **1969**, *269*, 623–626.
127. Mittas, J. Hypergroupes et hyperanneaux polysymétriques, C.R. *Acad. Sci. Paris* **1970**, *271*, 920–923.
128. Mittas, J. Contributions a la théorie des hypergroupes, hyperanneaux, et les hypercorps hypervalues, C.R. *Acad. Sci. Paris* **1971**, *272*, 3–6.
129. Mittas, J. Certains hypercorps et hyperanneaux définis à partir de corps et anneaux ordonnés. *Bull. Math. Soc. Sci. Math. R. S. Roum.* **1971**, *15*, 371–378.
130. Mittas, J. Sur les hyperanneaux et les hypercorps. *Math. Balk.* **1973**, *3*, 368–382.
131. Mittas, J. Contribution à la théorie des structures ordonnées et des structures valuées. *Proc. Acad. Athens* **1973**, *48*, 318–331.
132. Mittas, J. Hypercorps totalement ordonnes. *Sci. Ann. Polytech. Sch. Univ. Thessalon.* **1974**, *6*, 49–64.
133. Mittas, J. Espaces vectoriels sur un hypercorps. Introduction des hyperspaces affines et Euclidiens. *Math. Balk.* **1975**, *5*, 199–211.
134. Viro, O. On basic concepts of tropical geometry. *Proc. Steklov Inst. Math.* **2011**, *273*, 252–282. [CrossRef]
135. Viro, O. Hyperfields for Tropical Geometry I. Hyperfields and dequantization. *arXiv* **2010**, arXiv:1006.3034.
136. Baker, M.; Bowler, N. Matroids over hyperfields. *arXiv* **2017**, arXiv:1601.01204.
137. Jun, J. Geometry of hyperfields. *arXiv* **2017**, arXiv:1707.09348.
138. Lorscheid, O. Tropical geometry over the tropical hyperfield. *arXiv* **2019**, arXiv:1907.01037.
139. Connes, A.; Consani, C. The hyperring of adèle classes. *J. Number Theory* **2011**, *131*, 159–194. [CrossRef]
140. Connes, A.; Consani, C. From monoids to hyperstructures: In search of an absolute arithmetic. *arXiv* **2010**, arXiv:1006.4810.
141. Massouros, C.G. Constructions of hyperfields. *Math. Balk.* **1991**, *5*, 250–257.

142. Massouros, C.G. Methods of constructing hyperfields. *Int. J. Math. Math. Sci.* **1985**, *8*, 725–728. [CrossRef]
143. Massouros, C.G. A class of hyperfields and a problem in the theory of fields. *Math. Montisnigri* **1993**, *1*, 73–84.
144. Massouros, C.G. On the theory of hyperrings and hyperfields. *Algebra Logika* **1985**, *24*, 728–742. [CrossRef]
145. Nakassis, A. Recent results in hyperring and hyperfield theory. *Int. J. Math. Math. Sci.* **1988**, *11*, 209–220. [CrossRef]
146. Massouros, C.G. A Field Theory Problem Relating to Questions in Hyperfield Theory. *AIP Conf. Proc.* **2011**, *1389*, 1852–1855. [CrossRef]
147. Massouros, C.G. Free and cyclic hypermodules. *Ann. Mat. Pura Appl.* **1988**, *150*, 153–166. [CrossRef]
148. Vahedi, V.; Jafarpour, M.; Aghabozorgi, H.; Cristea, I. Extension of elliptic curves on Krasner hyperfields. *Comm. Algebra* **2019**, *47*, 4806–4823. [CrossRef]
149. Vahedi, V.; Jafarpour, M.; Cristea, I. Hyperhomographies on Krasner Hyperfields. *Symmetry* **2019**, *11*, 1442. [CrossRef]
150. Vahedi, V.; Jafarpour, M.; Hoskova-Mayerova, S.; Aghabozorgi, H.; Leoreanu-Fotea, V.; Bekesiene, S. Derived Hyperstructures from Hyperconics. *Mathematics* **2020**, *8*, 429. [CrossRef]