An Analytical Method for the Longitudinal Vibration of a Large-Diameter Pipe Pile in Radially Heterogeneous Soil Based on Rayleigh–Love Rod Model

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Abstract: Based on the Rayleigh–Love rod model and Novak’s plane-strain theory, an analytical method for the longitudinal vibration of a large-diameter pipe pile in radially heterogeneous soil is proposed. Firstly, the governing equations of the pile-soil system are established by taking both the construction disturbance effect and transverse inertia effect into account. Secondly, the analytical solution of longitudinal dynamic impedance at the pile top can be achieved by using Laplace transform and complex stiffness transfer techniques. Thirdly, the present analytical solution for dynamic impedance can also be performed in contrast with the existing solution to examine the correctness of the analytical method in this work. Further, the effect of pile Poisson’s ratio, pile diameter ratio as well as soil disturbed degree on the dynamic impedance are investigated. The results demonstrate that the Rayleigh–Love rod is appropriate for simulating the vibration of a large-diameter pipe pile in heterogeneous soils.

Keywords: pile vibration; analytical method; Rayleigh–Love rod model; dynamic impedance; complex stiffness transfer method; construction disturbance effect

1. Introduction

Pile foundations of the building and civil engineering structures often suffer earthquake loads, dynamic machine loads and traffic loads in a natural environment. The theory of the pile-soil interaction has played an important role in seismic design and dynamic design of foundation [1–3]. Many scholars have established lots of simplified computational models in combination with the knowledge of mathematics and mechanics to explore the longitudinal vibration characteristics of pipe piles in the complex geological conditions [4–14]. Based on the one-dimensional (1D) theoretical model, Randolph et al. [15,16] first proposed a new pile-soil coupling vibration model for the thin-wall pipe piles by taking soil plug into account, and then verified the rationality of the model by the experimental and numerical methods. Liu et al. [17,18] applied the large diameter thin wall pipe piles for natural soft-clay, and expounded the advantages of pipe piles applied to practical engineering. Ding’s research team [19,20] has developed some analytical methods for the longitudinal vibration of pipe pile by using Laplace transform and inverse Fourier transform technique, based on Winkler model and

three-dimensional (3D) continuous medium model. Further, Zheng et al. [21,22] took the longitudinal and radial displacement of soil into account, and then deduced novel analytical solutions of the vertical vibration frequency domain of the pipe pile by using the potential function method. Based on the 1D vibration theory, Ai et al. [23] researched the dynamic respond problems of pipe piles in a multilayered medium by means of the matrix method and analytical layer-element method. In order to consider the longitudinal and radial heterogeneity soil, Li et al. [24] introduced a new analytical approach for the complex impedance of pipe piles by using variable separation technique, and investigated the vertical dynamic response of the pipe pile. According to the plane-strain theory, Wu et al. [25] and Liu et al. [26] further developed a new pile-soil coupling vibration model by introducing an additional mass to take the soil plug into account, and then highlighted the importance of the apparent wave velocity in the results of non-destructive testing of tubular piles.

It is obvious that the above pile foundation models are all based on the classical Bernoulli–Euler theory, which is only suitable for the slender piles. However, the propagation of the stress wave in the pile shaft can be induced due to the dispersion effect, and there is a certain deviation for the utilization of the 1D stress-wave theory in the treatment of large-diameter piles. Based on this, it is more appropriate to adopt Rayleigh–Love rod to simulate the vertical vibration of large-diameter piles in the natural medium [27–30]. Hence, based on Novak’s model and the Rayleigh–Love rod model, Li et al. [31] and Wu et al. [32] developed new coupling vibration models for the large-diameter solid piles in radial heterogeneous and homogeneous soil, respectively. Afterwards, based on the 3D continuum model and Rayleigh–Love rod model, Lu et al. [33,34] and Li et al. [35] made an exploration to the vertical vibration for the large-diameter solid pile in a multilayer medium. Further, Zheng et al. [36] established a new coupled vibration model between a large-diameter tubular pile and core soil, and then emphasized a necessity of taking lateral inertia effect into account for the longitudinal vibration of large diameter tubular piles.

However, no work so far has focused on the longitudinal vibration of large-diameter tubular piles in heterogeneous medium by taking lateral inertia effect into account. In light of this, an analytical method for longitudinal vibration of a large-diameter pipe pile in radially heterogeneous soil is developed based on the Rayleigh–Love rod and Novak’s plane-strain models. Further, corresponding analytical solutions for the dynamic impedance of a large-diameter pipe pile are deduced by employing Laplace transform and complex stiffness transfer techniques. Finally, the influences of pile Poisson’s ratio, diameters ratio as well as the disturbance degree on the dynamic impedance are investigated by means of parametric analysis method. Besides, the present analytical solution can be reduced to investigate slender pile or pipe pile embedded in homogeneous soil with hysteretic-type damping, and also suitable for various vibration problems with respect to the 3D effect of wave propagation for the large-diameter pipe pile.

2. Mechanical Model and Basic Assumptions

A new simplified mechanical model of the pile-soil coupled vibration can be established as shown in Figure 1. A longitudinal height of pile-soil system is \( H \). The inner disturbed zone is dissected into \( n \) longitudinal ring-shaped homogeneous layer numbered \( 1, 2, \ldots, j, \ldots, n \). \( r_j \) is the radius of the \( j \)th sublayer. \( r_{n+1} \) is the radius of the inner disturbed zone. \( b \) is a radial thickness of inner disturbed zone. \( r_0 \) and \( r_1 \) represent the inner and outer radius of large-diameter pipe pile, respectively. \( p(t) \) represents the uniformly distributed excitation load. \( \delta p \) and \( k_p \) denote spring and damper coefficient of the pile toe, respectively. \( f(r) \) is a parabolic function [37,38].

Some basic assumptions are applied in this paper:
(1) A Rayleigh–Love rod model is introduced to simulate the pipe pile.
(2) A complex stiffness transfer technique can be used to transfer shear stress from soil to pile.
(3) There exists a small deformation when the pile is subjected to vertical vibration.
(4) In internal disturbance region, the interface is completely coupled. \( G_j^s \) and \( \eta_j \) satisfy the following expressions:

\[
G_j^s(r) = \begin{cases} 
G_{n+1}^s & r = r_1 \\
G_{n+1}^s \times f(r) & r_1 < r < r_{n+1} \\
G_{n+1}^s & r \geq r_{n+1} 
\end{cases} 
\]

\[
\eta_j(r) = \begin{cases} 
\eta_{n+1}^s & r = r_1 \\
\eta_{n+1}^s \times f(r) & r_1 < r < r_{n+1} \\
\eta_{n+1}^s & r \geq r_{n+1} 
\end{cases} 
\]

3. Governing Equations

According to the elastodynamic theory of Novak’s plane-strain conditions [39], an equilibrium equation of the \( j \)th disturbed layer is established as follows

\[
\left(G_j \frac{\partial^2}{\partial r^2} + \eta_j \frac{\partial^3}{\partial t \partial r^2}\right)u_j^p(r, t) + \frac{1}{r} \left(C_j \frac{\partial}{\partial r} + \eta_j \frac{\partial^3}{\partial t \partial r}\right)u_j^p(r, t) = \rho_j^s \frac{\partial^2}{\partial t^2} u_j^p(r, t) \tag{3}
\]

where \( u_j^p(r, t) \), \( \eta_j \) and \( \rho_j^s \) are the longitudinal displacement, viscous coefficient and density, respectively, of the surrounding medium.

An equilibrium equation of the core soil is also established:

\[
C_0^s \frac{\partial^2}{\partial r^2} u_0^p(r, t) + \eta_0^s \frac{\partial^3}{\partial t \partial r^2} u_0^p(r, t) + \frac{1}{r} \left[C_0^s \frac{\partial}{\partial r} u_0^p(r, t) + \eta_0^s \frac{\partial^3}{\partial t \partial r} u_0^p(r, t)\right] = \rho_0^s \frac{\partial^2}{\partial t^2} u_0^p(r, t) \tag{4}
\]

An equilibrium equation of large-diameter pipe pile shaft is given by

\[
EP \frac{\partial}{\partial z} \left( f_1^p(z, t) + \frac{1}{r} \frac{\partial}{\partial r} \left( f_1^p(z, t) + \frac{\partial}{\partial r} \left( f_1^p(z, t) \right) \right) \right) - 2 \pi r f_1^p(z, t) - 2 \pi r f_0^p(z, t) = 0 \tag{5}
\]

where \( u^p(z, t) \), \( \rho^p \), \( E \), \( v \), \( A^p \) and \( R^p \) are the longitudinal displacement, mass density, elastic modulus, Poisson’s ratio, cross-section area and radius of inertia of pipe pile, respectively. \( f_0^s \) and \( f_1^s \) denote shear stress exerted by the core and disturbed soil on large-diameter pipe pile shaft, respectively. \( A^p = \pi r_1^2 - \pi r_0^2 \), \( R^p = \sqrt{\frac{r_1^2 + r_0^2}{2}} \).
Stress and displacement continuity conditions at the interface between pile and the core soil are expressed as

\[ u^n_0(0, t) = u^P(0, t) \]  \hspace{1cm} (6)
\[ f^n_0 = \tau^n_0(\rho_n) \mid_{\rho_n = 0} \]  \hspace{1cm} (7)

In semi-infinite outer zone, the displacement tends to zero, i.e.,

\[ \lim_{r \to \infty} u^n_{n+1} = 0 \]  \hspace{1cm} (8)

The relationships of displacement and stress continuity at the interface between pile shaft and the disturbed medium are given by

\[ u^n_1(r_1, t) = u^P(r_1, t) \]  \hspace{1cm} (9)
\[ f^n_1 = -\tau^n_1(\rho_return) \mid_{\rho = r_1} \]  \hspace{1cm} (10)

The vibration of the pile can be written by

\[ \left[ \rho^P A^P (v^P R^P) \frac{\partial^2}{\partial r^2} u^P(z, t) + E P A^P \frac{\partial}{\partial z} u^P(z, t) \right]_{z = 0} = -p(t) \]  \hspace{1cm} (11)
\[ \left[ \rho^P A^P (v^P R^P) \frac{\partial^2}{\partial r^2} u^P(z, t) + E P A^P \frac{\partial}{\partial z} u^P(z, t) + (\delta^P_1 \frac{\partial}{\partial t} + k^P) u^P(z, t) \right]_{z = H} = 0 \]  \hspace{1cm} (12)

4. Solution of the Disturbed Soil

Applying Laplace transform to Equation (3), it can be transformed to

\[ (G^p_{ij} \frac{\partial^2}{\partial r^2} + \eta^p_{ij} \frac{\partial^2}{\partial r^2} ) U^p_i(r, s) + \frac{1}{r} (G^p_{ij} \frac{\partial}{\partial r} + \eta^p_{ij} \frac{\partial}{\partial r}) U^p_i(r, s) = \rho^p_{ij} s^2 U^p_i(r, s) \]  \hspace{1cm} (13)

where $U^p_i$ is Laplace transform of $u^p_i$, the transformation formula is $U(r, s) = \int_0^{+\infty} u(r, t)e^{-st} dt$, $s = i\omega$, $i = \sqrt{-1}$.

After further arrangement of Equation (13), it can be given by

\[ (\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r}) U^p_i(r, s) = (q^p_i)^2 U^p_i(r, s) \]  \hspace{1cm} (14)

where $q^p_i = \sqrt{\frac{\rho^p_{ij}}{G^p_{ij} + \eta^p_{ij}}}$.

Hence, the general solution of Equation (14) is expressed as

\[ U^p_i(r, s) = A^p_i K_0(q^p_i r) + B^p_i I_0(q^p_i r) \]  \hspace{1cm} (15)

where $I_0(q^p_i r)$ and $K_0(q^p_i r)$ are the first and second kind, respectively, modified Bessel functions of order zero; $A^p_i$ and $B^p_i$ are undetermined coefficients.

Substituting Equation (8) into Equation (15) and applying Laplace transform, it produces

\[ U^a_{n+1}(r, i\omega) = A^a_{n+1} K_0(q^a_{n+1} r) \]  \hspace{1cm} (16)

Therefore, the vertical shear stress at outer zone is given by

\[ \tau^a_{n+1} = (G^a_{n+1} + i\omega \eta^a_{n+1}) \frac{\partial U^a_{n+1}(r, i\omega)}{\partial r} = -q^a_{n+1} A^a_{n+1} (G^a_{n+1} + i\omega \eta^a_{n+1}) K_1(q^a_{n+1} r) \]  \hspace{1cm} (17)
The longitudinal shear stress of the \( j \)th layer is expressed as

\[
\tau_j^s = (G_j^s + \eta_j^s) \frac{\partial U_j^s(r, s)}{\partial r} = -q_j^s (G_j^s + \eta_j^s) [A_j^sK_1(q_j^s r) - B_j^s I_1(q_j^s r)]
\]  

(18)

Further, the longitudinal complex stiffness at the outer boundary and inner boundary of the \( j \)th layer are given by

\[
TT_{j+1}^s = -\frac{2\pi r_{j+1}q_j^s(r_{j+1})}{U_j^s(r_{j+1}, s)} = \frac{2\pi r_jq_j^s(G_j^s + \eta_j^s) [A_j^sK_1(q_j^s r_j) - B_j^s I_1(q_j^s r_j)]}{A_j^sK_0(q_j^s r_j) + B_j^s I_0(q_j^s r_j)}
\]

(19)

\[
TT_j^s = -\frac{2\pi r_jq_j^s(G_j^s + \eta_j^s) [A_j^sK_1(q_j^s r_j) - B_j^s I_1(q_j^s r_j)]}{A_j^sK_0(q_j^s r_j) + B_j^s I_0(q_j^s r_j)}
\]

(20)

Therefore, Equation (21) can be obtained by the condition of Equation (9), it yields

\[
TT_j^s = \left(\frac{2\pi r_jq_j^s(G_j^s + \eta_j^s)(M_j^sTT_{j+1}^s + E_j^s)}{N_j^sTT_{j+1}^s + F_j^s}\right)
\]

(21)

where \( E_j^s, F_j^s, M_j^s \) and \( N_j^s \) can be viewed in Appendix A.

5. Solutions of the Inner Soil

Performing Laplace transform to Equation (4), it can be transformed to

\[
(G_0^s \frac{\partial^2}{\partial r^2} + \eta_0^s \frac{\partial^2}{\partial r^2})U_0^s(r, s) + \frac{1}{r} (G_0^s \frac{\partial}{\partial r} + \eta_0^s \frac{\partial}{\partial r})U_0^s(r, s) = \rho_0^s \frac{\partial^2}{\partial r^2} U_0^s(r, s)
\]

(22)

After further arrangement of Equation (22), the expression is as follows

\[
\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r}\right)U_0^s(r, s) = (q_0^s)^2 U_0^s(r, s)
\]

(23)

where \( q_0^s = \sqrt{\rho_0^s (G_0^s + \eta_0^s)} \).

The solution of Equation (23) is expressed as

\[
U_0^s(r, s) = A_0^sK_0(q_0^s r) + B_0^s I_0(q_0^s r)
\]

(24)

Further, substituting Equation (6) into Equation (23), it produces

\[
U_0^s(r, s) = B_0^s I_0(q_0^s r)
\]

(25)

Therefore, combining Equations (7) and (25), the longitudinal shear stiffness at interface of soil-pile is given by

\[
TT_0 = -\frac{2\pi r_0 q_0^s(r_0)}{U_0} = -\frac{2\pi r_0 q_0^s(G_0^s + \eta_0^s) I_1(q_0^s r_0)}{I_0(q_0^s r_0)}
\]

(26)

6. Solutions of the Pipe Pile

Combining with \( TT_j^s \) and \( TT_0 \), Equation (5) can be transformed to

\[
\frac{\partial U_j^p(z, s)}{\partial z^2} = a^2 U_j^p(z, s)
\]

(27)
where \( a^2 = \frac{\rho^2 P \rho^2 + TT_1 - TT_0}{AP(\rho^2 + \rho^2(\rho^2 P)^2)} \).

The solution of Equation (27) is given by

\[
U_P(z, s) = \frac{M P e^{2az} + N P}{e^{asz}}
\]

where \( q_i^{*} = \sqrt{G_j^{*} + \eta_f^*} \).

Substituting Equation (28) into Equations (11) and (12) and applying Laplace transform, it produces

\[
M' = \frac{\xi P(s)}{E \rho A_p (\xi - 1)}
\]

\[
N' = \frac{P(s)}{E \rho A_p (\xi - 1)}
\]

where \( \xi = \frac{\rho^2 AP(\rho^2 P)^2 - \rho^2 (\rho^2 + \rho^2 (\rho^2 P)^2)^2}{a^2 (\rho^2 P)^2 + (\rho^2 + \rho^2 (\rho^2 P)^2)^2} \).

Hence, the displacement solution of the pile shaft can be obtained as

\[
U_P(z, s) = \frac{(\xi + 1)e^{-as}}{E \rho A_p (\xi - 1)} P(s)
\]

where \( P(s) \) is the Laplace transform of \( p(t) \).

Further, the longitudinal impedance function of the pile top is given by

\[
K_d = \frac{P(s)}{U_P} = \frac{E \rho A_p (\xi - 1)}{(1 + \xi)} = \frac{E \rho A_p}{H} K_d'
\]

\[
K_d' = \frac{\rho (\xi - 1)}{\xi^2 + 1}
\]

where \( \rho = aH \).

The dimensionless complex impedance \( K_d' \) can be further rewritten as

\[
K_d' = K_r + iK_i
\]

where \( K_r \) represents the stiffness factor; \( K_i \) denotes the damping factor.

7. Results and Discussions

Comprehensive parametric analyses are given insight into the dynamic impedance for a large-diameter pipe pile in radially heterogeneous medium based on a solution of Equation (34). Wang et al. [37] and El Naggar [40] have pointed out that the analytical solutions tend to be stable by setting \( n \geq 20 \). Therefore, in this paper, the value of \( n \) is equal to 20. In addition, the coefficient of disturbance degree \( \zeta \) [37] is defined as

\[
\zeta = \sqrt{G_{n+1}/\eta_{n+1}^*} = \sqrt{\eta_{j}^*/\eta_{n+1}^*} = V_{1n}^s/V_{n+1}^s
\]

when \( \zeta = 1 \), the soil is homogeneous; when \( \zeta > 1 \), the soil is strengthened; when \( \zeta < 1 \), the soil is weakened. The parameters of pile-soil system are listed: \( H = 6 \text{ m} \); \( r_1 = b = 0.5 \text{ m} \); \( r_0 = 0.38 \text{ m} \);
\( V_P = 3200 \text{ m/s} \); \( \rho_P = 2500 \text{ kg/m}^3 \); \( k_P = 1 \times 10^7 \text{ kN/m}^3 \); \( \delta_P = 1 \times 10^2 \text{ kN·s/m}^2 \); \( V_{m+1}^s = V_0^s = 50 \text{ m/s} \); \( \eta_{n+1}^* = \eta_0^* = 5 \text{ kN·s/m}^2 \); \( \rho_j^* = \rho_0^* = 2000 \text{ kg/m}^3 \); \( v_P = 0.25 \) and \( \zeta = 1.4 \).
7.1. Verification of the Solution

To verify the rationality of the analytical solution of a large-diameter pipe pile in radial heterogeneous medium based on Rayleigh–Love rod model, a degradation of present solution of the dimensionless complex impedance $K'_d$ given in Equation (34) is compared with the existing literature. By setting $v^p = 0$ and $\zeta^s = 1$, the Rayleigh–Love rod model in this paper was reduced to the Euler–Bernoulli rod model of Ref. [41]. It is apparent from Figure 2 that the degenerated solution of this work with different inner radius agreed well with the previous solution.

Figure 2. Contrast of the degradation of the present solution ($v^p = 0$, $\zeta^s = 1$) with the solution of Ref. [41]: (a) true stiffness and (b) equivalent damping.

7.2. Parametric Analyses

In the following analysis, the characteristic of the large-diameter pipe pile is conducted to study by taking a transverse inertia effect into account. According to the analytical solution of the method in this work, the Rayleigh–Love rod can be degenerated into the Euler–Bernoulli rod by setting $v^p = 0$. The effect of Poisson’s ratio on the dynamic impedance is shown in Figure 3. It is clear that the effect of Poisson’s ratio on the dynamic impedance becomes significant mainly in the high-frequency range, that is, both the resonance amplitude and frequency of the curves decrease as Poisson’s ratio increases in high-frequency range, and the amplitude is obviously reducing with a rising frequency. However, the effect of pile Poisson’s ratio is almost negligible in the low-frequency region. The results showed that the Rayleigh–Love rod model performs better than Euler–Bernoulli rod model when simulate large-diameter pipe pile embedded in heterogeneous medium.

Figure 4 shows the effect of the diameter ratio on the dynamic impedance at the head of the large-diameter pipe pile. As shown in Figure 4, the diameter ratio had a remarkable effect on the dynamic impedance curves. The resonance amplitude and frequency on the dynamic impedance curves increased significantly with a decrease of the diameter ratio. The resonance amplitude was the largest when the inner radius of the pipe pile was equal to zero, which means that the inner soil shows an obvious vibration reduction effect on the large-diameter pipe pile. Besides, it can be found that both the resonance amplitude and frequency at high-frequency reduced obviously on the dynamic impedance curves when transverse inertia effect was taken into account (i.e., $v^p = 0.25$), but the curves kept invariant at the low-frequency.

Figure 5 depicts the influences of the transverse inertia effect on the dynamic impedance with a different value of the disturbance degree coefficient. As shown in Figure 5, the disturbance degree had a remarkable effect on the dynamic impedance. Compared to the homogeneous soil (i.e., $\zeta^s = 1$), the resonance amplitude and frequency on the dynamic impedance curves increased obviously in the softening soil (i.e., $\zeta^s = 0.6$). In contrary, the resonance amplitude and frequency on the
dynamic impedance curves decreased in the hardening soil (i.e., $\zeta^s = 1.4$). Besides, when $v^P = 0.25$, the resonance amplitude and frequency on the dynamic impedance curves decreased gradually as the excitation frequency increased, and this trend became more evident at high-frequency.

Figure 3. Variation of dynamic impedance with pile Poisson’s ratio: (a) true stiffness and (b) equivalent damping.

Figure 4. Variation of dynamic impedance with the diameters ratio: (a) true stiffness and (b) equivalent damping.

Figure 5. Variation of dynamic impedance with the coefficient of the disturbance degree: (a) true stiffness and (b) equivalent damping.
8. Conclusions

According to the Rayleigh–Love rod model and elastodynamic theory of Novak’s plane-strain, an analytical method for the longitudinal vibration of the large-diameter pipe pile in radially inhomogeneous soil with viscous damping can be proposed by means of the Laplace transform and complex stiffness transfer techniques. The effect of Poisson’s ratio, the diameter ratio as well as the disturbance extent on the dynamic impedance is investigated. Calculation and analysis results show that:

1. The impact of pile Poisson’s ratio on the pile head’s dynamic impedance was most evident in the high-frequency region. The resonance amplitude and frequency of the pile head’s dynamic impedance curves decreased with an increase of pile Poisson’s ratio, and the tendency became more obvious with a rising frequency. The results demonstrated the rationality and necessity of introducing Rayleigh–Love rod model to simulate the large-diameter pipe pile.

2. The resonance amplitude and frequency on the dynamic impedance curves had a remarkable increase as a decrease of the diameter ratio. It is shown that the inner soil had an obvious vibration reduction effect on the large-diameter pipe pile.

3. In the softening (hardening) soil, the resonance amplitude and frequency on the dynamic impedance curves were larger (smaller) than those in the homogeneous soil. When the transverse inertia effect was considered, the change became more noticeable at the high frequency.

4. The degeneration of the present solution corresponded well to the previous solutions, and it indicates that Rayleigh–Love rod model was appropriate for the vertical vibration problem of large-diameter pipe pile. The present solution had given a significant insight into the vital relationship between the mechanical parameters of pipe pile and radially heterogeneous medium, and could overcome some limitations of the previous models based on the 1D classical rod theory. In addition, this method could be extensively applied into the complex geological environment due to the construction disturbance.

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Appendix A

The coefficients of $E^{\phi}_j$, $F^{\phi}_j$, $M^{\phi}_j$ and $N^{\phi}_j$ can be expressed by

$$E^{\phi}_j = 2\pi r_{j+1}(G^s + \eta^s s)q_j^s[K_1(q_j^s r_j)I_1(q_j^s r_{j+1}) - K_1(q_j^s r_{j+1})I_1(q_j^s r_j)]$$  \hspace{1cm} (A1)

$$F^{\phi}_j = 2\pi r_{j+1}(G^s + \eta^s s)q_j^s[K_0(q_j^s r_j)I_1(q_j^s r_{j+1}) - K_1(q_j^s r_{j+1})I_0(q_j^s r_j)]$$  \hspace{1cm} (A2)

$$M^{\phi}_j = K_0(q_j^s r_{j+1})I_1(q_j^s r_j) + K_1(q_j^s r_j)I_0(q_j^s r_{j+1})$$ \hspace{1cm} (A3)

$$N^{\phi}_j = -K_0(q_j^s r_{j+1})I_1(q_j^s r_j) + K_0(q_j^s r_j)I_0(q_j^s r_{j+1})$$ \hspace{1cm} (A4)

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