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Article

On the Beckenbach–Gini–Lehmer Means and Means Mappings

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Abstract: Beckenbacg–Gini–Lehmer type means and mean-type mappings generated by functions of several variables, for which the arithmetic mean is invariant, are introduced. Equality of means of that type, their homogeneity, and convergence of the iterates of the respective mean-type mappings are considered. An application to solving a functional equation is given.

Keywords: means; Gini means; mean-type mapping; invariant mean; functional equation; iteration

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1. Introduction

If I is an interval and φ is a positive function defined on I, then the two-variable function $M_{[\varphi]}$ defined on I^2 by

$$M_{[\varphi]}(x,y) = \frac{x\varphi(x) + y\varphi(y)}{\varphi(x) + \varphi(y)}, \qquad x, y \in I,$$

maps I^2 into I, and even more: it is a mean in I, that is, $\min(x,y) \le M_{[\varphi]}(x,y) \le \max(x,y)$ for all $x, y \in I$, and it is called a Gini mean, Beckenbach–Gini mean, or Lehmer mean ([1–3]) of a generator φ. In the sequel we shall write briefly, B–G–L mean. The function $M^*_{[\varphi]} : I^2 \to I$ defined by

$$M^*_{[\varphi]}(x,y) = \frac{y\varphi(x) + x\varphi(y)}{\varphi(x) + \varphi(y)}, \qquad x, y \in I,$$

is also a B–G–L mean in I, because $M^*_{\left[\frac{1}{\varphi}\right]} = M_{[\varphi]}$. The pair of means $M_{[\varphi]}$ and $M^*_{[\varphi]}$ has an interesting property. Namely, the arithmetic mean $A_2(x,y) = \frac{x+y}{2}$ is invariant with respect to the mean-type mapping $\left(M_{[\varphi]}, M^*_{[\varphi]}\right) : I^2 \to I^2$, i.e.,

$$A_2 \circ (M_{[\varphi]}, M^*_{[\varphi]}) = A_2;$$

and, if φ is continuous, the sequence $\left((M_{[\varphi]}, M^*_{[\varphi]})^n : n \in \mathbb{N}\right)$ of iterates of the mean-type mapping $\left(M_{[\varphi]}, M^*_{[\varphi]}\right)$ converges to the mean-type mapping (A_2, A_2), uniformly on compact sets ([4,5]). For this reason $M^*_{[\varphi]}$ is called complementary to $M_{[\varphi]}$ (and vice-versa) with respect to the arithmetic mean A_2 ([6]).

In this paper we show that these properties remain true for some broader classes of means which generalize the two-variable B–G–L means. In particular, in Section 3, modifying a k-variable counterpart of B–G–L mean

$$M_{[\varphi],k}^{[1]}(x_1, \ldots, x_k) = \frac{x_1\varphi(x_1) + \cdots + x_k\varphi(x_k)}{\varphi(x_1) + \cdots + \varphi(x_k)}$$

by cycling the variables $x_1, \ldots, x_k$ which are not arguments of φ, we construct the means $M_{[\varphi],k}^{[2]}, \ldots, M_{[\varphi],k}^{[k]}$ ("complementary" to $M_{[\varphi],k}^{[1]}$) such that the arithmetic mean A_k is invariant with respect to the mean-type mapping $\mathbf{M}_{[\varphi],k} = \left(M_{[\varphi],k}^{[1]}, \ldots, M_{[\varphi],k}^{[k]}\right)$ and the sequence $\left(\left(\mathbf{M}_{[\varphi],k}\right)^n : n \in \mathbb{N}\right)$ of iterates of $\mathbf{M}_{[\varphi],k}$ converges to $\mathbf{A}_k := \underbrace{(A_k, \ldots, A_k)}_{k-times}$ (Theorem 1). In Section 4 we show that similar facts hold true if a single-variable function φ is replaced by a suitable generator f of $k-1$ variables, which leads to a mean-type mapping $\mathbf{M}_{[f],k}$ (Theorem 3). In Section 5 we consider equality of means considered in Sections 3 and 4 (Theorems 4 and 5). In Section 6 we examine conditions under which two mean-type mappings constructed in Section 4 are equal (Theorem 6). Section 7 is devoted to the question of homogeneity of the mean $\mathbf{M}_{[f],k}$. In Section 8 we examine conditions under which the result obtained in Section 3 can be extended for the mean-type mappings of the form $\mathbf{M} = \left(M_{[\varphi_1],k}, \ldots, M_{[\varphi_k],k}\right)$. In the last section we give an application of Theorem 2 in solving some functional equations.

2. Preliminaries

In the sequel $I \subset \mathbb{R}$ denotes an interval and $k \in \mathbb{N}$, $k \geq 2$, a fixed number.

A function $M : I^k \to I$ is said to be a k-variable mean if, for all $x_1, \ldots, x_k \in I$,

$$\min\{x_1, \ldots, x_k\} \leq M(x_1, \ldots, x_k) \leq \max\{x_1, \ldots, x_k\}.$$

If for all $x_1, \ldots, x_k \in I$ such that $\min\{x_1, \ldots, x_k\} < \max\{x_1, \ldots, x_k\}$ (these inequalities are sharp), M is called a strict mean. A mean M is symmetric if

$$M(x_{\sigma(1)}, \ldots, x_{\sigma(k)}) = M(x_1, \ldots, x_k); \qquad x_1, \ldots, x_k \in I,$$

for every bijection $\sigma : \{1, \ldots, k\} \to \{1, \ldots, k\}$.

Let $I = (0, \infty)$. A mean $M : (0, \infty)^k \to (0, \infty)$ is called homogeneous, if

$$M(tx_1, \ldots, tx_k) = tM(x_1, \ldots, x_k), \quad t, x_1, \ldots, x_k > 0.$$

A function $\mathbf{M} : I^k \to I^k$, $\mathbf{M} = (M_1, \ldots, M_k)$, is called a mean-type mapping if, for each $i \in \{1, \ldots, k\}$, the function $M_i : I^k \to I$ is a mean. A mean-type mapping $\mathbf{M}$ is called strict if each of its coordinate functions M_i is a strict mean.

Let $\mathbf{M} : I^k \to I^k$, $\mathbf{M} = (M_1, \ldots, M_k)$, be a mean-type mapping. A mean $K : I^k \to I$ is called invariant with respect to a mean-type mapping $\mathbf{M}$, shortly, $\mathbf{M}$-invariant if $K \circ \mathbf{M} = \mathbf{K}$; i.e., if, for all $x_1, \ldots, x_k \in I$,

$$K(M_1(x_1, \ldots, x_k), \ldots, M_k(x_1, \ldots, x_k)) = K(x_1, \ldots, x_k).$$

It turns out that if $\mathbf{M} : I^k \to I^k$, $\mathbf{M} = (M_1, \ldots, M_k)$ *is a continuous and strict mean-type mapping, then there exists a unique continuous and* $\mathbf{M}$*-invariant mean* $K : I^k \to I$*. Moreover* K *is strict and the sequence*

$(\mathbf{M}^n : n \in \mathbb{N})$ *of iterates of the mean-type mapping* $\mathbf{M}$ *converges on* I^k *to the mean-type mapping* $\mathbf{K} = \underbrace{(K, \ldots, K)}_{k \text{ times}}$ (cf. [4,7,8] where a more general result is presented).

3. Mean-Type Mappings Generated by Functions of a Single Variable

In the sequel, for $k \in \mathbb{N}$, $k \geq 2$, the symbol A_k stands for the arithmetic mean

$$A_k(x_1, \ldots, x_k) := \frac{x_1 + \cdots + x_k}{k}, \quad x_1, \ldots, x_k \in \mathbb{R}.$$

The following extends the result of two-variable B–G–L means in [5].

Theorem 1. *Let* $k \in \mathbb{N}$, $k \geq 2$, *and* $\varphi : I \to (0, \infty)$ *be an arbitrary function. Then (i) For every* $i = 1, \ldots, k$ *the function* $M^{[i]}_{[\varphi],k} : I^k \to I$ *defined by*

$$M^{[i]}_{[\varphi],k}(x_1, \ldots, x_k) := \frac{\sum_{j=1}^{k-i+1} x_{i-1+j}\varphi(x_j) + \sum_{j=k-i+2}^{k} x_{j-k+i-1}\varphi(x_j)}{\varphi(x_1) + \cdots + \varphi(x_k)}$$

is a k-variable mean in I (here we adopt the convention, that any sum over the empty set of indices is zero); (ii) The arithmetic mean A_k *is invariant with respect to the mean-type mapping* $\mathbf{M}_{[\varphi],k} : I^k \to I^k$ *given by*

$$\mathbf{M}_{[\varphi],k} := \left(M^{[1]}_{[\varphi],k}, M^{[2]}_{[\varphi],k}, \ldots, M^{[k]}_{[\varphi],k} \right), \tag{1}$$

i.e., $A_k \circ \mathbf{M}_{[\varphi],k} = A_k$; *(iii) If* φ *is continuous then the sequence* $\left(\left(\mathbf{M}_{[\varphi],k} \right)^n : n \in \mathbb{N} \right)$ *of iterates of the mean-type map* $\mathbf{M}_{[\varphi],k}$ *converges uniformly on compact set to the mean-type mapping* $\mathbf{A}_k := \underbrace{(A_k, \ldots, A_k)}_{k-times}$.

Proof. (i) From the definition of $M^{[i]}_{[\varphi],k}$ we have

$$M^{[i]}_{[\varphi],k}(x_1, \ldots, x_k) = \sum_{j=1}^{k-i+1} \frac{\varphi(x_j)}{\varphi(x_1) + \cdots + \varphi(x_k)} x_{i-1+j} + \sum_{j=k-i+2}^{k} \frac{\varphi(x_j)}{\varphi(x_1) + \cdots + \varphi(x_k)} x_{j-k+i-1};$$

i.e., $M^{[i]}_{[\varphi],k}(x_1, \ldots, x_k)$ is a convex combination of the numbers $x_1, \ldots, x_k$ with positive coefficients summing up to 1; so $M^{[i]}_{[\varphi],k}$ is a mean. Result (ii) follows immediately from the obvious equality

$$\sum_{i=1}^{k} M^{[i]}_{[\varphi],k}(x_1, \ldots, x_k) = x_1 + \cdots + x_k.$$

Result (iii) is a consequence of (ii) and the main result of [4] (see also [8]). □

Remark 1. *Note that, under the conditions of the above result, we have*

$$M^{[1]}_{[\varphi],k}(x_1,\ldots,x_k) = \frac{x_1\varphi(x_1)+x_2\varphi(x_2)+\cdots+x_{k-1}\varphi(x_{k-1})+x_k\varphi(x_k)}{\varphi(x_1)+\cdots+\varphi(x_k)},$$
$$M^{[2]}_{[\varphi],k}(x_1,\ldots,x_k) = \frac{x_2\varphi(x_1)+x_3\varphi(x_2)+\cdots+x_k\varphi(x_{k-1})+x_1\varphi(x_k)}{\varphi(x_1)+\cdots+\varphi(x_k)},$$
$$\cdots$$
$$M^{[k]}_{[\varphi],k}(x_1,\ldots,x_k) = \frac{x_k\varphi(x_1)+x_1\varphi(x_2)+\cdots+x_{k-2}\varphi(x_{k-1})+x_{k-1}\varphi(x_k)}{\varphi(x_1)+\cdots+\varphi(x_k)},$$

so replacing cyclically x_1 by x_2, x_2 by x_3,…, x_{k-1} by x_k and x_k by x_1, from $M^{[1]}_{[\varphi],k}$ we get $M^{[2]}_{[\varphi],k}$. Similarly, from $M^{[2]}_{[\varphi],k}$ we get $M^{[3]}_{[\varphi],k}$, etc.

The mean $M^{[1]}_{[\varphi],k}$ is a natural extension of the two-variable B–G–L mean $M_{[\varphi],2}(x,y) = \frac{x\varphi(x)+y\varphi(y)}{\varphi(x)+\varphi(y)}$. The means $M_{[\varphi],2}$ and $M^{*}_{[\psi],2}$, where

$$M^{*}_{[\psi],2}(x,y) := \frac{x\psi(y)+y\psi(x)}{\psi(y)+\psi(x)}, \qquad x,y \in I,$$

are complementary with respect to the arithmetic mean $A_2(x,y) = \frac{x+y}{2}$; i.e.,

$$A_2 \circ (M_{[\varphi],2}, M^{*}_{[\psi],2}) = A_2,$$

if and only if $\varphi = \psi$ ([5]). Moreover the $M^{*}_{[\psi],2}$ is also a B–G–L mean. Note also that

$$M_{[\varphi],2}(x,y) = \frac{\varphi(x)}{\varphi(x)+\varphi(y)}x + \frac{\varphi(y)}{\varphi(x)+\varphi(y)}y,$$
$$M^{*}_{[\varphi],2}(x,y) = \frac{\varphi(x)}{\varphi(x)+\varphi(y)}y + \frac{\varphi(y)}{\varphi(x)+\varphi(y)}x,$$

so $M_{[\varphi],2}(x,y)$ and $M^{*}_{[\varphi],2}(x,y)$ are the weighted arithmetic means of x and y with cyclically replaced weights $\frac{\varphi(x)}{\varphi(x)+\varphi(y)}$ and $\frac{\varphi(y)}{\varphi(x)+\varphi(y)}$.

For this reason, the mean-type mapping $\mathbf{M}_{[\varphi],k}$ defined by (1) can be called a *B–G–L mean-type mapping of a generator φ*.

4. Mean-Type Mappings Generated by Functions of Several Variables

In this section we generalize Theorem 1 as follows.

Theorem 2. *Let $I \subset \mathbb{R}$ be an interval and $k \in \mathbb{N}$, $k \geq 2$, be fixed. Let $f : I^{k-1} \to (0,\infty)$ be an arbitrary function. Then (i) The functions $M^{[1]}_{[f],k},\ldots,M^{[k]}_{[f],k} : I^k \to I$ defined by*

$$M_{[f],k}^{[1]}(x_1,\ldots,x_k)=\frac{x_1f(x_2,\ldots,x_k)+x_2f(x_3,\ldots,x_k,x_1)+\cdots+x_kf(x_1,\ldots,x_{k-1})}{f(x_2,\ldots,x_k)+f(x_3,\ldots,x_k,x_1)+\cdots+f(x_1,\ldots,x_{k-1})}$$
$$M_{[f],k}^{[2]}(x_1,\ldots,x_k)=\frac{x_2f(x_2,\ldots,x_k)+x_3f(x_3,\ldots,x_k,x_1)+\cdots+x_1f(x_1,\ldots,x_{k-1})}{f(x_2,\ldots,x_k)+f(x_3,\ldots,x_k,x_1)+\cdots+f(x_1,\ldots,x_{k-1})}$$
$$\cdots\cdots\cdots\cdots$$
$$M_{[f],k}^{[k]}(x_1,\ldots,x_k)=\frac{x_kf(x_2,\ldots,x_k)+x_1f(x_3,\ldots,x_k,x_1)+\cdots+x_{k-1}f(x_1,\ldots,x_{k-1})}{f(x_2,\ldots,x_k)+f(x_3,\ldots,x_k,x_1)+\cdots+f(x_1,\ldots,x_{k-1})}$$

are means in I. (ii) The arithmetic mean A_k is invariant with respect to the mean-type mapping $\mathbf{M}_{[f],k}:I^k\to I^k$ defined by

$$\mathbf{M}_{[f],k}:=\left(M_{[f],k}^{[1]},\ldots,M_{[f],k}^{[k]}\right);$$

(iii) If f is continuous then the sequence $\left(\left(\mathbf{M}_{[f],k}\right)^n:n\in\mathbb{N}\right)$ of iterates of the mean-type map $\mathbf{M}_{[f],k}$ converges uniformly on compact sets to the mean-type mapping $\mathbf{A}_k:=\underbrace{(A_k,\ldots,A_k)}_{k-times}$.

Proof. (i) From the definition of $M_{[f],k}^{[1]}(x_1,\ldots,x_k)$ it is a convex combination of the numbers $x_1,\ldots,x_k$ with respective positive weights

$$\frac{f(x_2,\ldots,x_k)}{f(x_2,\ldots,x_k)+f(x_3,\ldots,x_k,x_1)+\cdots+f(x_1,\ldots,x_{k-1})},$$
$$\ldots,\frac{f(x_1,\ldots,x_{k-1})}{f(x_2,\ldots,x_k)+f(x_3,\ldots,x_k,x_1)+\cdots+f(x_1,\ldots,x_{k-1})}$$

summing up to 1; so $M_{[f],k}^{[i]}$ is a mean. We omit similar arguments for the remaining functions. Result (ii) follows from an easy to see equality

$$\sum_{i=1}^{k}M_{[f],k}^{[i]}(x_1,\ldots,x_k)=x_1+\cdots+x_k.$$

Result (iii) is a consequence of (ii) and the main result of [4] (see also [8]). □

As suggested in the previous section, the mean-type mapping $\mathbf{M}_{[f],k}$ is referred to as a *B–G–L mean-type mapping of the generator f*.

Remark 2. *The above result remains true on replacing $\mathbf{M}_{[f],k}=\left(M_{[f],k}^{[1]},\ldots,M_{[f],k}^{[k]}\right)$ by each of the following $k-1$ mean-type mappings:*

$$\left(M_{[f],k}^{[2]},\ldots,M_{[f],k}^{[k]},M_{[f],k}^{[1]}\right),\quad\left(M_{[f],k}^{[3]},\ldots,M_{[f],k}^{[k]},M_{[f],k}^{[1]},M_{[f],k}^{[2]}\right),$$
$$\ldots,\quad\left(M_{[f],k}^{[k]},M_{[f],k}^{[1]},M_{[f],k}^{[2]},\ldots,M_{[f],k}^{[k-1]}\right),$$

which arises from $\mathbf{M}_{[f],k}$ by the cyclic translation of its coordinate means.

5. Equality $\mathbf{M}_{[f],k} = \mathbf{M}_{[\varphi],k}$

In this section we examine when the means defined in the previous two sections coincide (and therefore we omit writing upper indexes at $M^{[i]}_{[f],k}$ and $M^{[i]}_{[\varphi],k}$). We begin with the following

Remark 3. *Let $\varphi, \psi : I \to (0, \infty)$. Then*

$$M_{[\varphi],2}(x, y) = M^{*}_{[\psi],2}(x, y), \qquad x, y \in I,$$

if, and only if, for a positive constant c,

$$\psi(x) = \frac{c}{\varphi(x)}, \qquad x \in I.$$

This remark is an easy consequence of the definitions of the involved means. Its 3-dimensional counterpart reads as follows:

Theorem 3. *Let $f : I^2 \to (0, \infty)$ be a symmetric function and $\varphi : I \to (0, \infty)$. Then*

$$M_{[\varphi],3}(x, y, z) = M_{[f],3}(x, y, z), \qquad x, y, z \in I, \tag{2}$$

if, and only if, there is $c > 0$ such that

$$f(x, y) = \frac{c}{\varphi(x)\varphi(y)}, \qquad x, y \in I.$$

Proof. Assume that equality (2) is satisfied. Setting $z := y$ in (2), making use of the definitions of the means $M_{[\varphi],3}$ and $M_{[f],3}$, and the symmetry of f, we obtain that

$$\frac{x\varphi(x) + 2y\varphi(y)}{\varphi(x) + 2\varphi(y)} = \frac{xf(y, y) + 2yf(x, y)}{f(y, y) + 2f(x, y)}, \qquad x, y \in I.$$

Defining $D : I \to (0, \infty)$ by

$$D(x) := f(x, x), \qquad x \in I,$$

we can write this equation in the form

$$\frac{x\varphi(x) + 2y\varphi(y)}{\varphi(x) + 2\varphi(y)} = \frac{xD(y) + 2yf(x, y)}{D(y) + 2f(x, y)}, \qquad x, y \in I.$$

Hence, we get

$$f(x, y) = \frac{D(y)\varphi(y)}{\varphi(x)}, \qquad x, y \in I. \tag{3}$$

The symmetry of f implies that

$$\frac{D(y)\varphi(y)}{\varphi(x)} = \frac{D(x)\varphi(x)}{\varphi(y)}, \qquad x, y \in I,$$

which is equivalent to

$$D(x)[\varphi(x)]^2 = D(y)[\varphi(y)]^2, \qquad x, y \in I.$$

Consequently, there is a constant $c > 0$ such that

$$D(x) = \frac{c}{[\varphi(x)]^2}, \qquad x \in I.$$

Hence, applying (3)

$$f(x,y) = \frac{c}{\varphi(x)\varphi(y)}, \qquad x,y \in I.$$

Since the reversed implication is easy to verify, the proof is completed. □

In a similar way, one can prove more general

Theorem 4. *Let $I \subset \mathbb{R}$ be an interval, $k \in \mathbb{N}$, $k \geq 3$ fixed. Suppose that a function $f : I^{k-1} \to (0,\infty)$ is symmetric and $\varphi : I \to (0,\infty)$. Then*

$$M_{[\varphi],k}(x_1,\ldots,x_k) = M_{[f],k}(x_1,\ldots,x_k), \qquad x_1,\ldots,x_k \in I, \tag{4}$$

if, and only if, there is $c > 0$ such that

$$f(x_1,\ldots,x_{k-1}) = \frac{c}{\varphi(x_1)\cdot\ldots\cdot\varphi(x_{k-1})}, \qquad x_1,\ldots,x_{k-1} \in I.$$

Proof. (Proof an idea). Suppose that (4) is satisfied. Putting in (4)

$$x_{k-2} := x_{k-1},\ x_{k-3} := x_{k-1},\ \ldots,\ x_2 := x_{k-1},$$

by the symmetry of f, we get

$$f(x_1,x_{k-1},\ldots,x_{k-1})\varphi(x_1) = f(x_{k-1},x_{k-1},\ldots,x_{k-1})\varphi(x_{k-1}), \quad x_1,x_{k-1} \in I,$$

and, consequently,

$$f(x_1,x_{k-1},\ldots,x_{k-1}) = \frac{f(x_{k-1},x_{k-1},\ldots,x_{k-1})\varphi(x_{k-1})}{\varphi(x_1)}, \quad x_1,x_{k-1} \in I.$$

Returning to the one before last substitution $x_{k-3} = \cdots = x_3 = x_{k-1}$, by the symmetry of f and the above, we obtain

$$f(x_1,x_2,x_{k-1},\ldots,x_{k-1}) = \frac{f(x_{k-1},x_{k-1},\ldots,x_{k-1})\left[\varphi(x_{k-1})\right]^2}{\varphi(x_1)\varphi(x_2)}, \quad x_1,x_2,x_{k-1} \in I,$$

and finally, repeating this procedure $k-1$-times, we come back to (4) getting

$$f(x_1,\ldots x_{k-2},x_{k-1}) = \frac{f(x_{k-1},x_{k-1},\ldots,x_{k-1})\left[\varphi(x_{k-1})\right]^{k-2}}{\varphi(x_1)\cdot\cdots\cdot\varphi(x_{k-2})}, \tag{5}$$

for all $x_1,\ldots x_{k-2},x_{k-1} \in I$. Since f is symmetric, we have

$$\frac{f(x_{k-1},\ldots,x_{k-1})\left[\varphi(x_{k-1})\right]^{k-2}}{\varphi(x_1)\cdot\ldots\cdot\varphi(x_{k-2})} = \frac{f(x_{\sigma(k-1)},\ldots,x_{\sigma(k-1)})\left[\varphi\left(x_{\sigma(k-1)}\right)\right]^{k-2}}{\varphi(x_{\sigma(1)})\cdot\ldots\cdot\varphi(x_{\sigma(k-2)})},$$

whence

$$f(x_{k-1}, x_{k-1}, \ldots, x_{k-1}) \left[\varphi\left(x_{k-1}\right)\right]^{k-1} = f(x_{\sigma(k-1)}, x_{\sigma(k-1)}, \ldots, x_{\sigma(k-1)}) \left[\varphi\left(x_{\sigma(k-1)}\right)\right]^{k-1}$$

for every bijection $\sigma : \{1, \ldots, k-1\} \to \{1, \ldots, k-1\}$ and $x_1, ...x_{k-2}, x_{k-1} \in I$. Thus, there is a constant $c > 0$ such that

$$f(x_{k-1}, x_{k-1}, \ldots, x_{k-1}) = \frac{c}{\left[\varphi\left(x_{k-1}\right)\right]^{k-1}}, \qquad x_{k-1} \in I,$$

which, by (5), gives

$$f(x_1, \ldots, x_{k-1}) = \frac{c}{\varphi(x_1) \cdot ... \cdot \varphi(x_{k-1})}, \qquad x_1, \ldots, x_{k-1} \in I.$$

The reversed implication is obvious. □

Remark 4. *It is easy to show that the family of B–G–L means of the form*

$$M_{[\varphi],3}(x, y, z) = \frac{x\varphi(x) + y\varphi(y) + z\varphi(z)}{\varphi(x) + \varphi(y) + \varphi(z)}$$

with the single variable generators $\varphi : (0, \infty) \to (0, \infty)$ *does not contain the geometric mean* $G(x, y, z) = (xyz)^{1/3}$. *Taking however* $f : (0, \infty)^2 \to (0, \infty)$, *given by*

$$f(x, y) = (xy)^{1/3}(x^{1/3} + y^{1/3}), \qquad x, y > 0,$$

we obtain

$$M_f(x, y, z) = (xyz)^{1/3} = G(x, y, z), \qquad x, y, z > 0.$$

Theorems 2 and 3 and this remark show, in particular, that the family of means generated by the several variables functions f is essentially richer than $M_{[\varphi]}$.

6. Equality $\mathbf{M}_{[f],k} = \mathbf{M}_{[g],k}$

Theorem 5. *Suppose that* $f, g : I^{k-1} \to (0, \infty)$ *are continuous functions. Then* $\mathbf{M}_{[f],k} = \mathbf{M}_{[g],k}$ *if, and only if,* $g = cf$ *for some constant* $c > 0$.

For the simplicity of notations, we prove this result assuming that $k = 3$.

In view of Theorem 2, if $f : I^2 \to (0, \infty)$ then $\mathbf{M}_{[f],3} = \left(M^{[1]}_{[f],3}, M^{[2]}_{[f],3}, M^{[3]}_{[f],3}\right)$ where

$$M^{[1]}_{[f],3}(x, y, z) = \frac{xf(y, z) + yf(z, x) + zf(x, y)}{f(y, z) + f(z, x) + f(x, y)}, \tag{6}$$

$$M^{[2]}_{[f],3}(x, y, z) = \frac{yf(y, z) + zf(z, x) + xf(x, y)}{f(y, z) + f(z, x) + f(x, y)}, \tag{6a}$$

$$M^{[3]}_{[f],3}(x, y, z) = \frac{zf(y, z) + xf(z, x) + yf(x, y)}{f(y, z) + f(z, x) + f(x, y)}, \tag{6b}$$

for $x, y, z \in I,$ and we have the following

Remark 5. *If $f : I^2 \to (0, \infty)$, then $\mathbf{M}_{[f],3} = \left(M_{[f],3}^{[1]}, M_{[f],3}^{[2]}, M_{[f],3}^{[3]} \right)$ and*

$$M_{[f],3}^{[1]}(x,y,z) = \lambda_1 x + \lambda_2 y + \lambda_3 z, \quad M_{[f],3}^{[2]}(x,y,z) := \lambda_1 y + \lambda_2 z + \lambda_3 x,$$
$$M_{[f],3}^{[3]}(x,y,z) := \lambda_1 z + \lambda_2 x + \lambda_3 y$$

where

$$\lambda_1 = \frac{f(y,z)}{f(y,z) + f(z,x) + f(x,y)}, \quad \lambda_2 = \frac{f\,(z,x)}{f(y,z) + f(z,x) + f(x,y)},$$
$$\lambda_3 = \frac{f(x,y)}{f(y,z) + f(z,x) + f(x,y)},$$

are positive, and

$$\lambda_1 + \lambda_2 + \lambda_3 = 1.$$

The arithmetic mean $A_3\,(x,y,z) := \frac{x+y+z}{3}$ is invariant with respect to the mean-type mapping $\mathbf{M}_{[f],3}$. Moreover, if f is continuous then the sequence $\left(\left(\mathbf{M}_{[f],3} \right)^n : n \in \mathbb{N} \right)$ of iterates of the mean-type map $\mathbf{M}_{[f],3}$ converges uniformly on compact sets to the mean-type mapping $\mathbf{A}_3 := (A_3, A_3, A_3)$.

Proof of Theorem 6 for $k = 3$. Assume that $\mathbf{M}_{[f],3} = \mathbf{M}_{[g],3}$. Then $M_{[f],3}^{[1]} = M_{[g],3}^{[1]}$, $M_{[f],3}^{[2]} = M_{[g],3}^{[2]}$, $M_{[f],3}^{[3]} = M_{[g],3}^{[3]}$; that is, in view of Remark 5, we have

$$\lambda_1 x + \lambda_2 y + \lambda_3 z = \mu_1 x + \mu_2 y + \mu_3 z$$
$$\lambda_1 y + \lambda_2 z + \lambda_3 x = \mu_1 y + \mu_2 z + \mu_3 x$$
$$\lambda_1 z + \lambda_2 x + \lambda_3 y = \mu_1 z + \mu_2 x + \mu_3 y$$

where

$$\mu_1 = \frac{g(y,z)}{g(y,z) + g(z,x) + g(x,y)}, \quad \mu_2 = \frac{g\,(z,x)}{g(y,z) + g(z,x) + g(x,y)},$$
$$\mu_3 = \frac{g(x,y)}{g(y,z) + g(z,x) + g(x,y)}.$$

Hence, taking into account that $\lambda_1 + \lambda_2 + \lambda_3 = 1 = \mu_1 + \mu_2 + \mu_3$, we see that $(\lambda_1 - \mu_1)$, $(\lambda_2 - \mu_2)$, $(\mu_1 + \mu_2) - (\lambda_1 + \lambda_1)$ satisfy the system of linear equations

$$x\,(\lambda_1 - \mu_1) + y\,(\lambda_2 - \mu_2) + z\,[(\mu_1 + \mu_2) - (\lambda_1 + \lambda_1)] = 0$$
$$y\,(\lambda_1 - \mu_1) + z\,(\lambda_2 - \mu_2) + x\,[(\mu_1 + \mu_2) - (\lambda_1 + \lambda_1)] = 0$$
$$z\,(\lambda_1 - \mu_1) + x\,(\lambda_2 - \mu_2) + y\,[(\mu_1 + \mu_2) - (\lambda_1 + \lambda_1)] = 0.$$

Since

$$\det \begin{bmatrix} x & y & z \\ y & z & x \\ z & x & y \end{bmatrix} = 3xyz - x^3 - y^3 - z^3,$$

and the interior of the set where this determinant disappears is empty, the continuity of f and g implies that $\lambda_1 - \mu_1 = 0$ and $\lambda_2 - \mu_2 = 0$. Hence

$$\frac{\mu_1}{\lambda_1} = \frac{\mu_2}{\lambda_2},$$

and, by the above formulas,

$$\frac{g(y,z)}{f(y,z)} = \frac{g(z,x)}{f(z,x)}, \qquad x,y,z \in I,$$

which shows that the second variable's function $\frac{g}{f}$ does not depend on the first variable, as the left-hand side does not depend on y, and it does not depend on the second variable, as the left-hand side does not depend on x. Thus $\frac{g}{f} = c$ for a real constant $c > 0$. The converse implication is obvious. □

Let us observe that under the assumption that the functions f and g are symmetric the above theorem remains true if we replace $\mathbf{M}_{[f],k} = \mathbf{M}_{[g],k}$ by $M^{[1]}_{[f],3} = M^{[1]}_{[g],3}$. Namely, we have the following.

Remark 6. *Let $I \subset \mathbb{R}$ be an interval and let $f, g : I^2 \to (0,\infty)$ be symmetric functions. Then*

$$\frac{xf(y,z) + yf(x,z) + zf(x,y)}{f(y,z) + f(x,z) + f(x,y)} = \frac{xg(y,z) + yg(x,z) + zg(x,y)}{g(y,z) + g(x,z) + g(x,y)}, \quad x,y,z \in (0,\infty), \tag{7}$$

if, and only if, $g = cf$ for a positive constant c.

Proof. Assume that equality (7) is satisfied. Then, after simple modifications, we get

$$[f(x,z)g(y,z) - f(y,z)g(x,z)](x-y) + [f(x,y)g(y,z) - f(y,z)g(x,y)](x-z)$$
$$+ [f(x,y)g(x,z) - f(x,z)g(x,y)](y-z) = 0, \qquad x,y,z \in (0,\infty). \tag{8}$$

Putting $z := x$ in (8) we obtain

$$f(x,x)[g(y,x) + g(x,y)] = g(x,x)[f(y,x) + f(x,y)], \qquad x,y \in (0,\infty);$$

therefore, by symmetry of f and g, we get

$$\frac{g(x,y)}{f(x,y)} = \frac{g(x,x)}{f(x,x)}, \qquad x,y \in (0,\infty). \tag{9}$$

Since the left-hand side is symmetric with respect to x and y, and the right-hand side does not depend on y, it follows that

$$\frac{g(x,x)}{f(x,x)} = \frac{g(y,y)}{f(y,y)}, \qquad x,y \in (0,\infty),$$

and consequently, there is a real constant $c > 0$ such that

$$\frac{g(x,x)}{f(x,x)} = c, \qquad y \in (0,\infty),$$

which, by virtue of (9), gives the required claim. Since the reversed implication is obvious, the proof is completed. □

7. Homogeneity of the Mean $M_{[f],k}$

In this section we do not consider the mean-type mappings—therefore, similarly to in Section 5, we omit writing upper indexes.

Theorem 6. *Let $f : (0, \infty)^2 \to (0, \infty)$ be a symmetric function. The following conditions are equivalent: (i) The mean $M_{[f],3}$ is positively homogeneous; i.e.,*

$$M_{[f],3}(tx, ty, tz) = tM_{[f],3}(x, y, z), \qquad x, y, z, t > 0;$$

(ii) There exists a multiplicative function $m : (0, \infty) \to (0, \infty)$ such that

$$f(tx, ty) = m(t)f(x, y), \qquad x, y, t > 0.$$

Proof. We first prove that condition (i) implies (ii). Writing the homogeneity condition of $M_{[f],3}$ with the aid of the formula (6) for $I = (0, \infty)$, and making simple modifications, we obtain

$$\begin{aligned} &[f(tz, tx)f(y, z) - f(ty, tz)f(z, x)](y - x) \\ &+ [f(tx, ty)f(y, z) - f(ty, tz)f(x, y)](z - x) \end{aligned}$$

$$+[f(tx, ty)f(z, x) - f(tz, tx)f(x, y)](z - y) = 0.$$

Setting here $z := y$ gives, for all $x, y, t > 0$,

$$\begin{aligned} &[f(ty, tx)f(y, y) - f(ty, ty)f(y, x)](y - x) \\ &+ [f(tx, ty)f(y, y) - f(ty, ty)f(x, y)](y - x) = 0. \end{aligned}$$

Hence, by the symmetry of f, we obtain

$$f(tx, ty)f(y, y) = f(ty, ty)f(x, y), \qquad x, y, t > 0, x \neq y.$$

Since x and y play a symmetric role, we get

$$f(tx, ty)f(x, x) = f(tx, tx)f(x, y), \qquad x, y, t > 0, x \neq y. \tag{10}$$

Dividing these two relations by sides gives

$$\frac{f(y, y)}{f(x, x)} = \frac{f(ty, ty)}{f(tx, tx)}, \qquad x, y, t > 0,$$

and, consequently,

$$\frac{f(tx, tx)}{f(x, x)} = \frac{f(ty, ty)}{f(y, y)}, \qquad x, y, t > 0.$$

This relation implies that for every $t > 0$ there is an $m(t) > 0$ such that

$$f(tx, tx) = m(t)f(x, x), \qquad x, t > 0.$$

An easy argument shows that $m : (0, \infty) \to (0, \infty)$ is multiplicative. Hence, making use of (10), we obtain

$$f(tx, ty) = m(t)f(x, y), \qquad x, y, t > 0.$$

Since the converse implication is obvious, the proof is complete. □

Theorem 7. *Let $\varphi : (0, \infty) \to (0, \infty)$ and $k \geq 2$. Then (i) The mean $M_{[\varphi],k} : (0, \infty)^k \to (0, \infty)$ is positively homogeneous; i.e.,*

$$M_{[\varphi],k}(tx_1, \ldots, tx_k) = tM_{[\varphi],k}(x_1, \ldots, x_k), \qquad x_1, \ldots, x_k, t > 0,$$

if, and only if, there exists a multiplicative function $m : (0, \infty) \to (0, \infty)$ such that

$$\varphi(tx) = m(t)\varphi(x), \qquad x, t > 0;$$

(ii) If φ is measurable or the graph φ is not a dense set in $(0, \infty)^2$, then there is a $p > 0$ such that

$$\varphi(x) = \varphi(1)x^p, \qquad x > 0;$$

moreover, $M_\varphi = M_{[p],k}$ where

$$M_{[p],k}(x_1, \ldots, x_k) := \frac{x_1^{p+1} + \cdots + x_k^{p+1}}{x_1^p + \cdots + x_k^p}, \qquad x_1, \ldots, x_k, t > 0.$$

The means $M_{[p],k}$ have the following interesting property.

Remark 7. *Let $k, m \in \mathbb{N}$, $2 \leq m < k$.*

1. *If $p \geq 0$ then*

$$\lim_{x_{m+1} \to 0, \ldots, x_k \to 0} M_{[p],k}(x_1, \ldots, x_k) = M_{[p],k}(x_1, \ldots, x_m), \qquad x_1, \ldots, x_k > 0.$$

2. *if $p < 0$ then*

$$\lim_{x_{m+1} \to 0, \ldots, x_k \to 0} M_{[p],k}(x_1, \ldots, x_k) = 0, \qquad x_1, \ldots, x_k > 0.$$

The first property in this remark can be generalized as follows.

Proposition 1. *Let $I \subset \mathbb{R}$ be an open interval and let $a \in [-\infty, +\infty]$ be one of its endpoints. If $\varphi : I \to (0, \infty)$ is such that*

$$\lim_{x \in I,\, x \to a} \varphi(x) = 0,$$

then for $k, m \in \mathbb{N}$, $2 \leq m < k$,

$$\lim_{x_{m+1} \to a, \ldots, x_k \to a} M_{[\varphi],k}(x_1, \ldots, x_k) = M_{[\varphi],k}(x_1, \ldots, x_m), \qquad x_1, \ldots, x_k \in I.$$

8. Invariance of the Arithmetic Mean with Respect to the Mean-Type Mapping with Coordinate B–G–L Means of Different Single Variable Generators

In this section we consider the problem of invariance of the arithmetic mean A_k with respect to the B–G–L mean-type mappings $(M_{[\varphi_1]}, \ldots, M_{[\varphi_k]})$.

In the case $k = 2$ we have the following (cf. [5]).

Theorem 8. *Let $I \subset \mathbb{R}$ be an interval and $\varphi, \psi : I \to (0, \infty)$. Then A_2 is $(M_{[\varphi]}, M_{[\psi]})-$ invariant if, and only if, there is a constant $c > 0$ such that*

$$\varphi(x)\psi(x) = c, \qquad x \in I.$$

The next result shows that in the case $k \geq 3$ the situation is completely different.

Theorem 9. *Let $I \subset \mathbb{R}$ be an interval and suppose that $\varphi_j : I \to (0, \infty),\ j = 1, \ldots, k$, are twice differentiable. If $k \geq 3$ then the following conditions are equivalent:*

1. *A_k is invariant with respect to the mean-type mapping $(M_{[\varphi_1],k}, \ldots, M_{[\varphi_k],k})$;*
2. *There are $c_j \in \mathbb{R}$, $c_j > 0$ $(j = 1, \ldots, k)$, such that $\varphi_j = c_j$ $(j = 1, \ldots, k)$;*
3. *$M_{[\varphi_1]} = \cdots = M_{[\varphi_k]} = A_k$.*

Proof. Suppose that A_k is $(M_{[\varphi_1]}, \ldots, M_{[\varphi_k]})$-invariant; i.e., that

$$\sum_{i=1}^{k} \frac{\sum_{j=1}^{k} x_j \varphi_i(x_j)}{\sum_{j=1}^{k} \varphi_i(x_j)} = \sum_{i=1}^{k} x_i, \quad x_1, \ldots, x_k \in I.$$

Put $x := x_1$ and $y := x_2$. Differentiating both sides of this equation first with respect to x gives

$$\sum_{i=1}^{k} \frac{[\varphi_i(x) + x\varphi_i'(x)] \sum_{j=1}^{k} \varphi_i(x_j) - \varphi_i'(x) \sum_{j=1}^{k} x_j \varphi_i(x_j)}{\left(\sum_{j=1}^{k} \varphi_i(x_j)\right)^2} = 1,$$

and with respect to y gives

$$\sum_{i=1}^{k} \frac{[\varphi_i(y) + y\varphi_i'(y)] \sum_{j=1}^{k} \varphi_i(x_j) - \varphi_i'(y) \sum_{j=1}^{k} x_j \varphi_i(x_j)}{\left(\sum_{j=1}^{k} \varphi_i(x_j)\right)^2} = 1.$$

Subtracting the respective sides of these equalities and then dividing the obtained difference by $x - y$ we get

$$\sum_{i=1}^{k} \frac{\frac{\varphi_i(x) + x\varphi_i'(x) - \varphi_i(y) - y\varphi_i'(y)}{x-y} \sum_{j=1}^{k} \varphi_i(x_j) - \frac{\varphi_i'(x) - \varphi_i'(y)}{x-y} \sum_{j=1}^{k} x_j \varphi_i(x_j)}{\left(\sum_{j=1}^{k} \varphi_i(x_j)\right)^2} = 0.$$

Letting here $y \to x$ gives

$$\sum_{i=1}^{k} \frac{[2\varphi_i'(x) + x\varphi_i''(x)] \sum_{j=1}^{k} \varphi_i(x_j) - \varphi_i''(x) \sum_{j=1}^{k} x_j \varphi_i(x_j)}{\left(\sum_{j=1}^{k} \varphi_i(x_j)\right)^2} = 0,$$

for all $x_1 = x_2 = x \in I$ and all $x_3, \ldots, x_k \in I$. Setting here

$$x_1 = \cdots = x_k = x,$$

we get

$$\frac{2}{k} \sum_{i=1}^{k} \frac{\varphi_i'(x)}{\varphi_i(x)} = 0, \qquad x \in I,$$

which easily implies that, for some $c \in \mathbb{R}$,

$$\prod_{i=1}^{k} \varphi_i(x) = c, \qquad x \in I.$$

For the simplicity of notation we confine the remaining part of the proof to the case $k = 3$. Let $\varphi, \psi, \gamma : I \to (0, \infty)$ be twice-differentiable. According to what we have shown, the invariance of A_3 with respect to the mapping $(M_{[\varphi]}, M_{[\psi]}, M_{[\gamma]})$, i.e., the relation

$$M_{[\varphi],3}(x,y,z) + M_{[\psi],3}(x,y,z) + M_{[\gamma],3}(x,y,z) = x + y + z, \quad x, y, z \in I, \tag{11}$$

implies that there is a constant $c > 0$ such that

$$\varphi(x)\psi(x)\gamma(x) = c, \qquad x \in I,$$

and consequently,

$$\gamma(x) = \frac{c}{\varphi(x)\psi(x)}, \qquad x \in I. \tag{12}$$

Replacing $\gamma(x), \gamma(y), \gamma(z)$ in (11) by the suitable values given by this formula, and then by substituting $z = y$, we obtain, for all $x, y \in I$,

$$\frac{x\varphi(x) + 2y\varphi(y)}{\varphi(x) + 2\varphi(y)} + \frac{x\psi(x) + 2y\psi(y)}{\psi(x) + 2\psi(y)} + \frac{x\varphi(y)\psi(y) + 2y\varphi(x)\psi(x)}{\varphi(y)\psi(y) + 2\varphi(x)\psi(x)} = x + 2y,$$

which reduces to

$$\varphi(x)^2\psi(x)^2 + \varphi(x)\varphi(y)\psi(y)^2 + \varphi(y)^2\psi(x)\psi(y) = 3\varphi(x)\varphi(y)\psi(x)\psi(y), \tag{13}$$

for all $x, y \in I$. Similarly, from (11) and (12), substituting $z = x$, we obtain

$$\frac{2x\varphi(x) + y\varphi(y)}{2\varphi(x) + \varphi(y)} + \frac{2x\psi(x) + y\psi(y)}{2\psi(x) + \psi(y)} + \frac{2x\varphi(y)\psi(y) + y\varphi(x)\psi(x)}{2\varphi(y)\psi(y) + \varphi(x)\psi(x)} = 2x + y,$$

which reduces to

$$\varphi(x)^2\psi(x)\psi(y) + \varphi(x)\varphi(y)\psi(x)^2 + \varphi(y)^2\psi(y)^2 = 3\varphi(x)\varphi(y)\psi(x)\psi(y), \tag{14}$$

for all $x, y \in I$. Subtracting (13) and (14) by sides gives

$$[\varphi(x) - \varphi(y)][\psi(y) - \psi(x)][\varphi(x)\psi(x) - \varphi(y)\psi(y)] = 0, \quad x, y \in I.$$

It follows that, for all $x, y \in I$,

$$\text{either } \varphi(x) = \varphi(y) \text{ or } \psi(y) = \psi(x) \text{ or } \varphi(x)\psi(x) = \varphi(y)\psi(y).$$

Suppose, for instance, that $\varphi(x) = \varphi(y)$. Then from (13) we get

$$\varphi(x)^2[\psi(x) - \psi(y)]^2 = 0,$$

and, consequently, $\psi(x) = \psi(y)$ as well as $\varphi(x)\psi(x) = \varphi(y)\psi(y)$. Repeating this reasoning in two remaining cases we infer that for each $x, y \in I$, we have $\varphi(y) = \varphi(x)$ and $\psi(y) = \psi(x)$. Thus the functions φ, ψ and γ are constant. This completes the proof. □

Remark 8. *Note that having proved formula (13) one can finish the proof with a shorter argument. Let us fix an arbitrary $x_0 \in I$. Without any loss of generality we may assume that*

$$\varphi(x_0) = \psi(x_0) = 1.$$

Hence, setting $y = x_0$ in (13), we obtain

$$\psi(x) = \frac{\varphi(x)}{3\varphi(x) - \varphi(x)^2 - 1}, \qquad x \in I.$$

Using this formula to eliminate the function ψ in (13) and then setting $x = x_0$ gives

$$(\varphi(y) - 1)[2\varphi(y)^2 - 2\varphi(y) + 1] = 0, \qquad y \in I,$$

and, consequently, $\varphi(y) = 1$ for all $y \in I$. In the same way one can show that ψ is constant.

An advantage of argument given in the proof is that, with mainly notational difficulties, it can be generalized to the arbitrary $k > 3$.

9. An Application

Making use of Theorem 2 we prove the following.

Theorem 10. *Let $f : I^{k-1} \to (0, \infty)$ be an arbitrary continuous function where $k \in \mathbb{N}$, $k \geq 2$. Assume that a function $\Phi : I^k \to \mathbb{R}$ is continuous on the diagonal*

$$\Delta\left(I^k\right) := \left\{x = (x_1, \dots, x_k) \in I^k : x_1 = \dots = x_k\right\}.$$

Then the function Φ is invariant with respect to the mean-type mapping $\mathbf{M}_{[f],k}$; that is, Φ satisfies the functional equation

$$\Phi = \Phi \circ \left(M_{[f],k}^{[1]}, \dots, M_{[f],k}^{[k]}\right);$$

if and only if there is a continuous single variable function $\varphi : I \to \mathbb{R}$ such that

$$\Phi = \varphi \circ A_k,$$

where A_k is the k-variable arithmetic mean.

Proof. Assume first that $\Phi : I^k \to \mathbb{R}$ is invariant with respect to $\mathbf{M}_{[f],k}$, that is $\Phi = \Phi \circ \mathbf{M}_{[f],k}$. Hence, by induction, we get

$$\Phi = \Phi \circ \left(\mathbf{M}_{[f],k}\right)^n, \quad n \in \mathbb{N},$$

where $\left(\mathbf{M}_{[f],k}\right)^n$ is the n-th iterate of $\mathbf{M}_{[f],k}$. By virtue of Theorem 2, the sequence of mean-type mappings $\left(\left(\mathbf{M}_{[f],k}\right)^n : n \in \mathbb{N}\right)$ converges to the mean-type mapping $\mathbf{A}_k := \underbrace{(A_k, \dots, A_k)}_{k-times}$. Since Φ is continuous on $\Delta\left(I^k\right)$, we hence get

$$\Phi = \lim_{n\to\infty} \Phi \circ \left(\left(\mathbf{M}_{[f],k}\right)^n\right) = \Phi \circ \left(\lim_{n\to\infty} \left(\mathbf{M}_{[f],k}\right)^n\right) = \Phi \circ \mathbf{A}_k = \Phi \circ (A_k, \dots, A_k).$$

Setting

$$\varphi(t) := \Phi \circ (t, \ldots, t), \quad t \in I,$$

we conclude that $\Phi = \varphi \circ A_k$. To prove the converse implication, take an arbitrary continuous function $\varphi : I \to \mathbb{R}$ and put $\Phi := \varphi \circ A_k$. Then Φ is continuous on the diagonal and, for all $x = (x_1, \ldots, x_k) \in I^k$, making use of the invariance of A_k with respect to the mean-type mapping $\mathbf{M}_{[f],k}$, we have

$$\begin{aligned}\Phi \circ \mathbf{M}_{[f],k}(x) &= (\varphi \circ A_k)\left(\mathbf{M}_{[f],k}(x)\right) = \varphi\left(A_k\left(\mathbf{M}_{[f],k}(x)\right)\right) = \varphi(A_k(x)) \\ &= \varphi\left(\frac{x_1 + \cdots + x_k}{k}\right) = \Phi(x),\end{aligned}$$

which completes the proof. □

Remark 9. *The assumption of the continuity of Φ can be omitted if Φ is a mean (see [8]).*

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