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Contact Metric Spaces and pseudo-Hermitian Symmetry

Jong Taek Cho

Department of Mathematics, Chonnam National University, Gwangju 61186, Korea; jtcho@chonnam.ac.kr

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Abstract: We prove that a contact strongly pseudo-convex CR (Cauchy–Riemann) manifold M^{2n+1} , $n \geq 2$, is locally pseudo-Hermitian symmetric and satisfies $\nabla_{\xi}h = \mu h\phi$, $\mu \in \mathbb{R}$, if and only if M is either a Sasakian locally ϕ -symmetric space or a non-Sasakian (k, μ) -space. When $n = 1$, we prove a classification theorem of contact strongly pseudo-convex CR manifolds with pseudo-Hermitian symmetry.

Keywords: contact almost CR (Cauchy–Riemann) manifold; generalized Tanaka–Webster connection; pseudo-Hermitian symmetric space

MSC: 53C15; 53C25; 53D10

1. Introduction

For a contact manifold $(M^{2n+1}; \eta)$, we have the two fundamental structures associated with the contact form η : the Riemannian metric g and the Levi form related with an endomorphism J on $D(=\text{Ker of } \eta)$ such that $J^2 = -I$. Between the two associated structures we have a one-to-one correspondence by the equation

$$g = L + \eta \otimes \eta,$$

where L denotes the (0,2)-tensor field on M which naturally extends the Levi form. When the integrability of the almost complex structure J on D is assumed, we call it an (*integrable*) CR structure. A Sasakian manifold is a contact Riemannian manifold $(M; \eta, g)$ satisfying a normality condition (see, Section 2.1). Then, we may describe the Sasakian structure as a contact strongly pseudo-convex CR (Cauchy–Riemann) structure whose characteristic vector field is a Killing vector field for its associated Riemannian metric.

In studying a contact manifold from the Riemannian view point, the question of contact metric manifolds $(M; \eta, g)$ satisfying (Cartan’s) local symmetry, that is, whose Riemannian curvature tensor R satisfies

$$\nabla R = 0,$$

where ∇ denotes the Levi-Civita connection of g , has a long history (cf. Section 7.5 in Reference [1]). (For Sasakian manifolds, the problem was settled already by M. Okumura [2].) At last, E. Boeckx and the present author [3] have proved that a locally symmetric contact metric space is locally isometric to either the unit sphere or the unit tangent sphere bundle of the Euclidean space with its standard contact metric structure.

This result says that the local symmetry is too strong a condition to impose on contact manifolds. In this context, T. Takahashi [4] introduced the notion of *Sasakian locally ϕ -symmetric spaces* as the analogue of locally Hermitian symmetric spaces. A Sasakian manifold is said to be locally ϕ -symmetric if the Riemannian curvature tensor R satisfies

$$(*) \quad g((\nabla_X R)(Y, Z)V, U) = 0$$

for all vector fields X, Y, Z, V and U orthogonal to ζ . Making a generalization of this definition, in Reference [5], the authors call a contact metric manifold locally ϕ -symmetric if it satisfies the same curvature condition (*) as in the Sasakian case.

Taking a look at contact manifolds from the point of view of its pseudo-Hermitian structure, then we have a canonical affine connection, distinct from the Levi-Civita connection of an associated metric g . Indeed, the *generalized Tanaka-Webster connection* $\hat{\nabla}$ on a strongly pseudo-convex almost CR manifold is invariant under pseudo-homothetic transformations (which preserve ϕ). In previous works [6–11], the generalized Tanaka-Webster connection has been playing an important part when we studied the interplay between the contact Riemannian structure and the contact strongly pseudo-convex almost CR structure. In Reference [10] (for the CR-integrable case in Reference [6]), we defined *locally pseudo-Hermitian symmetric spaces* to be strongly pseudo-convex almost CR manifolds whose pseudo-Hermitian curvature tensor \hat{R} satisfies

$$L((\hat{\nabla}_X \hat{R})(Y, Z)V, U) = 0$$

for all vector fields X, Y, Z, V and U orthogonal to ζ . Then, in Section 3 of the present paper, we prove that a contact strongly pseudo-convex CR manifold M^{2n+1} , $n \geq 2$, is locally pseudo-Hermitian symmetric and satisfies $\nabla_{\zeta} h = \mu h \phi$, $\mu \in \mathbb{R}$, if and only if M is either a Sasakian locally ϕ -symmetric space or a non-Sasakian (k, μ) -space (Theorem 2). In Section 4, we treat the three-dimensional case. In particular, we prove a classification theorem of three-dimensional contact strongly pseudo-convex CR manifolds with pseudo-Hermitian symmetry (Theorem 4). Moreover, we give non-homogeneous examples of locally pseudo-Hermitian symmetric contact strongly pseudo-convex CR manifolds.

2. Preliminaries

2.1. Contact Riemannian Structures

First of all, we recall some basic notions and formulas in contact Riemannian geometry. All manifolds treated in this paper are assumed to be connected and of class C^∞ .

Definition 1 ([1]). *A contact manifold $(M^{2n+1}; \eta)$ is an odd-dimensional manifold equipped with a global 1-form η such that $\eta \wedge (d\eta)^n \neq 0$ everywhere.*

For a contact form η , we have a unique vector field ζ , which is called the *characteristic vector field* or the *Reeb vector field*, satisfying $\eta(\zeta) = 1$ and $i_{\zeta} d\eta = 0$. Here, i_{ζ} denotes the interior product operator by ζ . Then we have also a Riemannian metric g and a $(1, 1)$ -tensor field ϕ such that

$$\eta(X) = g(X, \zeta), \quad \phi^2 X = -X + \eta(X)\zeta, \quad d\eta(X, Y) = g(X, \phi Y), \tag{1}$$

for all $X, Y \in \mathfrak{X}(M)$, where $\mathfrak{X}(M)$ denotes the Lie algebra of all smooth vector fields on M . From (1), we have easily that

$$\eta \circ \phi = 0, \quad \phi \zeta = 0, \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y). \tag{2}$$

A manifold M equipped with structure tensors (η, ζ, ϕ, g) satisfying (1) is said to be a *contact Riemannian manifold* or *contact metric manifold* and is denoted by $M = (M; \eta, g)$. We define a $(1, 1)$ -tensor field h on M by $2h = \mathcal{L}_{\zeta} \phi$, where \mathcal{L}_{ζ} denotes Lie differentiation with respect to ζ . The operator h is self-adjoint and satisfies

$$h\zeta = 0, \quad h\phi = -\phi h, \tag{3}$$

$$\nabla_X \zeta = -\phi X - \phi h X, \tag{4}$$

where ∇ is the Levi-Civita connection. From (3) and (4) we see that any flow line of ξ is a geodesic. Furthermore, we know that $\nabla_{\xi}\phi = 0$ in general (cf. p. 67 in Reference [1]). From the second equation of (3) it follows also that

$$(\nabla_{\xi}h)\phi = -\phi(\nabla_{\xi}h). \tag{5}$$

The Riemannian curvature tensor R is defined by

$$R(X, Y)Z = \nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) - \nabla_{[X, Y]}Z$$

for all $X, Y, Z \in \mathfrak{X}(M)$. The characteristic Jacobi operator ℓ is the symmetric $(1, 1)$ -tensor defined by $\ell(X) = R(X, \xi)\xi$. Then, from the definition of R , by using (4) we have

$$\ell = -\phi^2 + \phi\nabla_{\xi}h - h^2. \tag{6}$$

Using (6), together with (3) and (5), then we have

$$2\nabla_{\xi}h = \ell\phi - \phi\ell. \tag{7}$$

A contact metric manifold is called a *K-contact manifold* if ξ is a Killing vector field, or equivalently, h vanishes. Given $M = (M; \eta, g)$, an almost complex structure \tilde{J} on $M \times \mathbb{R}$ is naturally defined by

$$\tilde{J}(X, f \frac{d}{dt}) = (\phi X - f\xi, \eta(X) \frac{d}{dt}),$$

where $X \in \mathfrak{X}(M)$, $t \in \mathbb{R}$ and $f \in \mathcal{F}(M \times \mathbb{R})$, where $\mathcal{F}(M \times \mathbb{R})$ denotes the algebra of differentiable functions on $M \times \mathbb{R}$. If the almost complex structure \tilde{J} is integrable, M is said to be *normal* or *Sasakian*. Note that any 3-dimensional *K-contact* manifold is already Sasakian, but in higher dimension it does not hold any more. A Sasakian structure is characterized by the following equation:

$$(\nabla_X\phi)Y = g(X, Y)\xi - \eta(Y)X, \tag{8}$$

or equiivalently

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y \tag{9}$$

for all $X, Y \in \mathfrak{X}(M)$. We refer to Blair’s book [1] for more details.

2.2. Strongly Pseudo-Convex almost CR Structures

In this subsection, we begin by a brief reviewing of pseudo-Hermitian CR structure.

Definition 2 ([12]). Let M be a $(2n + 1)$ -dimensional manifold and TM be its tangent bundle. A CR structure on M is the subbundle $\mathcal{H} \subset \mathbb{C}TM = TM \otimes \mathbb{C}$ satisfying the following:

- each fiber \mathcal{H}_p , $p \in M$, is of complex dimension n
- $\mathcal{H} \cap \bar{\mathcal{H}} = \{0\}$, where $\bar{\mathcal{H}}$ denotes the complex conjugation of \mathcal{H} ,
- $[\mathcal{H}, \mathcal{H}] \subset \mathcal{H}$ (integrability).

We call $(M; \mathcal{H})$ a CR manifold (of hypersurface type).

Given a CR manifold $(M; \mathcal{H})$, we have a unique subbundle $D = \text{Re}\{\mathcal{H} \oplus \bar{\mathcal{H}}\}$, which is maximally holomorphic subbundle of TM , and have a unique bundle isomorphism $J : D \rightarrow D$ such that $J^2 = -I$ and $\mathcal{H} = \{X - iJX | X \in \Gamma(D)\}$. Such (D, J) is called the real representation of \mathcal{H} . Let $E \subset T^*M$ be the conormal bundle of D . If M is oriented, then E is a trivial bundle, and hence it admits globally defined a nowhere zero 1-form η such that $\text{Ker}(\eta) = D$. The *Levi form* is defined by

$$L : \Gamma(D) \times \Gamma(D) \rightarrow \mathcal{F}(M), \quad L(X, Y) = -d\eta(X, JY).$$

If the Levi form is non-degenerate (positive or negative definite, resp.), then (η, L) is called a *non-degenerate (strongly pseudo-convex, resp.) pseudo-Hermitian CR structure*. Then there exists a unique globally defined nowhere vanishing vector field ξ such that $\eta(\xi) = 1$ and $i_\xi d\eta = 0$. In particular, for a strongly pseudo-convex pseudo-Hermitian CR structure (η, L) , we can extend the Levi form canonically to a Riemannian metric on M defined by

$$g_\eta = L + \eta \otimes \eta,$$

where $i_\xi L = 0$. It is called the *Webster metric* (cf. Reference [13]).

Let's return to a contact Riemannian manifold $M = (M; \eta, g)$. At each point $p \in M$, the subspace $D_p (= \text{Ker of } \eta_p)$ of the tangent space $T_p M$ of M gives the decomposition $T_p M = D_p \oplus \{\xi\}_p$ (direct sum). Then $D : p \rightarrow D_p$ defines a $2n$ -dimensional distribution orthogonal to ξ , which is called the *contact distribution* or the *contact subbundle*. The associated pseudo-Hermitian structure is naturally provided by the subbundle $\mathcal{H} = \{X - iJX : X \in \Gamma(D)\} \subset \mathbb{C}TM$, where $J = \phi|_D$, the restriction of ϕ to D . In fact, we see that each fiber $\mathcal{H}_p, p \in M$, is of complex dimension $n, \mathcal{H} \cap \bar{\mathcal{H}} = \{0\}$ and $\mathbb{C}D = \mathcal{H} \oplus \bar{\mathcal{H}}$. Moreover, M satisfies

$$[JX, JY] - [X, Y] \in \Gamma(D) \text{ (or } [JX, Y] + [X, JY] \in \Gamma(D)) \tag{10}$$

for all $X, Y \in \Gamma(D)$. Such (η, J) is called an *pseudo-Hermitian almost CR structure*. It should be notable that the CR-integrability does not hold in general for a contact metric manifold. In terms of the structure tensors, CR-integrability condition is equivalent to the condition $\Omega = 0$, where Ω is the $(1, 2)$ -tensor field on M defined by

$$\Omega(X, Y) = (\nabla_X \phi)Y - g(X + hX, Y)\xi + \eta(Y)(X + hX) \tag{11}$$

for all $X, Y \in \mathfrak{X}(M)$ (see Reference [14], [Proposition 2.1]). From (8) and (11), we see that the associated pseudo-Hermitian structure of a Sasakian manifold is strongly pseudo-convex and CR-integrable. The same is true for all three-dimensional contact metric spaces [6].

Definition 3 ([15]). *A pseudo-homothetic or D_a -homothetic transformation of a contact metric manifold is a diffeomorphism f on M such that $f^*\eta = a\eta, f^*\xi = \frac{1}{a}\xi, \phi \circ f_* = f_* \circ \phi, f^*g = ag + a(a - 1)\eta \otimes \eta$, where a is a positive constant.*

A *pseudo-homothetic or D_a -homothetic deformation* $(\bar{\eta}, \bar{\xi}, \bar{\phi}, \bar{g})$ of a given contact metric structure (η, ξ, ϕ, g) is defined by

$$\bar{\eta} = a\eta, \bar{\xi} = \frac{1}{a}\xi, \bar{\phi} = \phi, \bar{g} = ag + a(a - 1)\eta \otimes \eta. \tag{12}$$

From (12), we have $\bar{h} = (1/a)h$. We remark that CR-integrability of the associated pseudo-Hermitian structure is preserved under pseudo-homothetic transformations. In fact, by direct computations, we have that $\Omega = 0$ implies $\bar{\Omega} = 0$ [6].

3. The pseudo-Hermitian Symmetric Spaces

In this section, we study on the pseudo-Hermitian curvature tensor and symmetric properties.

Definition 4 ([14]). *The generalized Tanaka-Webster connection $\hat{\nabla}$ on a contact strongly pseudo-convex almost CR manifold $M = (M; \eta, L)$ is defined by*

$$\hat{\nabla}_X Y = \nabla_X Y + \eta(X)\phi Y + (\nabla_X \eta)(Y)\xi - \eta(Y)\nabla_X \xi$$

for all $X, Y \in \mathfrak{X}(M)$.

Using (4), we may rewrite $\hat{\nabla}$ as

$$\hat{\nabla}_X Y = \nabla_X Y + A(X, Y). \tag{13}$$

Here, A is $(1, 2)$ -tensor field defined by

$$A(X, Y) = \eta(X)\phi Y + \eta(Y)(\phi X + \phi hX) - g(\phi X + \phi hX, Y)\xi. \tag{14}$$

Then $\hat{\nabla}$ has the torsion

$$\hat{T}(X, Y) = 2g(X, \phi Y)\xi + \eta(Y)\phi hX - \eta(X)\phi hY. \tag{15}$$

In particular, for the K -contact case we get $A(X, Y) = \eta(X)\phi Y + \eta(Y)\phi X - g(\phi X, Y)\xi$.

Proposition 1 ([14]). *The generalized Tanaka-Webster connection $\hat{\nabla}$ on a contact Riemannian manifold $M = (M; \eta, g)$ is the unique linear connection satisfying the following conditions:*

- (i) $\hat{\nabla}\eta = 0, \hat{\nabla}\xi = 0;$
- (ii) $\hat{\nabla}g = 0;$
- (iii-1) $\hat{T}(X, Y) = 2L(X, JY)\xi, X, Y \in \Gamma(D);$
- (iii-2) $\hat{T}(\xi, \phi Y) = -\phi\hat{T}(\xi, Y), Y \in \Gamma(D);$
- (iv) $(\hat{\nabla}_X \phi)Y = \Omega(X, Y), X, Y \in \Gamma(TM).$

In the case $\Omega = 0$, that is, CR-integrability holds, $\hat{\nabla}$ reduces to the Tanaka-Webster connection, which is defined on a non-degenerate integrable pseudo-Hermitian manifold [16,17]. So, the above definition is useful especially for the non-integrable case.

Proposition 2 ([6]). *The generalized Tanaka-Webster connection is invariant under a pseudo-homothetic transformation.*

The generalized Tanaka-Webster curvature tensor \hat{R} is defined by: $\hat{R}(X, Y)Z = \hat{\nabla}_X(\hat{\nabla}_Y Z) - \hat{\nabla}_Y(\hat{\nabla}_X Z) - \hat{\nabla}_{[X, Y]}Z$. We call sometimes \hat{R} the pseudo-Hermitian curvature tensor. Then we have

Corollary 1 ([6]). *The pseudo-Hermitian curvature tensor \hat{R} and its covariant derivative $\hat{\nabla}\hat{R}$ are pseudo-homothetically invariant.*

Proposition 3 ([6]).

$$\hat{R}(X, Y)Z = -\hat{R}(Y, X)Z, \quad L(\hat{R}(X, Y)Z, W) = -L(\hat{R}(X, Y)W, Z).$$

In order to make a development of the curvature tensor \hat{R} , from (3) and (13), we first compute

$$\begin{aligned} (\hat{\nabla}_X h)Y &= (\nabla_X h)Y + A(X, hY) - hA(X, Y) \\ &= (\nabla_X h)Y + 2\eta(X)\phi hY + g((\phi h + \phi h^2)X, Y)\xi \\ &\quad + \eta(Y)(\phi hX + \phi h^2 X) \end{aligned} \tag{16}$$

for all $X, Y \in \mathfrak{X}(M)$. We put $P(X, Y) = (\nabla_X h)Y - (\nabla_Y h)X$. Then, from the definition of \hat{R} and (13), taking into account $\hat{\nabla}\eta = 0, \hat{\nabla}\xi = 0, \hat{\nabla}g = 0, \hat{\nabla}\phi = \Omega$ and (16), together with (1)–(3) and (14), we have

$$\hat{R}(X, Y)Z = R(X, Y)Z + B(X, Y)Z, \tag{17}$$

where we have put

$$\begin{aligned}
 B(X, Y)Z &= -g\left(\Omega(X, Y) - \Omega(Y, X) + \Omega(X, hY) - \Omega(Y, hX) + \phi P(X, Y), Z\right)\xi \\
 &\quad - \eta(X)(\Omega(Y, Z) + g(Y + hY, Z)\xi) \\
 &\quad + g(\phi Y + \phi hY, Z)(\phi X + \phi hX) - g(\phi X + \phi hX, Z)(\phi Y + \phi hY) \\
 &\quad - 2g(\phi X, Y)\phi Z
 \end{aligned}$$

for any $X \in \mathfrak{X}(M)$ and any vector fields $Y, Z \perp \xi$ (cf. Reference [10]). Define the pseudo-Hermitian Ricci curvature tensor $\hat{\rho}$ by $\hat{\rho}(X, Y) = \text{tr} \{V \mapsto \hat{R}(V, X)Y\}$, $X, Y \perp \xi$. Then from (17) we have

$$\hat{\rho}(X, Y) = \rho(X, Y) + \sum_{I=1}^{2n+1} g(B(e_I, X)Y, e_I), \tag{18}$$

where $\{e_I\}$, $I = 1, 2, \dots, 2n + 1$, is an orthonormal frame and ρ denotes the Ricci curvature for the Levi-Civita connection. The pseudo-Hermitian scalar curvature \hat{r} is defined by $\hat{r} = \sum_{i=1}^{2n} \hat{\rho}(e_i, e_i)$, where $\{e_i, e_{2n+1} = \xi\}$ ($i = 1, \dots, 2n$) is an adapted orthonormal frame. We recall

Definition 5 ([10]). *Let $(M; \eta, L)$ be a pseudo-Hermitian strongly pseudo-convex almost CR manifold. Then M is said to be a locally pseudo-Hermitian symmetric space if M satisfies*

$$L((\hat{\nabla}_X \hat{R})(Y, Z)U, V) = 0$$

for all vector fields X, Y, Z, U, V orthogonal to ξ .

Then we obtain the following results.

Proposition 4 ([10]). *A locally pseudo-Hermitian symmetric space has constant pseudo-Hermitian scalar curvature \hat{r} .*

Proposition 5 ([6]). *A Sasakian manifold is locally pseudo-Hermitian symmetric if and only if it is locally ϕ -symmetric.*

Since $g((\hat{\nabla}_X \hat{R})(\xi, Y)Z, \xi) = 0$, we find that

$$(\hat{\nabla}_X \hat{\rho})(Y, Z) = \sum_{i=1}^{2n} g(\hat{\nabla}_X \hat{R})(e_i, Y)Z, e_i, \tag{19}$$

where $\{e_i, e_{2n+1} = \xi\}$ ($i = 1, \dots, 2n$) is an adapted orthonormal frame. Now, we introduce a very useful class of contact metric manifolds when we study their pseudo-Hermitian geometry. This is the so-called (k, μ) -spaces [18] defined by the condition:

$$R(X, Y)\xi = (kId + \mu h)(\eta(Y)X - \eta(X)Y), \tag{20}$$

where $k, \mu \in \mathbb{R}$. It includes Sasakian spaces for $k = 1$ and $h = 0$. Popular examples of non-Sasakian (k, μ) -spaces are provided by the unit tangent sphere bundles of spaces of constant curvature c ($k = c(2 - c)$ and $\mu = -2c$) when $c \neq 1$. (Due to Tashiro’s result [19], we know that the standard contact metric structure is Sasakian only when $c = 1$.) Very recently, the present author [13] proved a complete classification theorem of non-Sasakian (k, μ) -spaces which are realized as real hypersurfaces in the complex quadric Q^{n+1} , its dual Q^{*n+1} , or the complex Euclidean space \mathbb{C}^{n+1} . Other remarkable properties of (k, μ) -spaces [18] are as follows:

- such a class is invariant under pseudo-homothetic transformations,

- the associated pseudo-Hermitian structure is CR-integrable.

From the second property, we obtain a formula of $\nabla\phi$ by (11). Also for ∇h , we have an explicit expression:

$$\begin{aligned}
 (\nabla_X h)Y &= ((1 - k)g(X, \phi Y) - g(X, \phi h Y))\zeta \\
 &\quad - \eta(Y)((1 - k)\phi X + \phi h X) - \mu\eta(X)\phi h Y
 \end{aligned}
 \tag{21}$$

for all $X, Y \in \mathfrak{X}(M)$ [18]. From (21), we have that a (k, μ) -space is an η -parallel contact metric space, which means that it satisfies $g((\nabla_X h)Y, Z) = 0$ for all vector fields $X, Y, Z \perp \zeta$. E. Boeckx and the present author proved that the converse holds also:

Theorem 1 ([20]). *An η -parallel contact metric space is either a K-contact space or a (k, μ) -space.*

In Reference [6], we obtained the following property of non-Sasakian (k, μ) -spaces.

Proposition 6. *A non-Sasakian (k, μ) -space is a locally pseudo-Hermitian symmetric space.*

Now, we prove the following theorem.

Theorem 2. *A contact strongly pseudo-convex CR manifold M^{2n+1} , $n \geq 2$, is locally pseudo-Hermitian symmetric and satisfies $\nabla_{\zeta} h = \mu h\phi$, $\mu \in \mathbb{R}$, if and only if M is either a Sasakian locally ϕ -symmetric space or a non-Sasakian (k, μ) -space.*

Proof of Theorem 2. First of all, use (3), (4) and the assumption $\nabla_{\zeta} h = \mu h\phi$ to get

$$\begin{aligned}
 P(X, \zeta) &= (\nabla_X h)\zeta - (\nabla_{\zeta} h)X \\
 &= (1 - \mu)h\phi X - h^2\phi X.
 \end{aligned}
 \tag{22}$$

From (18) and (22), we compute

$$g(\hat{S}X, Y) = g(SX, Y) - \mu g(hX, Y) + 2g(X, Y)
 \tag{23}$$

for any vector fields $X, Y \perp \zeta$, where S and \hat{S} are the $(1, 1)$ -tensors defined by $\rho(X, Y) = g(SX, Y)$, $\hat{\rho}(X, Y) = g(\hat{S}X, Y)$, respectively. Using (23) we have

$$g((\hat{S}\phi - \phi\hat{S})X, Y) = g((S\phi - \phi S)X, Y) - 2\mu g(h\phi X, Y)
 \tag{24}$$

for any vector fields $X, Y \perp \zeta$. Here, we recall a useful formula which holds in general for a contact strongly pseudo-convex CR manifold (cf. Reference [21]):

$$S\phi - \phi S = \ell\phi - \phi\ell + 4(n - 1)h\phi - \eta \otimes \phi S\zeta + (\eta \circ S\phi) \otimes \zeta.$$

Since $\nabla_{\zeta} h = \mu h\phi$, together with (7), we have

$$g((S\phi - \phi S)X, Y) = (2\mu + 4(n - 1))g(h\phi X, Y)
 \tag{25}$$

for any vector fields $X, Y \perp \zeta$. Hence, from (24) and (25) we have

$$g((\hat{S}\phi - \phi\hat{S})X, Y) = 4(n - 1)g(h\phi X, Y)
 \tag{26}$$

for any vector fields $X, Y \perp \zeta$.

Suppose that $(M; \eta, L)$ is locally pseudo-Hermitian symmetric. Then from (19) we have

$$g((\hat{\nabla}_U \hat{S})X, Y) = 0 \tag{27}$$

for any vector fields $U, X, Y \perp \xi$. Differentiating (26) covariantly for $\hat{\nabla}$, then using $\hat{\nabla}\phi = 0$ and (27) we have

$$(n - 1) g((\hat{\nabla}_U h)\phi X, Y) = 0 \tag{28}$$

for any vector fields U, X, Y orthogonal to ξ . From (28) we have either $n = 1$ or h is η -parallel (for $\hat{\nabla}$ and ∇). Indeed, from (16) we are easily aware that η -parallel h for ∇ and for $\hat{\nabla}$ are equivalent each other. Thus, owing to Proposition 5 and Theorem 1, we have that M is either a Sasakian locally ϕ -symmetric space or a non-Sasakian (k, μ) -space when $n \geq 2$. In order to prove the converse, we first note that a Sasakian manifold ($h = 0$) and non-Sasakian (k, μ) -space satisfies $\nabla_{\xi}h = \mu h\phi$ (see (21)) automatically. Moreover, by virtue of Propositions 5 and 6, we find that the converse holds. Therefore, we have completed the proof. \square

Recently, A. Ghosh and R. Sharma [22] introduced the so-called Jacobi (k, μ) -contact manifolds, which generalize the (k, μ) -spaces. Namely, a Jacobi (k, μ) -contact manifold is a contact metric manifold whose characteristic Jacobi operator ℓ satisfies

$$\ell = -k\phi^2 + \mu h. \tag{29}$$

There are a lot of examples of Jacobi (k, μ) -contact manifolds which are not (k, μ) -spaces (cf. Reference [23]). We note that Jacobi (k, μ) -contact metric structures are also invariant under a pseudo-homothetic transformation. From (7), together with (29) we have $\nabla_{\xi}h = \mu h\phi$.

Corollary 2. *Let $(M^{2n+1}; \eta, L)$, $n \geq 2$, be a contact strongly pseudo-convex CR manifold. Then M is locally pseudo-Hermitian symmetric Jacobi (k, μ) -space if and only if M is either a Sasakian locally ϕ -symmetric space or a non-Sasakian (k, μ) -space.*

4. The Contact Strongly Pseudo-Convex 3-Manifolds

In this section, we study three-dimensional locally pseudo-Hermitian symmetric contact metric spaces. Let $(M; \eta, g)$ be a three-dimensional contact metric manifold. Let U_1 be the open set defined by $h \neq 0$ and U_2 be the maximal open subset on which h is identically zero. The union $U_1 \cup U_2$ is open and dense subset of M . Suppose that M is non-Sasakian. Then U_1 is non-empty and fix a point x in U_1 . Then we have a local orthonormal frame field $\{\xi, e, \phi e\}$ on a neighborhood of $x \in U_1$ such that $h(e) = \lambda e$, $h(\phi e) = -\lambda \phi e$ for some positive function λ . The covariant derivative is then of the following form (see Reference [24], [Lemma 2.1]):

$$\begin{aligned} \nabla_{\xi}\xi &= 0, & \nabla_{\xi}e &= -ue, & \nabla_{\xi}\phi e &= ue, \\ \nabla_e\xi &= -(\lambda + 1)\phi e, & \nabla_{\phi e}\xi &= (1 - \lambda)e, \\ \nabla_e e &= p\phi e, & \nabla_{\phi e}\phi e &= qe, \\ \nabla_e\phi e &= -pe + (\lambda + 1)\xi, \\ \nabla_{\phi e} e &= -q\phi e + (\lambda - 1)\xi, \end{aligned} \tag{30}$$

where u is a smooth function and we have put $p = \frac{1}{2\lambda}(\phi e(\lambda) + \sigma(e))$, $q = \frac{1}{2\lambda}(e(\lambda) + \sigma(\phi e))$. Here, $\sigma = \rho(\xi, \cdot)$ and ρ denotes the Ricci curvature tensor. By using (14) we also calculate the (1,2)-tensor field A :

$$\begin{aligned} A(\xi, \xi) &= 0, & A(e, \xi) &= (1 + \lambda)\phi e, & A(\phi e, \xi) &= -(1 - \lambda)e, \\ A(\xi, e) &= \phi e, & A(e, e) &= 0, & A(\phi e, e) &= (1 - \lambda)\xi, \\ A(\xi, \phi e) &= -e, & A(e, \phi e) &= -(1 + \lambda)\xi, & A(\phi e, \phi e) &= 0. \end{aligned} \tag{31}$$

Using (30) and (31), we get the following expressions for $\hat{\nabla}$ from (13):

$$\begin{aligned} \hat{\nabla}_\xi \xi &= 0, & \hat{\nabla}_\xi e &= (1 - u)\phi e, & \hat{\nabla}_\xi \phi e &= -(1 - u)e, \\ \hat{\nabla}_e \xi &= 0, & \hat{\nabla}_e e &= p\phi e, & \hat{\nabla}_e \phi e &= -pe, \\ \hat{\nabla}_{\phi e} \xi &= 0, & \hat{\nabla}_{\phi e} e &= -q\phi e, & \hat{\nabla}_{\phi e} \phi e &= qe. \end{aligned} \tag{32}$$

From these equations, we calculate the pseudo-Hermitian curvature tensor:

$$\begin{aligned} \hat{R}(e, \phi e)e &= \hat{\nabla}_e \hat{\nabla}_{\phi e} e - \hat{\nabla}_{\phi e} \hat{\nabla}_e e - \hat{\nabla}_{[e, \phi e]} e \\ &= -(e(q) + \phi e(p) - (p^2 + q^2) + 2(1 - u))\phi e, \\ \hat{R}(e, \phi e)\phi e &= (e(q) + \phi e(p) - (p^2 + q^2) + 2(1 - u))e. \end{aligned} \tag{33}$$

Then we have the holomorphic sectional curvature \hat{H} for $\hat{\nabla}$ and the pseudo-Hermitian scalar curvature \hat{r} :

$$\hat{H} = e(q) + \phi e(p) - (p^2 + q^2) + 2(1 - u), \quad \hat{r} = 2\hat{H}. \tag{34}$$

Thus we have the following theorem.

Theorem 3. *A three-dimensional contact strongly pseudo-convex CR manifold is a locally pseudo-Hermitian symmetric space if and only if it is of constant holomorphic sectional curvature \hat{H} .*

Proof of Theorem 3. Suppose that M is locally pseudo-Hermitian symmetric. Then, since $h = 0$ on U_2 , we see that it has Sasakian structure. Then by Proposition 5 it is locally ϕ -symmetric, moreover, due to a result in Reference [25] we have that it has a constant holomorphic sectional curvature $\hat{H}(= H + 3)$ on U_2 , where H denotes the holomorphic sectional curvature for ∇ . Next, from (32) and (33) we compute on U_1 :

$$\begin{aligned} (\hat{\nabla}_e \hat{R})(e, \phi e)e &= \hat{\nabla}_e(\hat{R}(e, \phi e)e) - \hat{R}(\hat{\nabla}_e e, \phi e)e \\ &\quad - \hat{R}(e, \hat{\nabla}_e \phi e)e - \hat{R}(e, \phi e)\hat{\nabla}_e e \\ &= -e(e(q) + \phi e(p) - (p^2 + q^2) + 2(1 - u))\phi e, \\ (\hat{\nabla}_{\phi e} \hat{R})(e, \phi e)e &= -\phi e(e(q) + \phi e(p) - (p^2 + q^2) + 2(1 - u))\phi e. \end{aligned}$$

Since M is locally pseudo-Hermitian symmetric, \hat{H} satisfies

$$e(\hat{H}) = \phi e(\hat{H}) = 0.$$

We also have, using the second condition of (1):

$$0 = [e, \phi e](\hat{H}) = \eta([e, \phi e])\xi(\hat{H}) = -2d\eta(e, \phi e)\xi(\hat{H}) = 2\xi(\hat{H}).$$

These computations imply that \hat{H} is constant on U_1 . Since $U_1 \cup U_2$ is dense and M is connected, \hat{H} is constant on the whole M . In addition, we can easily show that the converse also holds. After all, we conclude the proof of Theorem 3. \square

In a 3-dimensional Riemannian manifold, we have the curvature tensor R by the following form:

$$R(X, Y)Z = \rho(Y, Z)X - \rho(X, Z)Y + g(Y, Z)SX - g(X, Z)SY - \frac{r}{2}\{g(Y, Z)X - g(X, Z)Y\} \tag{35}$$

for all $X, Y, Z \in \mathfrak{X}(M)$, where r is the (Riemannian) scalar curvature of the manifold. From (17), we have $\hat{H} = H + 3 - \frac{1}{2}\text{tr } h^2$. But, using (35), we have a holomorphic sectional curvature for ∇ : $H = g(R(e, \phi e)\phi e, e) = \rho(e, e) + \rho(\phi e, \phi e) - \frac{r}{2}$. Then, we have again $\hat{H} = \frac{1}{2}(r - \rho(\xi, \xi)) + 2$, where we have used $\rho(\xi, \xi) = 2 - \text{tr } h^2$ (Corollary 7.1 in Reference [1]). Together with Theorem 3, we have

Corollary 3. *A three-dimensional contact metric space is a locally pseudo-Hermitian symmetric space if and only if it is a Sasakian space form on U_2 and $r - \rho(\xi, \xi)$ is constant on U_1 . Here, a Sasakian space form is a Sasakian space of constant holomorphic sectional curvature H .*

For a Jacobi (k, μ) -contact manifold, use (29) to obtain $\rho(\xi, \xi) = \text{tr } \ell = 2k$. A three-dimensional Sasakian manifold is locally ϕ -symmetric if and only if the scalar curvature is constant (cf. Reference [5]).

Corollary 4. *A three-dimensional Jacobi (k, μ) -contact manifold is locally pseudo-Hermitian symmetric if and only if the scalar curvature is constant.*

In a previous paper, we have proved the following fact.

Proposition 7 ([10]). *A 3-dimensional homogeneous contact metric space is locally pseudo-Hermitian symmetric space.*

Now, we prove the following theorem.

Theorem 4. *Let $(M; \eta, g)$ be a three-dimensional contact metric space with constant scalar curvature r and vertical Ricci curvature $\sigma(X)$ for $X \in \Gamma(D)$. Then M is a locally pseudo-Hermitian symmetric space if and only if it is a Sasakian space form or a Lie group with left-invariant contact metric structure.*

Proof of Theorem 4. Suppose that M is a locally pseudo-Hermitian symmetric space with constant scalar curvature r . We divide our proof into two cases: U_1 is empty or not. If U_1 is empty, then $M = U_2$ and M is a Sasakian space form by Corollary 3. Now, we treat the case that U_1 is not empty, and we deliver our argument on U_1 . Then, from Proposition 4 we have \hat{r} is a constant. But, from (18), $\hat{r} = r - \rho(\xi, \xi) + 4$. Hence, we have that $\rho(\xi, \xi) = 2 - 2\lambda^2$ is a constant, which gives that $\lambda (> 0)$ is a constant. From this, we have $M = U_1$. We compute

$$\rho(e, \xi) = 2\lambda p, \quad \rho(\phi e, \xi) = 2\lambda q. \tag{36}$$

Since a vertical Ricci curvature $\sigma(X)$ for $X \in \Gamma(D)$ is constant, from (36) we have that p and q are constants, and from (34) we find that u is also a constant. Then, using (30) we get

$$[e, \phi e] = -pe + q\phi e + 2\xi, \quad [\phi e, \xi] = (1 - \lambda - u)e, \quad [\xi, e] = (1 + \lambda - u)\phi e.$$

Here we use the Jacobi identity:

$$[[e, \phi e], \xi] + [[\phi e, \xi], e] + [[\xi, e], \phi e] = 0.$$

Then, since p and q are constants, using again (30) we have $(u + \lambda - 1)q = 0$ and $(-u + \lambda + 1)p = 0$. Since $\lambda \neq 0$, we have $pq = 0$. We divide our arguments into three cases:

(i) $p = 0, q \neq 0$: Then $u = 1 - \lambda$ and

$$[e, \phi e] = q\phi e + 2\xi, [\phi e, \xi] = 0, [\xi, e] = 2\lambda\phi e.$$

(ii) $q = 0, p \neq 0$: Then $u = 1 + \lambda$ and

$$[e, \phi e] = -pe + 2\xi, [\phi e, \xi] = -2\lambda e, [\xi, e] = 0.$$

Then, owing to Perrone’s result [26] (or Milnor’s result [27]) for the above two cases M is locally isometric to a non-unimodular Lie group with left-invariant contact metric structure.

(iii) $p = q = 0$:

$$[e, \phi e] = 2\xi, [\phi e, \xi] = (1 - \lambda - u)e, [\xi, e] = (1 + \lambda - u)\phi e.$$

Using Perrone’s result again, M is locally isometric to a unimodular Lie group with left-invariant contact metric structure. Conversely, by Propositions 5 we know that any Sasakian space form is locally pseudo-Hermitian symmetric space and by Proposition 7 we also know that a Lie group is locally pseudo-Hermitian symmetric space. Therefore, we have completed the proof. \square

We have non-homogeneous locally pseudo-Hermitian symmetric spaces.

Example 1. ([28]) Let $M = \{(x, y, z) \in \mathbb{R}^3(x, y, z) \mid x \neq 0\}$ be a 3-dimensional contact manifold with the contact form $\eta = xydx + dz$. The Reeb vector field is given by $\xi = \partial/\partial z$. For a positive constant β , take a global frame field

$$e_1 = -\frac{2}{x} \frac{\partial}{\partial y}, \quad e_2 = \frac{\partial}{\partial x} - \frac{\beta z}{x} \frac{\partial}{\partial y} - xy \frac{\partial}{\partial z}, \quad e_3 = \xi,$$

and define a Riemannian metric g such that $\{e_1, e_2, e_3\}$ is orthonormal with respect to it. Define ϕ by $\phi e_1 = e_2, \phi e_2 = -e_1$ and $\phi \xi = 0$. Then we find that (η, g) is a Jacobi (k, μ) -contact structure, where $k = 1 - \frac{\beta^2}{16}, \mu = 2 + \frac{\beta}{2}$. We compute $\hat{H} = -\frac{\beta}{2} (< 0)$, which implies that $M_\beta = (M; \eta, L)$ is a locally pseudo-Hermitian symmetric space. Moreover, we obtain that the components of the Ricci operator are $\rho_{13} = \rho_{31} = -\frac{\beta}{2x}$, that is, vertical Ricci curvatures are not constant. Note that M_β is a locally ϕ -symmetric space if and only if $\beta = 4$ (cf. Reference [29]).

Example 2. In $\mathbb{R}^3(x, y, z)$, we define a contact 1-form η by

$$\eta = dx + \frac{2y}{1+z^2} dz.$$

For any smooth function $f(y, z)$ of variables y, z , we define a global frame field $\{e_1, e_2, e_3\}$ by

$$e_1 = -2y \frac{\partial}{\partial x} + (2x - fy) \frac{\partial}{\partial y} + (1 + z^2) \frac{\partial}{\partial z}, \quad e_2 = \frac{\partial}{\partial y}, \quad \xi = \frac{\partial}{\partial x}.$$

Define a Riemannian metric g such that $\{e_1, e_2, e_3 = \xi\}$ is orthonormal with respect to it. As usual, the endomorphism field ϕ is defined by $\phi e_1 = e_2, \phi e_2 = -e_1$ and $\phi e_3 = 0$. Then we see that $M_f = (\mathbb{R}^3; \eta, g)$ is a Jacobi $(0, 0)$ -contact space, but its vertical Ricci curvature $\sigma(\phi e) = 2f$ are not constant, in general. Then we have locally pseudo-Hermitian symmetric spaces of constant holomorphic sectional curvature (for $\hat{\nabla}$) $\hat{H} = c$ for all $c \in \mathbb{R}$ by the following examples:

- $f = z$: M_f has a constant holomorphic sectional curvature $\hat{H} = 3$. It is not locally ϕ -symmetric in the sense of Reference [5].
- $f = \alpha$ (const.): M_f has a constant holomorphic sectional curvature $\hat{H} = 2 + \alpha^2$. It is curvature homogeneous. A locally ϕ -symmetric space occurs only when $\alpha = 0$.

- $f = \alpha \tan(\alpha \arctan z)$, $\alpha > 0$ (const.) and if M_f is restricted to the set $\{(x, y, z) \in \mathbb{R}^3 \mid -\frac{\pi}{2\alpha} < \arctan z < \frac{\pi}{2\alpha}\}$: it has a constant holomorphic sectional curvature $\hat{H} = 2 - \alpha^2$. It is not locally ϕ -symmetric.

Remark 1. Since the class of Jacobi (k, μ) -contact manifolds is invariant under pseudo-homothetic transformations, by applying a pseudo-homothetic transformation to Examples 1 and 2, we have more Jacobi (k, μ) -contact manifolds which are locally pseudo-Hermitian symmetric.

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