Ulam Type Stability of $A$-Quadratic Mappings in Fuzzy Modular $\ast$-Algebras

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Received: 28 August 2020; Accepted: 17 September 2020; Published: 21 September 2020

Abstract: In this paper, we find the solution of the following quadratic functional equation
\[
\sum_{1 \leq i < j \leq n} Q(x_i - x_j) = \sum_{i=1}^n Q\left(\sum_{j \neq i} x_j - (n-1)x_i\right),
\]
which is derived from the gravity of the $n$ distinct vectors $x_1, \cdots, x_n$ in an inner product space, and prove that the stability results of the $A$-quadratic mappings in $\mu$-complete convex fuzzy modular $\ast$-algebras without using lower semicontinuity and $\beta$-homogeneous property.

Keywords: fuzzy modular $\ast$-algebras; modular $\ast$-algebras; $A$-quadratic derivation; $\Delta_2$-condition; $\beta$-homogeneous property

1. Introduction

A concept of stability in the case of homomorphisms between groups was formulated by S.M. Ulam [1] in 1940 in a talk at the University of Wisconsin. Let $(G_1, \ast)$ be a group and let $(G_2, \circ, d)$ be a metric group with the metric $d(\cdot, \cdot)$. Given $\epsilon > 0$, does there exist a $\delta(\epsilon) > 0$ such that if a mapping $h : G_1 \to G_2$ satisfies the inequality
\[
d(h(x \ast y), h(x) \circ h(y)) < \delta
\]
for all $x, y \in G_1$, then there is a homomorphism $H : G_1 \to G_2$ with
\[
d(h(x), H(x)) < \epsilon
\]
for all $x \in G_1$?

The first affirmative answer to the question of Ulam was given by Hyers [2,3] for the Cauchy functional equation in Banach spaces as follows: Let $X$ and $Y$ be Banach spaces. Assume that $f : X \to Y$ satisfies
\[
\|f(x + y) - f(x) - f(y)\| \leq \epsilon
\]
for all $x, y \in X$ and for some $\epsilon \geq 0$. Then, there exists a unique additive mapping $T : X \to Y$ such that
\[
\|f(x) - T(x)\| \leq \epsilon
\]
for all $x \in X$. A number of mathematicians were attracted to this result and stimulated to investigate the stability problems of various (functional, differential, difference, integral) equations in some spaces [4–11].

defined between probabilistic modular spaces in the probabilistic sense. After then, Shen and Chen [15]
following the idea of probabilistic modular spaces and the definition of fuzzy metric spaces based on
George and Veeramani’s sense [16], applied fuzzy concept to the classical notions of modular spaces.
Using Khamsi’s fixed point theorem in modular spaces [17], Wongkum and Kumam [18] proved
the stability of sextic functional equations in fuzzy modular spaces equipped necessarily with lower
semicontinuity and $\beta$-homogeneous property.

In a recent paper [11], Ulam stability of the following additive functional equation
\[
\sum_{1 \leq i < j \leq n} f \left( \frac{\sum_{j=1}^{m} x_{ij}}{m} + \frac{\sum_{l=1}^{n-m} x_{il}}{m} \right) = \frac{n-m+1}{n} \left( \frac{n}{m} \right) \sum_{i=1}^{n} f(x_i).
\]
was investigated in modular algebras without using the lower semicontinuity and Fatou property.

In the present paper, concerning the stability problem for the following functional equation
\[
n \sum_{1 \leq i < j \leq n} f(x_i - x_j) = \sum_{i=1}^{n} f \left( \sum_{j \neq i} x_j - (n-1)x_i \right)
\]
which is derived from the gravity of the $n$-distinct vectors in an inner product space, we investigate
the stability problem for $A$-quadratic mappings in $\mu$-complete convex fuzzy modular $*$-algebras of the
following functional equation without using lower semicontinuity and $\beta$-homogeneous property.

2. Preliminaries

Proposition 1. Let $X_1, X_2, \ldots, X_n$ $(n \geq 3)$ be distinct vectors in a finite $n$-dimensional Euclidean space $E$.
Putting $G := \frac{\sum_{i=1}^{n} X_i}{n}$, the gravity of the $n$ distinct vectors, then we get the following identity
\[
\sum_{1 \leq i < j \leq n} \|X_i - X_j\|^2 = n \sum_{i=1}^{n} \|X_i - G\|^2,
\]
which is equivalent to the equation
\[
n \sum_{1 \leq i < j \leq n} \|X_i - X_j\|^2 = \sum_{i=1}^{n} \left\| \sum_{j \neq i} X_j - (n-1)X_i \right\|^2
\]
for any distinct vectors $X_1, X_2, \ldots, X_n$.

Employing the above equality (1), we introduce the new functional equation:
\[
n \sum_{1 \leq i < j \leq n} Q(x_i - x_j) = \sum_{i=1}^{n} Q \left( \sum_{j \neq i} x_j - (n-1)x_i \right)
\]
for a mapping $Q : U \to V$ and for all vectors $x_1, \ldots, x_n \in U$, where $U$ and $V$ are linear spaces and $n \geq 3$ is a positive integer.

From now on, we introduce some basic definitions of fuzzy modular $*$-algebras.

Definition 1. [18] A triangular norm (briefly, t-norm) is a function $\odot : [0, 1] \times [0, 1] \to [0, 1]$ satisfies the
following conditions:

1. $\odot$ is commutative, associative;
2. $a \odot 1 = a$;
3. $a \odot b \leq c \odot d$, whenever $a, b, c, d \in [0, 1]$ with $a \leq b, c \leq d$. 

Three common examples of the t-norm are (1) \( a \circ_M b = \min\{a, b\} \); (2) \( a \circ_P b = a \cdot b \); (3) \( a \circ_L b = \max\{a + b - 1, 0\} \). For more example, we refer to [19]. Throughout this paper, we denote that

\[
\prod_{i=1}^{n} x_i := x_1 \circ \cdots \circ x_n
\]

for all \( x_1, \ldots, x_n \in [0, 1] \).

**Definition 2.** [18] Let \( X \) be a complex vector space and \( \circ \) a t-norm, and \( \mu : X \times (0, \infty) \rightarrow [0, 1] \) be a function.

(a) The triple \((X, \mu, \circ)\) is said to be a fuzzy modular space if, for each \( x, y \in X \) and \( s, t > 0 \) and \( \alpha, \beta \in [0, \infty) \) with \( \alpha + \beta = 1 \),

(FM1) \( \mu(x, t) > 0 \);

(FM2) \( \mu(x, t) = 1 \) for all \( t > 0 \) if and only if \( x = \theta \);

(FM3) \( \mu(x, t) = \mu(-x, t) \);

(FM4) \( \mu(ax + \beta y, s + t) \geq \mu(x, s) \circ \mu(y, t) \);

(FM5) the mapping \( t \rightarrow \mu(x, t) \) is continuous at each fixed \( x \in X \);

(b) alternatively, if (FM-4) is replaced by

(FM4-1) \( \mu(ax + \beta y, s + t) \geq \mu(x, s) \circ \mu(y, t) \), (where \( \alpha, \beta \neq 0 \));

then we say that \((X, \mu, \circ)\) is a convex fuzzy modular.

Now, we extend the properties (FM4) and (FM4-1) in real fields to complex scalar field acting on the space \( X \), as follows:

(FM4) \( \mu(ax + \beta y, s + t) \geq \mu(x, s) \circ \mu(y, t) \); for \( \alpha, \beta \in \mathbb{C} \) with \( |\alpha| + |\beta| = 1 \),

(FM4-1) \( \mu(ax + \beta y, s + t) \geq \mu(x, s) \circ \mu(y, t) \) for \( \alpha, \beta \in \mathbb{C} \) with \( |\alpha| + |\beta| = 1 \).

Next, we introduce the concept of fuzzy modular algebras based on the definition of fuzzy normed algebras [20,21]. If \( X \) is algebra with fuzzy modular \( \mu \) subject to \( \mu(xy, st) \geq \mu(x, s) \circ \mu(y, t) \) for all \( x, y \in X \) and \( s, t \in (0, \infty) \), then we say \((X, \mu, \circ)\) is called a fuzzy modular algebra. In addition, a fuzzy modular algebra \( X \) is a fuzzy modular 
\( \ast \)-algebra if the fuzzy modular \( \mu \) satisfies \( \mu(z^\ast, t) = \mu(z, t) \) for all \( z \in X, t > 0 \).

**Example 1.** Let \((X, \rho)\) be a modular \( \ast \)-algebra [22] and \( \circ \) defined by \( a \circ b := a \circ_M b \). For every \( t \in (0, \infty) \), define \( \mu(x, t) = \frac{t}{1 + \rho(x)} \) for all \( x \in X \). Then, \((X, \mu, \circ)\) is a (convex) fuzzy modular \( \ast \)-algebra.

**Definition 3.** (1) We say that \((X, \mu, \circ)\) is \( \beta \)-homogeneous if, for every \( x \in X, t > 0 \) and \( \lambda \in \mathbb{R}\setminus\{0\} \),

\[
\mu(\lambda x, t) = \mu\left(x, \frac{t}{|\lambda|^\beta}\right), \quad \text{where } \beta \in (0, 1].
\]

(2) Let \( n \in \mathbb{N} \). We say that \((X, \mu, \circ)\) satisfies \( \Delta_n \)-condition if there exist \( \kappa_n \geq n \) such that

\[
\mu(nx, t) \geq \mu\left(x, \frac{t}{\kappa_n}\right), \quad \forall x \in X.
\]

**Remark 1.** Let \((X, \mu, \circ)\) be \( \beta \)-homogeneous for some fixed \( \beta \in (0, 1] \). Then, we observe that

\[
\mu(2x, t) = \mu\left(x, \frac{t}{2^\beta}\right) \geq \mu\left(x, \frac{t}{\kappa_2}\right)
\]

for all \( x \in X \) and all \( \kappa_2 \geq 2 \geq |2|^{\beta} \). Thus, \( \beta \)-homogeneous property implies \( \Delta_2 \)-condition.
Example 2. Let $\rho : \mathbb{R} \to \mathbb{R}$, $\mu : \mathbb{R} \times (0, \infty) \to (0, 1]$ be defined by $\rho(x) = x^2$ and $\mu(x, t) = \frac{t}{1 + pt^2}$. Then, we can check that $(\mu, \circ_M)$ is a convex fuzzy modular on $\mathbb{R}$ but $(\mathbb{R}, \mu, \circ_M)$ does not satisfy $\beta$-homogeneous property. Let $\kappa_2 \geq 4$. Then,

$$\mu(2x, t) = \frac{t}{1 + \rho(2x)} = \mu\left(x, \frac{t}{4}\right) \geq \mu\left(x, \frac{t}{\kappa_2}\right)$$

for all $x \in \mathbb{R}$. Thus, $(\mathbb{R}, \mu, \circ)$ satisfies $\Delta_2$-condition with $\kappa_2 \geq 4$ but is not $\beta$-homogeneous.

Definition 4. Let $(X, \mu, \circ)$ be a fuzzy modular space and $\{x_n\}$ be a sequence in $X$.

1. $\{x_n\}$ is said to be $\mu$-convergent to a point $x \in X$ if for any $t > 0$,

$$\mu(x - x_n, t) \to 1$$

as $n \to \infty$.

2. $\{x_n\}$ is called $\mu$-Cauchy if for each $\varepsilon > 0$ and each $t > 0$, there exists $n_1$ such that, for all $n \geq n_1$ and all $p > 0$, we have $\mu(x_{n+p} - x_n) < 1 - \varepsilon$.

3. If each Cauchy sequence is convergent, then the fuzzy modular space is said to be complete.

3. Fuzzy Modular Stability for $A$-Quadratic Mappings

First of all, we find out the general solution of (1.3) in the class of mappings between vector spaces.

Theorem 1. Let $U$ and $V$ be vector spaces. A mapping $Q : U \to V$ satisfies the functional Equation (2) for each positive integer $n \geq 2$ if and only if there exists a symmetric biadditive mapping $B : U \times U \to V$ such that $Q(x) = B(x, x)$ for all $x \in U$.

Proof. Let $Q$ satisfy Equation (2). One finds that $Q(0) = 0$ and $Q(ax) = a^2Q(x)$ by changing $(x, y)$ to $(0, 0)$ and $(x, 0)$ in (3), respectively, where $a := n - 1$ is a positive integer with $a \geq 2$. Putting $x_1 := x$, $x_2 := y$ and $x_i := 0$ for all $i = 3, \cdots, n$ in (2), we get

$$Q(x - ay) + Q(ax - y) + (a - 1)Q(x + y) = (a + 1)Q(x - y) + (a^2 - 1)\left[Q(x) + Q(y)\right]$$

for all $x, y \in U$. Using [23] [Theorem 1], we obtain that $Q$ is a generalized polynomial map of degree at most 4. Therefore,

$$Q(x) = A_0 + A_1(x) + A_2(x, x) + A_3(x, x, x) + A_4(x, x, x, x)$$

for all $x \in U$, where $A_k : U^k \to V$ is a $k$-additive symmetric map ($k = 1, \cdots, 4$) and $A_0 \in V$. Since $a$ is an integer, we get

$$(a^2 - 1)A_0 + (a^2 - a)A_1(x) + (a^2 - a^3)A_2(x, x) + (a^2 - a^4)A_4(x, x, x, x) = 0$$

for all $x \in U$ by $Q(ax) = a^2Q(x)$. This yields that $Q(x) = A_2(x, x)$ for all $x \in U$. □

Let $A$ be a complex $*$-algebra with unit and let $M$ be a left $A$-module. We call a mapping $Q : M \to A$ an $A$-quadratic mapping if both relations $Q(ax) = aQ(x)a^*$ and $Q(x + y) + Q(x - y) = 2Q(x) + 2Q(y)$ are fulfilled for all $a \in A, x, y \in M$ [24]. For the sake of convenience, we define the following:
Theorem 2. Let Equation (2) and the inequality for all $x \in M$ and $t > 0$, where

$$\Phi(x, t) \coloneqq \max_{(i,j) \in \mathcal{J}} \left\{ \mu' \left( \epsilon_i(-x), \frac{(n-1)^2t}{6} \right) \circ \mu' \left( \epsilon_i(x), \frac{(n-1)^2nt}{6(n^2 - (i+1)n + 1)} \right) \circ \mu' \left( \epsilon_{i+1}(x), \frac{(n-1)^2nt}{6(n^2 - (i+1)n + 1)} \right) \right\}. $$

Proof. Define a mapping $g : M \to A$ by $g(x) := f(x) + \frac{(n-1)f(0)}{2}$ for all $x \in M$. Then, for each $x \in M$, the following equation is obtained:

$$g((n-1)x) - (n-1)^2g(x) = D_{u}f_{i}(-x) + \left( \frac{n^2 - (i+1)n + 1}{n} \right)[D_{1}f_{i}(x) - D_{1}f_{i+1}(x)]$$

for all $i = 1, \cdots, n - 1$ and for all $j = 1, \cdots, n$, where

$$D_{u}f_{i}(x) = D_{1}f(0, \cdots, 0, x, 0, \cdots, 0).$$
For each fixed \((i, j) \in J\), one obtains from \(\sum_{k=1}^{m} \frac{1+2\left(\frac{t}{(n-1)^{2k}}\right)}{(n-1)^{2k}} \leq 1\) that

\[
\mu\left(g(x) - \frac{g((n-1)^{m}x)}{(n-1)^{2m}}, t\right) \geq \mu\left(\sum_{k=1}^{m} \frac{(n-1)^{2k}}{(n-1)^{2k}} \left(\epsilon_{i}(x) - \frac{g((n-1)^{k}x)}{(n-1)^{2k}}\right)\right)
\geq \prod_{k=1}^{m} \left(\frac{\mu'(\epsilon_{i}(x), \frac{\epsilon_{i}(x)}{\frac{\epsilon_{i}(x)}{x}})}{3 \cdot 2k^{k-1}}\right) \left(\frac{\mu'(\epsilon_{i+1}(x), \frac{\epsilon_{i+1}(x)}{\frac{\epsilon_{i+1}(x)}{x}})}{3 \cdot 2k^{k-1}}\right)
\geq \mu\left(\left(\epsilon_{i}(x) - \frac{g((n-1)^{m}x)}{(n-1)^{2m}}\right) \cdot \frac{\epsilon_{i}(x)}{6(n^{2} - (i+1)n + 1)}\right)
\geq \mu\left(\left(\epsilon_{i+1}(x) - \frac{g((n-1)^{m}x)}{(n-1)^{2m}}\right) \cdot \frac{\epsilon_{i+1}(x)}{6(n^{2} - (i+1)n + 1)}\right)
\]

for all \(t > 0\) and \(x \in M, m \in \mathbb{N}\). Then, it follows from the above inequality that

\[
\mu\left(g(x) - \frac{g((n-1)^{m}x)}{(n-1)^{2m}}, t\right) \geq \Phi(x, t)
\]

for all \(x \in M\) and \(t > 0\). Therefore, we prove from this relation that, for any integers \(m, p,\)

\[
\mu\left(\frac{g((n-1)^{m}x)}{(n-1)^{2m}} - \frac{g((n-1)^{m+p}x)}{(n-1)^{2(m+p)}}, t\right) \geq \mu\left(\frac{g((n-1)^{m}x)}{(n-1)^{2m}} - \frac{g((n-1)^{m}x)}{(n-1)^{2m}}\right)
\geq \Phi((n-1)^{m}x, (n-1)^{2m}), t) \geq \Phi\left(x, \frac{(n-1)^{2m}}{\beta}\right)
\]

for all \(t > 0, x \in M\). Since the right-hand side of the above inequality tends to 1 as \(m \to \infty\), the sequence \(\{\frac{g((n-1)^{m}x)}{(n-1)^{2m}}\}\) is \(\mu\)-Cauchy and thus converges in \(A\). Hence, we may define a mapping \(Q : M \to A\) as

\[
Q(x) := \mu - \lim_{m \to \infty} \frac{g((n-1)^{m}x)}{(n-1)^{2m}} \left(\lim_{m \to \infty} \mu\left(Q(x) - \frac{g((n-1)^{m}x)}{(n-1)^{2m}}, t\right) = 1\right)
\]

for all \(x \in M\) and \(t > 0\). In addition, we claim that the mapping \(Q\) satisfies (2). For this purpose, we calculate the following inequality:

\[
\mu\left(\frac{D_{n}Q(x_{1}, \ldots, x_{n})}{L}, t\right) \geq \prod_{1 \leq i < j \leq n} \left(\mu\left(Q(ux_{i} - ux_{j}) - \frac{g((n-1)^{m}(ux_{i} - ux_{j}))}{(n-1)^{2m}}\right) \cdot \frac{Lt}{2^{k/n}}\right)
\geq \prod_{1 \leq i < j \leq n} \left(\mu\left(Q(\sum_{j=1}^{n} x_{j} - nx_{j})u^{*} - \frac{ug((n-1)^{m}(\sum_{j=1}^{n} x_{j} - nx_{j}))u^{*}}{(n-1)^{2m}}\right) \cdot \frac{Lt}{2^{k/n}}\right)
\]

for all \(x \in M\) and \(t > 0\).
for all $x \in M$, $u \in \mathcal{U}(A)$, $m \in \mathbb{N}$, $t > 0$, where $L := \frac{n^3 - n^2 + 2n + 2}{2}$. This means that $\mathcal{D}_u Q(x_1, \cdots, x_n) = 0$ for all $x_1, \cdots, x_n \in M, u \in \mathcal{U}(A)$. Hence, the mapping $Q$ satisfies (2) and so $Q((n - 1)x) = (n - 1)^2 Q(x)$ for all $x \in M$. It follows that
\[
\mu\left( Q(x) - g(x), t \right) \geq \mu\left( \frac{Q((n - 1)x)}{(n - 1)^2} - \frac{g((n - 1)^{m+1}x)}{(n - 1)^{2m+2}} + \sum_{k=1}^{m} \frac{(n - 1)^2g((n - 1)^{k-1}x) - g((n - 1)^{k}x)}{(n - 1)^{2k}}, t \right) \\
\geq \Phi\left( (n - 1)x, \frac{(n - 1)^2t}{2} \right) \circ \prod_{k=1}^{m} \Phi\left( x, \frac{(n - 1)^{2k}t}{2^k} \right)
\]
for all $x \in M, t > 0$.

To prove the uniqueness, let $Q'$ be another mapping satisfying (2) and
\[
\mu\left( g(x) - Q'(x), t \right) \geq \Phi\left( x, \frac{(n - 1)^2}{2^\beta - t} \right)
\]
for all $x \in M$. Thus, we have
\[
\mu\left( \frac{1}{2}(Q(x) - Q'(x)), t \right) \geq \mu\left( \frac{Q((n - 1)^{m}x) - g((n - 1)^{m}x)}{(n - 1)^{2m}}, t \right) \\
\quad \circ \mu\left( \frac{g((n - 1)^{m}x)}{(n - 1)^{2m}} - Q'((n - 1)^{m}x), t \right) \\
\geq \Phi\left( x, \frac{(n - 1)^{2m}}{2^\beta - t} \right)
\]
for all $x \in M, t > 0$. Taking the limit as $m \to \infty$, then we conclude that $Q(x) = Q'(x)$ for all $x \in M$.

Under the assumption that either $f$ is measurable or $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in M$, the quadratic mapping $Q$ satisfies $Q(tx) = t^2 Q(x)$ for all $x \in M$ and for all $t \in \mathbb{R}$ by the same reasoning as the proof of [25]. That is, $Q$ is $\mathbb{R}$-quadratic. Let $P := \frac{u^4 - 2n^3 + 3a^2 - 3n + 14}{4}$. Putting $x_1 := -(n - 1)^4 x$ and $x_i := 0$ for all $i = 2, \cdots, n$ in (4) and dividing the resulting inequality by $(n - 1)^{2k}$, we have
\[
\mu\left( \frac{1}{P} (n(n - 1)Q(-ux) - uQ((n - 1)x)u^* - (n - 1)uQ(-x)u^*), 4t \right) \\
\geq \mu\left( Q(-ux) - \frac{g(-u(n - 1)^k x)}{(n - 1)^{2k}}, n(n - 1) \right) \\
\quad \circ \mu\left( uQ(-(n - 1)x)u^* - \frac{g(-A \cdot (n - 1)^k x)}{(n - 1)^{2k}}, Pt \right) \\
\quad \circ \mu\left( uQ(x)u^* - \frac{g((n - 1)^k x)}{(n - 1)^{2k} u^*}, \frac{Pt}{n(n - 1)} \right) \\
\quad \circ \mu\left( \mathcal{D}_u f(-(n - 1)^k x, 0, \cdots, 0), (n - 1)^{2k} Pt \right) \\
\quad \circ \mu\left( f(0), \frac{4(n - 1)^{2k} Pt}{(n - 2)(n - 1)n(n + 1)} \right)
\]
for all $x \in M, u \in \mathcal{U}(A), t > 0$. Taking $k \to \infty$ and using the evenness of $Q$, we obtain that $Q(ux) = uQ(x)u^*$ for all $x \in M$ and for each $u \in \mathcal{U}(A)$. The last relation is also true for $u = 0$. 
Now, let a be a nonzero element in \(A\) and \(K\) a positive integer greater than \(4\|a\|\). Then, we have \(\|a\| < \frac{1}{4} < 1 - \frac{3}{2}\). By [26] [Theorem 1], there exist three elements \(u_1, u_2, u_3 \in \mathcal{U}(A)\) such that \(3^2 = u_1 + u_2 + u_3\). Thus, we calculate in conjunction with [27] [Lemma 2.1] that

\[
Q(ax) = Q\left(\frac{K(a^2)}{3^2}x\right) = \left(\frac{K}{3}\right)^2 Q(u_1x + u_2x + u_3x)
\]

\[
= \left(\frac{K}{3}\right)^2 B(u_1x + u_2x + u_3x, u_1x + u_2x + u_3x)
\]

\[
= \left(\frac{K}{3}\right)^2 (u_1 + u_2 + u_3)Q(x, x)(u_1 + u_2 + u_3)
\]

\[
= \left(\frac{K}{3}\right)^2 \frac{a}{K} Q(x)3^2 \frac{a^*}{K} = aQ(x)a^*
\]

for all \(a \in A(a \neq 0)\) and for all \(x \in M\). Thus, the unique \(\mathbb{R}\)-quadratic mapping \(Q\) is also \(A\)-quadratic, as desired. This completes the proof. \(\square\)

**Corollary 1.** Let \((A, \rho)\) be a \(\rho\)-complete convex modular \(*\)-algebra with norm \(\|\cdot\|\) and \(M\) be a left \(A\)-module, \(\mathcal{U}(A)\) the unitary group of \(A\). Assume that there exist two mappings \(f \in \mathcal{M}^A\) and \(\epsilon \in \mathbb{R}^{M_n}\) such that

\[
\rho(D_u f(x_1, \cdots, x_n)) \leq \epsilon(x_1, \cdots, x_n),
\]

\[
\epsilon((n-1)x_1, \cdots, (n-1)x_n) \leq \beta \epsilon(x_1, \cdots, x_n)
\]

for all \((x_1, \cdots, x_n) \in X^n, u \in \mathcal{U}(A)\), where \(2 \leq 2\beta < (n-1)^2\), and either \(f\) is measurable or \(f(tx)\) is continuous in \(t \in \mathbb{R}\) for each fixed \(x \in M\). Then, there exists a unique mapping \(Q \in \mathcal{M}_A(M, A)\) which satisfies Equation (2) and the inequality

\[
\rho\left(f(x) + \frac{(n-1)f(0)}{2} - Q(x)\right) \leq \frac{12\beta}{(n-1)^4} \min_{(i,j) \in J} \left\{ \max_{i, j \in J} \left\{ \frac{(n^2 - (i+1)n + 1)}{n} \epsilon_i(x), \frac{(n^2 - (i+1)n + 1)}{n} \epsilon_{i+1}(x) \right\} \right\}
\]

for all \(x \in M\).

**Proof.** Let \(X = \mathbb{R}\) with the fuzzy modular \(\mu' : X \times (0, \infty) \to \mathbb{R}\) as

\[
\mu'(z, t) = \frac{t}{t + |z|}
\]

for all \(z \in \mathbb{R}, t > 0\). In addition, define the following convex fuzzy modular \(\mu\) as

\[
\mu(y, t) = \frac{t}{t + \rho(y)}
\]

for all \(y \in M, t > 0\). As noted in Example 1, \((A, \mu, \circ_M)\) is a \(\mu\)-complete convex fuzzy modular \(*\)-algebra and \((\mathbb{R}, \mu', \circ_M)\) is a fuzzy modular space. The result follows from the fact that (4) and (5) are equivalent to (7) and (8), respectively. \(\square\)

**Corollary 2.** Let \((A, \|\cdot\|)\) be a Banach \(*\)-algebra and \(M\) be a left \(A\)-module and \(\theta > 0, p \in (0, 2 - \log_2 2)\). Assume that there exists a mapping \(f \in \mathcal{M}^A\) such that

\[
\|D_u f(x_1, \cdots, x_n)\| \leq \theta(\|x_1\|^p + \cdots + \|x_n\|^p)
\]
for all \((x_1, \cdots, x_n) \in X^n, u \in \mathcal{U}(\mathcal{A})\), and either \(f\) is measurable or \(f(tx)\) is continuous in \(t \in \mathbb{R}\) for each fixed \(x \in M\). Then, there exists a unique quadratic mapping \(Q \in Q_A(M, \mathcal{A})\) which satisfies Equation (2) and the inequality

\[
\|f(x) + \frac{(n - 1)f(0)}{2} - Q(x)\| \leq \frac{12}{(n - 1)^4 - \epsilon} \|x\|^p
\]

for all \(x \in M\), where \(\epsilon\) is a real number defined by

\[
\epsilon := \begin{cases} 
\min \left\{ \frac{n^2 - (i + 1)n + 1}{n} \geq 1 \mid i = 1, \cdots, n - 1 \right\}, & \text{if } n > 3, \\
\text{if } n = 3.
\end{cases}
\]

**Proof.** Letting \(\epsilon(x_1, \cdots, x_n) := \theta(\|x_1\|^p + \cdots + \|x_n\|^p)\), \(\beta := (n - 1)^p\) and applying Corollary 1, we obtain the desired result, as claimed. \(\square\)

Next, we provide an alternative stability theorem of Theorem 2 equipped with \(\Delta_{n-1}\)-condition in \(\mu\)-complete convex fuzzy modular \(*\)-algebras.

**Theorem 3.** Let \((\mathcal{A}, \mu, \sigma)\) be a \(\mu\)-complete convex fuzzy modular \(*\)-algebra with \(\Delta_{n-1}\)-condition and norm \(\|\cdot\|\) and \(M\) be an \(\mathcal{A}\)-left module, \((X, \mu', \sigma)\) fuzzy modular space. Assume that there exist two mappings \(f \in \mathcal{A}^M\) and \(\epsilon \in X^M\) such that

\[
\mu(\mathcal{D}_nf(x_1, \cdots, x_n), t) \geq \mu'(\epsilon(x_1, \cdots, x_n), t),
\]

\[
\mu'(\epsilon\left(\frac{x_1}{n-1}, \cdots, \frac{x_n}{n-1}\right), t) \geq \mu'(\epsilon(x_1, \cdots, x_n), \gamma t)
\]

for all \((x_1, \cdots, x_n) \in X^n, u \in \mathcal{U}(\mathcal{A})\), where \((n - 1)^2\gamma > 2\kappa_{n-1}^2\), and either \(f\) is measurable or \(f(tx)\) is continuous in \(t \in \mathbb{R}\) for each fixed \(x \in M\). Then, there exists a unique mapping \(Q \in Q_A(M, \mathcal{A})\) which satisfies Equation (2) and the inequality

\[
\mu(f(x) - Q(x), t) \geq \Psi\left(x, \frac{(n - 1)t}{2\kappa_{n-1}}\right)
\]

for all \(x \in M, t > 0\), where

\[
\Psi(x, t) = \max_{(i,j) \in J} \left\{ \mu'(\epsilon_i(-x), \gamma(n - 1)^2t \frac{\epsilon_i(x)}{6\kappa_{n-1}}) \circ \mu'(\epsilon_i(x), \frac{\epsilon_i(x)}{6\kappa_{n-1}}) \circ \mu'(\epsilon_{i+1}(x), \frac{\gamma(n - 1)^2t \epsilon_i(x)}{6\kappa_{n-1}}) \right\}.
\]

**Proof.** Letting \((x_1, \cdots, x_n) := (0, \cdots, 0)\) in (9) and using it, we get

\[
\mu'(\epsilon(0, \cdots, 0), t) \geq \mu'(\epsilon(0, \cdots, 0), \gamma^mt)
\]

for all \(t > 0, m \in \mathbb{N}\). Thus, \(\epsilon(0, \cdots, 0) = 0\) and

\[
\mu\left(\frac{n(n - 1)}{2}f(0), t\right) = \mu(\mathcal{D}_n\delta(0, \cdots, 0), t) \geq \mu'(\epsilon(0, \cdots, 0), t) = 1
\]
for all \( t > 0 \), which implies \( f(0) = 0 \). From Equation (6), we get the following equality

\[
f(x) - (n - 1)^2f\left(\frac{x}{n-1}\right) = D_1f_i\left(-\frac{x}{n-1}\right) + \left(\frac{n^2 - (i + 1)n + 1}{n}\right)\left[D_1f_i\left(\frac{x}{n-1}\right) - D_1f_{i+1}\left(\frac{x}{n-1}\right)\right]
\]

for all \((i, j) \in J\). Using (11) and \( \Delta_{n-1} \)-condition of \( \mu \), one gets

\[
\mu\left(f(x) - (n - 1)^2m\left(\frac{x}{(n-1)^m}\right), t\right) \\
\geq \mu\left(\sum_{k=1}^{m} \frac{(n - 1)^{4k-2}}{(n-1)^{2k}} \left(f\left(\frac{x}{(n-1)^k}\right) - (n - 1)^2f\left(\frac{x}{(n-1)^k}\right)\right), \sum_{k=1}^{m} \frac{t}{3^k}\right) \\
\geq \prod_{k=1}^{m} \mu'\left(\epsilon_i(-x), \left(\frac{\gamma(n - 1)^2}{2 \kappa_{n-1}}\right)^k \cdot \frac{\kappa_{n-1}^2 t}{3}\right) \\
\circ \mu'\left(\epsilon_i(x), \left(\frac{\gamma(n - 1)^2}{2 \kappa_{n-1}}\right)^k \cdot \frac{\kappa_{n-1}^2 n t}{3(n^2 - (i + 1)n + 1)}\right) \\
\circ \mu\left(\epsilon_{i+1}(x), \left(\frac{\gamma(n - 1)^2}{2 \kappa_{n-1}}\right)^k \cdot \frac{\kappa_{n-1}^2 n t}{3(n^2 - (i + 1)n + 1)}\right) \\
= \mu'\left(\epsilon_i(-x), \frac{\gamma(n - 1)^2 t}{6 \kappa_{n-1}^2 (n^2 - (i + 1)n + 1)}\right) \\
\circ \mu'\left(\epsilon_{i+1}(x), \frac{\gamma(n - 1)^2 nt}{6 \kappa_{n-1}^2 (n^2 - (i + 1)n + 1)}\right)
\]

for all \( x \in M, t > 0, (i, j) \in J \). This relation leads to

\[
\mu\left(f(x) - (n - 1)^2m\left(\frac{x}{(n-1)^m}\right), t\right) \geq \Psi(x, t)
\]

(12)

for all \( x \in M \) and \( t > 0 \). Now, replacing \( x \) by \( \frac{x}{(n-1)^m} \) in (12), we have

\[
\mu\left((n - 1)^2m\left(\frac{x}{(n-1)^m}\right) - (n - 1)^{2m+2p}\left(\frac{x}{(n-1)^{m+p}}\right), t\right) \\
\geq \mu\left(f\left(\frac{x}{(n-1)^m}\right) - (n - 1)^{2p}\left(\frac{x}{(n-1)^{m+p}}\right), \frac{t}{\kappa_{n-1}^2}\right) \\
\geq \Psi\left(\frac{x}{(n-1)^m}, \frac{t}{\kappa_{n-1}^2}\right) \geq \Psi\left(x, \frac{\gamma}{\kappa_{n-1}^2}\right)
\]

which converges to zero as \( m \to \infty \). Thus, \{ \( (n - 1)^2m\left(\frac{x}{(n-1)^m}\right) \) \} is \( \mu \)-Cauchy for all \( x \in M \), and so it is \( \mu \)-convergent in \( A \) since the space \( A \) is \( \mu \)-complete. Thus, we may define a mapping \( Q : M \to A \) as

\[
Q(x) := \mu - \lim_{m \to \infty} (n - 1)^2m\left(\frac{x}{(n-1)^m}\right)
\]

\( \iff \lim_{m \to \infty} (n - 1)^2m\mu\left(Q(x) - f\left(\frac{x}{(n-1)^m}\right), t\right) = 1 \)
for all \( x \in M \) and all \( t > 0 \). Using \( \Delta_{n-1} \)-condition and convexity of \( \mu \), we find the following inequality

\[
\mu\left(f(x) - Q(x), t\right) \geq \mu\left(f(x) - (n-1)^2mf\left(x\left(\frac{x}{(n-1)^2m}\right), \frac{(n-1)t}{2k_{n-1}}\right), t\right)
\]

\[
\circ \mu\left((n-1)^2mf\left(x\left(\frac{x}{(n-1)^2m}\right), \frac{(n-1)t}{2k_{n-1}}\right), Q(x), \frac{(n-1)t}{2k_{n-1}}\right)
\]

\[
\geq \Psi\left(x, \frac{(n-1)t}{2k_{n-1}}\right)
\]

for all \( x \in M, t > 0 \) and for enough large \( m \in \mathbb{N} \). By the similar way of the proof of Theorem 2, we get \( Q \) is \( \mathcal{A} \)-quadratic functional equation.

To prove the uniqueness, let \( T \) be another \( \mathcal{A} \)-quadratic mapping satisfying (10). Then, we get \( T((n-1)^m x) = (n-1)^2mT(x) \) for all \( x \in M \) and all \( m \in \mathbb{N} \). Thus, we have

\[
\mu\left(T(x) - Q(x), \frac{t}{2m}\right) \geq \mu\left(T\left(x\left(\frac{x}{(n-1)^m}\right)\right) - f\left(x\left(\frac{x}{(n-1)^m}\right), \frac{t}{2m}\right), x\right)
\]

\[
\circ \mu\left(f\left(x\left(\frac{x}{(n-1)^m}\right), \frac{t}{2m}\right) - Q\left(x\left(\frac{x}{(n-1)^m}\right), \frac{t}{2m}\right), x\right)
\]

\[
\geq \Psi\left(x, \frac{(n-1)t}{2k_{n-1}}\right)
\]

Taking the limit as \( m \to \infty \), then we conclude that \( T(x) = Q(x) \) for all \( x \in M \). This completes the proof. \( \square \)

**Corollary 3.** Let \( (\mathcal{A}, \rho) \) be a \( \rho \)-complete convex modular \( * \)-algebra with \( \Delta_{n-1} \)-condition and norm \( \| \cdot \| \). Assume that there exist two mappings \( f \in \mathcal{A}^M \) and \( e \in \mathbb{R}^M \) such that

\[
\rho(D_nf(x_1, \ldots, x_n)) \leq e(x_1, \ldots, x_n),
\]

\[
e\left(\frac{x_1}{n-1}, \ldots, \frac{x_n}{n-1}\right) \leq \frac{1}{\gamma}e(x_1, \ldots, x_n)
\]

for all \( (x_1, \ldots, x_n) \in X^n \), \( n \in \mathcal{U}(\mathcal{A}) \), where \( \gamma(n-1)^2 > 2k_{n-1}^3 \) and either \( f \) is measurable or \( f(tx) \) is continuous in \( t \in \mathbb{R} \) for each fixed \( x \in M \). Then, there exists a unique mapping \( Q \in \mathcal{Q}_\mathcal{A}(M, \mathcal{A}) \) which satisfies Equation (2) and the inequality

\[
\rho(f(x) - Q(x)) \leq \frac{12k_{n-1}^3}{\gamma(n-1)^3} \min_{(i, j) \in J} \left\{ \max_{\varepsilon_i(-x)} \left\{ \frac{(n^2 - (i+1)n + 1)\varepsilon_i(x)}{n}, \frac{(n^2 - (i+1)n + 1)\varepsilon_{i+1}(x)}{n} \right\} \right\}
\]

for all \( x \in M \).

**4. Conclusions**

We have studied a quadratic functional equation from the gravity of the \( n \)-distinct vectors and obtained the solution of the quadratic functional equation and investigated the stability results of a \( \mathcal{A} \)-quadratic mapping on \( \mu \)-complete convex fuzzy modular \( * \)-algebras without using \( \beta \)-homogeneous property and lower semicontinuity. Furthermore, as corollaries, we have presented the stability results of the \( \mathcal{A} \)-quadratic mapping in \( \rho \)-complete convex modular \( * \)-algebras and Banach \( * \)-algebras, respectively.
Author Contributions: Conceptualization, H.-Y.S.; Data curation, H.-M.K. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Acknowledgments: The authors would like to thank the referees for giving useful suggestions and for the improvement of this manuscript. This research was supported by Chungnam National University.

Conflicts of Interest: The authors declare no conflict of interest.

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