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Ulam Type Stability of $\mathcal{A}$-Quadratic Mappings in Fuzzy Modular $*$-Algebras

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Abstract: In this paper, we find the solution of the following quadratic functional equation $n\sum_{1\le i<j\le n} Q\left(x_i - x_j\right) = \sum_{i=1}^{n} Q\left(\sum_{j\neq i} x_j - (n-1)x_i\right)$, which is derived from the gravity of the n distinct vectors $x_1, \cdots, x_n$ in an inner product space, and prove that the stability results of the $\mathcal{A}$-quadratic mappings in μ-complete convex fuzzy modular $*$-algebras without using lower semicontinuity and β-homogeneous property.

Keywords: fuzzy modular $*$-algebras; modular $*$-algebras; $\mathcal{A}$-quadratic derivation; Δ_2-condition; β-homogeneous property

1. Introduction

A concept of stability in the case of homomorphisms between groups was formulated by S.M. Ulam [1] in 1940 in a talk at the University of Wisconsin. Let $(G_1, *)$ be a group and let $(G_2, \diamond, d)$ be a metric group with the metric $d(\cdot,\cdot)$. Given $\epsilon > 0$, does there exist a $\delta(\epsilon) > 0$ such that if a mapping $h : G_1 \to G_2$ satisfies the inequality

$$d(h(x * y), h(x) \diamond h(y)) < \delta$$

for all $x, y \in G_1$, then there is a homomorphism $H : G_1 \to G_2$ with

$$d(h(x), H(x)) < \epsilon$$

for all $x \in G_1$?

The first affirmative answer to the question of Ulam was given by Hyers [2,3] for the Cauchy functional equation in Banach spaces as follows: Let X and Y be Banach spaces. Assume that $f : X \to Y$ satisfies

$$\|f(x+y) - f(x) - f(y)\| \le \varepsilon$$

for all $x, y \in X$ and for some $\varepsilon \ge 0$. Then, there exists a unique additive mapping $T : X \to Y$ such that

$$\|f(x) - T(x)\| \le \varepsilon$$

for all $x \in X$. A number of mathematicians were attracted to this result and stimulated to investigate the stability problems of various(functional, differential, difference, integral) equations in some spaces [4–11].

In 2007, Nourouzi [12] presented probabilistic modular spaces related to the theory of modular spaces. Fallahi and Nourouzi [13,14] investigated the continuity and boundedness of linear operators defined between probabilistic modular spaces in the probabilistic sense. After then, Shen and Chen [15] following the idea of probabilistic modular spaces and the definition of fuzzy metric spaces based on George and Veeramani's sense [16], applied fuzzy concept to the classical notions of modular spaces. Using Khamsi's fixed point theorem in modular spaces [17], Wongkum and Kumam [18] proved the stability of sextic functional equations in fuzzy modular spaces equipped necessarily with lower semicontinuity and β-homogeneous property.

In a recent paper [11], Ulam stability of the following additive functional equation

$$\sum_{\substack{1\le i_1<\cdots<i_m\le n\\ 1\le k_l(\ne i_j,\forall j\in\{1,\cdots,m\})\le n}} f\left(\frac{\sum_{j=1}^m x_{i_j}}{m}+\sum_{l=1}^{n-m} x_{k_l}\right)=\frac{n-m+1}{n}\binom{n}{m}\sum_{i=1}^n f(x_i).$$

was investigated in modular algebras without using the lower semicontinuity and Fatou preperty.

In the present paper, concerning the stability problem for the following functional equation

$$n\sum_{1\le i<j\le n} f\left(x_i-x_j\right)=\sum_{i=1}^n f\left(\sum_{j\ne i} x_j-(n-1)x_i\right)$$

which is derived from the gravity of the n-distinct vectors in an inner product space, we investigate the stability problem for $\mathcal{A}$-quadratic mappings in μ-complete convex fuzzy modular $*$-algebras of the following functional equation without using lower semicontinuity and β-homogeneous property.

2. Preliminaries

Proposition 1. *Let $X_1, X_2, \cdots, X_n$ $(n\ge 3)$ be distinct vectors in a finite n-dimensional Euclidean space E. Putting $G := \frac{\sum_{i=1}^n X_i}{n}$, the gravity of the n distinct vectors, then we get the following identity*

$$\sum_{1\le i<j\le n}\|\overrightarrow{X_iX_j}\|^2=n\sum_{i=1}^n\|\overrightarrow{X_iG}\|^2,$$

which is equivalent to the equation

$$n\sum_{1\le i<j\le n}\left\|X_i-X_j\right\|^2=\sum_{i=1}^n\left\|\sum_{j\ne i}X_j-(n-1)X_i\right\|^2 \tag{1}$$

for any distinct vectors $X_1, X_2, \cdots, X_n$.

Employing the above equality (1), we introduce the new functional equation:

$$n\sum_{1\le i<j\le n} Q\left(x_i-x_j\right)=\sum_{i=1}^n Q\left(\sum_{j\ne i} x_j-(n-1)x_i\right) \tag{2}$$

for a mapping $Q: U\to V$ and for all vectors $x_1,\cdots,x_n\in U$, where U and V are linear spaces and $n\ge 3$ is a positive integer.

From now on, we introduce some basic definitions of fuzzy modular $*$-algebras.

Definition 1. *[18] A triangular norm (briefly, t-norm) is a function $\circ : [0,1]\times[0,1]\to[0,1]$ satisfies the following conditions:*

(1) $\circ$ is commutative, associative;
(2) $a\circ 1=a$;
(3) $a\circ b\le c\circ d$, whenever $a,b,c,d\in[0,1]$ with $a\le b, c\le d$.

Three common examples of the t-norm are (1) $a \circ_M b = \min\{a, b\}$; (2) $a \circ_p b = a \cdot b$; (3) $a \circ_L b = \max\{a + b - 1, 0\}$. For more example, we refer to [19]. Throughout this paper, we denote that

$$\prod_{i=1}^{n} x_i := x_1 \circ \cdots \circ x_n$$

for all $x_1, \cdots, x_n \in [0, 1]$.

Definition 2. *[18] Let X be a complex vector space and $\circ$ a t-norm, and $\mu : X \times (0, \infty) \to [0, 1]$ be a function.*

(a) The triple $(X, \mu, \circ)$ is said to be a fuzzy modular space if, for each $x, y \in X$ and $s, t > 0$ and $\alpha, \beta \in [0, \infty)$ with $\alpha + \beta = 1$,

(FM1) $\mu(x, t) > 0$;
(FM2) $\mu(x, t) = 1$ for all $t > 0$ if and only if $x = \theta$;
(FM3) $\mu(x, t) = \mu(-x, t)$;
(FM4) $\mu(\alpha x + \beta y, s + t) \geq \mu(x, s) \circ \mu(y, t)$;
(FM5) the mapping $t \to \mu(x, t)$ is continuous at each fixed $x \in X$;

(b) alternatively, if (FM-4) is replaced by

(FM4-1) $\mu(\alpha x + \beta y, s + t) \geq \mu(x, \frac{s}{\alpha}) \circ \mu(y, \frac{t}{\beta})$, (where $\alpha, \beta \neq 0$);

then we say that $(X, \mu, \circ)$ is a convex fuzzy modular.

Now, we extend the properties (FM4) and (FM4-1) in real fields to complex scalar field acting on the space X, as follows:

(FM4)′ $\mu(\alpha x + \beta y, s + t) \geq \mu(x, s) \circ \mu(y, t)$; for $\alpha, \beta \in \mathbb{C}$ with $|\alpha| + |\beta| = 1$,
(FM4-1)′ $\mu(\alpha x + \beta y, s + t) \geq \mu(x, \frac{s}{\alpha}) \circ \mu(y, \frac{t}{\beta})$ for $\alpha, \beta \in \mathbb{C}$ with $|\alpha| + |\beta| = 1$.

Next, we introduce the concept of fuzzy modular algebras based on the deifnition of fuzzy normed algebras [20,21]. If X is algebra with fuzzy modular μ subject to $\mu(xy, st) \geq \mu(x, s) \circ \mu(y, t)$ for all $x, y \in X$ and $s, t \in (0, \infty)$, then we say $(X, \mu, \circ)$ is called a fuzzy modular algebra. In addition, a fuzzy modular algebra X is a fuzzy modular $*$-algebra if the fuzzy modular μ satisfies $\mu(z^*, t) = \mu(z, t)$ for all $z \in X, t > 0$.

Example 1. *Let (X, ρ) be a modular $*$-algebra ([22]) and $\circ$ defined by $a \circ b := a \circ_M b$. For every $t \in (0, \infty)$, define $\mu(x, t) = \frac{t}{t + \rho(x)}$ for all $x \in X$. Then, $(X, \mu, \circ)$ is a (convex) fuzzy modular $*$-algebra.*

Definition 3. *(1). We say that $(X, \mu, \circ)$ is β-homogeneous if, for every $x \in X, t > 0$ and $\lambda \in \mathbb{R} \backslash \{0\}$,*

$$\mu(\lambda x, t) = \mu\Big(x, \frac{t}{|\lambda|^{\beta}}\Big), \quad \text{where } \beta \in (0, 1].$$

(2). Let $n \in \mathbb{N}$. We say that $(X, \mu, \circ)$ satisfies Δ_n-condition if there exist $\kappa_n \geq n$ such that

$$\mu(nx, t) \geq \mu\Big(x, \frac{t}{\kappa_n}\Big), \quad \forall x \in X.$$

Remark 1. *Let $(X, \mu, \circ)$ be β-homogeneous for some fixed $\beta \in (0, 1]$. Then, we observe that*

$$\mu(2x, t) = \mu\Big(x, \frac{t}{2^{\beta}}\Big) \geq \mu\Big(x, \frac{t}{\kappa_2}\Big)$$

for all $x \in X$ and all $\kappa_2 \geq 2 \geq |2|^{\beta}$. Thus, β-homogeneous property implies Δ_2-condition.

Example 2. *Let $\rho : \mathbb{R} \to \mathbb{R}, \mu : \mathbb{R} \times (0, \infty) \to (0, 1]$ be defined by $\rho(x) = x^2$ and $\mu(x, t) = \frac{t}{t+\rho(x)}$. Then, we can check that $(\mu, \circ_M)$ is a convex fuzzy modular on $\mathbb{R}$ but $(\mathbb{R}, \mu, \circ_M)$ does not satisfy β-homogeneous property. Let $\kappa_2 \geq 4$. Then,*

$$\mu(2x, t) = \frac{t}{t + \rho(2x)} = \mu\left(x, \frac{t}{4}\right) \geq \mu\left(x, \frac{t}{\kappa_2}\right)$$

for all $x \in \mathbb{R}$. Thus, $(\mathbb{R}, \mu, \circ)$ satisfies Δ_2-condition with $\kappa_2 \geq 4$ but is not β-homogeneous.

Definition 4. *Let $(X, \mu, \circ)$ be a fuzzy modular space and $\{x_n\}$ be a sequence in X_ρ.*
(1). $\{x_n\}$ is said to be μ-convergent to a point $x \in X$ if for any $t > 0$,

$$\mu(x - x_n, t) \to 1$$

as $n \to \infty$.
(2). $\{x_n\}$ is called μ-Cauchy if for each $\varepsilon > 0$ and each $t > 0$, there exists n_1 such that, for all $n \geq n_1$ and all $p > 0$, we have $\mu(x_{n+p} - x_n) > 1 - \varepsilon$.
(3). If each Cauchy sequence is convergent, then the fuzzy modular space is said to be complete.

3. Fuzzy Modular Stability for $\mathcal{A}$-Quadratic Mappings

First of all, we find out the general solution of (1.3) in the class of mappings between vector spaces.

Theorem 1. *Let U and V be vector spaces. A mapping $Q : U \to V$ satisfies the functional Equation (2) for each positive integer $n > 2$ if and only if there exists a symmetric biadditive mapping $B : U \times U \to V$ such that $Q(x) = B(x, x)$ for all $x \in U$.*

Proof. Let Q satisfy Equation (2). One finds that $Q(0) = 0$ and $Q(ax) = a^2Q(x)$ by changing (x, y) to $(0, 0)$ and $(x, 0)$ in (3), respectively, where $a := n - 1$ is a positive integer with $a \geq 2$. Putting $x_1 := x$, $x_2 := y$ and $x_i := 0$ for all $i = 3, \cdots, n$ in (2), we get

$$\begin{aligned} &Q(x - ay) + Q(ax - y) + (a - 1)Q(x + y) \\ &\quad = (a + 1)Q(x - y) + (a^2 - 1)[Q(x) + Q(y)] \end{aligned} \tag{3}$$

for all $x, y \in U$. Using [23] [Theorem 1], we obtain that Q is a generalized polynomial map of degree at most 4. Therefore,

$$Q(x) = A_0 + A_1(x) + A_2(x, x) + A_3(x, x,) + A_4(x, x, x, x)$$

for all $x \in U$, where $A_k : U^k \to V$ is a k-additive symmetric map ($k = 1, \cdots, 4$) and $A_0 \in V$. Since a is an integer, we get

$$(a^2 - 1)A_0 + (a^2 - a)A_1(x) + (a^2 - a^3)A_3(x, x, x) + (a^2 - a^4)A_4(x, x, x, x) = 0$$

for all $x \in U$ by $Q(ax) = a^2Q(x)$. This yields that $Q(x) = A_2(x, x)$ for all $x \in U$. □

Let $\mathcal{A}$ be a complex $*$-algebra with unit and let M be a left $\mathcal{A}$-module. We call a mapping $Q : M \to \mathcal{A}$ an $\mathcal{A}$-quadratic mapping if both relations $Q(ax) = aQ(x)a^*$ and $Q(x + y) + Q(x - y) = 2Q(x) + 2Q(y)$ are fulfilled for all $a \in \mathcal{A}, x, y \in M$ [24]. For the sake of convenience, we define the following:

$$\begin{aligned}
\mathcal{D}_u f(x_1,\cdots,x_n) &:= n\sum_{1\le i<j\le n} f(ux_i-ux_j)-\sum_{i=1}^{n} uf\Big(\sum_{j\neq i}x_j-(n-1)x_i\Big)u^*,\\
\varepsilon_i(x) &:= \varepsilon(0,\cdots,0,\underbrace{x}_{i-th},0,\cdots,0),\\
\mathcal{J} &:= \begin{cases}\{1,\cdots,n-1\}\times\{1,\cdots,n\}, & \text{if} \quad n>3,\\ \{2\}\times\{1,2,3\} & \text{if} \quad n=3.\end{cases}
\end{aligned}$$

In addition, let $\circ$ be defined by minimum t-norm and $\mathcal{A}^M$ be the set of all mapping from M to $\mathcal{A}$, $Q_{\mathcal{A}}(M,\mathcal{A})$ be the set of all $\mathcal{A}$-quadratic mappings from M to $\mathcal{A}$.

Now, we present a stability of the $\mathcal{A}$-quadratic mapping concerning Equation (2) in μ-complete convex fuzzy modular $*$-algebras without using β-homogeneous properties.

Theorem 2. *Let $(\mathcal{A},\mu,\circ)$ be μ-complete convex fuzzy modular $*$-algebra with norm $\|\cdot\|$ and M be a left $\mathcal{A}$-module, $(X,\mu',\circ)$ fuzzy modular space, $\mathcal{U}(\mathcal{A})$ the unitary group of $\mathcal{A}$. Assume that there exist two mappings $f\in\mathcal{A}^M$ and $\varepsilon\in X^{M^n}$ such that*

$$\begin{aligned}
\mu(\mathcal{D}_u f(x_1,\cdots,x_n),t) &\ge \mu'(\varepsilon(x_1,\cdots,x_n),t), \qquad (4)\\
\mu'\Big(\varepsilon((n-1)x_1,\cdots,(n-1)x_n),t\Big) &\ge \mu'\Big(\varepsilon(x_1,\cdots,x_n),\frac{t}{\beta}\Big)
\end{aligned}$$

for all $(x_1,\cdots,x_n)\in X^n, u\in\mathcal{U}(\mathcal{A})$, where $2\le 2\beta<(n-1)^2$, and either f is measurable or $f(tx)$ is continuous in $t\in R$ for each fixed $x\in M$. Then, there exists a unique mapping $Q\in Q_{\mathcal{A}}(M,\mathcal{A})$ that satisfies Equation (2) and the inequality

$$\mu\Big(f(x)+\frac{(n-1)f(0)}{2}-Q(x),t\Big)\ge\Phi\Big(x,\frac{(n-1)^2t}{2\beta}\Big) \qquad (5)$$

for all $x\in M$ and $t>0$, where

$$\begin{aligned}
\Phi(x,t) \;:=\; \max_{(i,j)\in\mathcal{J}}\Big\{&\mu'\Big(\varepsilon_j(-x),\frac{(n-1)^2t}{6}\Big)\circ\mu'\Big(\varepsilon_i(x),\frac{(n-1)^2nt}{6(n^2-(i+1)n+1)}\Big)\\
&\circ\mu'\Big(\varepsilon_{i+1}(x),\frac{(n-1)^2nt}{6(n^2-(i+1)n+1)}\Big)\Big\}.
\end{aligned}$$

Proof. Define a mapping $g:M\to\mathcal{A}$ by $g(x):=f(x)+\frac{(n-1)f(0)}{2}$ for all $x\in M$. Then, for each $x\in M$, the following equation is obtained:

$$g((n-1)x)-(n-1)^2g(x)=\mathcal{D}_u f_j(-x)+\Big(\frac{n^2-(i+1)n+1}{n}\Big)[\mathcal{D}_1 f_i(x)-\mathcal{D}_1 f_{i+1}(x)] \qquad (6)$$

for all $i=1,\cdots,n-1$ and for all $j=1,\cdots,n$, where

$$\mathcal{D}_1 f_i(x)=\mathcal{D}_1 f(0,\cdots,0,\underbrace{x}_{i-th},0,\cdots,0).$$

For each fixed $(i,j) \in \mathcal{J}$, one obtains from $\sum_{k=1}^{m} \frac{1+2(\frac{n^2-(i+1)n+1}{n})}{(n-1)^{2k}} \leq 1$ that

$$\begin{aligned} \mu\Big(g(x) - \frac{g((n-1)^m x)}{(n-1)^{2m}}, t\Big) &\geq \mu\Big(\sum_{k=1}^{m} \frac{(n-1)^2 g((n-1)^{k-1}x) - g((n-1)^k x)}{(n-1)^{2k}}, \sum_{k=1}^{m} \frac{t}{2^k}\Big) \\ &\geq \prod_{k=1}^{m} \Big(\mu'(\varepsilon_j(-x), \frac{(n-1)^{2k} t}{3\cdot 2^k \beta^{k-1}}) \\ &\quad \circ \mu'\Big(\varepsilon_i(x), \frac{(n-1)^{2k} nt}{3\cdot 2^k \beta^{k-1}(n^2-(i+1)n+1)}\Big) \\ &\quad \circ \mu'\Big(\varepsilon_{i+1}(x), \frac{(n-1)^{2k} nt}{3\cdot 2^k \beta^{k-1}(n^2-(i+1)n+1)}\Big)\Big) \\ &= \mu'\Big(\varepsilon_j(-x), \frac{(n-1)^2 t}{6}\Big) \circ \mu'\Big(\varepsilon_i(x), \frac{(n-1)^2 nt}{6(n^2-(i+1)n+1)}\Big) \\ &\quad \circ \mu'\Big(\varepsilon_{i+1}(x), \frac{(n-1)^2 nt}{6(n^2-(i+1)n+1)}\Big) \end{aligned}$$

for all $t > 0$ and $x \in M$, $m \in \mathbb{N}$. Then, it follows from the above inequality that

$$\mu\Big(g(x) - \frac{g((n-1)^m x)}{(n-1)^{2m}}, t\Big) \geq \Phi(x,t)$$

for all $x \in M$ and $t > 0$. Therefore, we prove from this relation that, for any integers m, p,

$$\begin{aligned} \mu\Big(\frac{g((n-1)^m x)}{(n-1)^{2m}} - \frac{g((n-1)^{m+p} x)}{(n-1)^{2(m+p)}}, t\Big) &\geq \mu\Big(g((n-1)^m x) - \frac{g((n-1)^p \cdot (n-1)^m x)}{(n-1)^p}, (n-1)^{2m} t\Big) \\ &\geq \Phi((n-1)^m x, (n-1)^{2m} t) \geq \Phi\Big(x, \Big(\frac{(n-1)^2}{\beta}\Big)^m t\Big) \end{aligned}$$

for all $t > 0, x \in M$. Since the right-hand side of the above inequality tends to 1 as $m \to \infty$, the sequence $\{\frac{g((n-1)^m x)}{(n-1)^{2m}}\}$ is μ-Cauchy and thus converges in $\mathcal{A}$. Hence, we may define a mapping $Q : M \to \mathcal{A}$ as

$$Q(x) := \mu - \lim_{m\to\infty} \frac{g((n-1)^m x)}{(n-1)^{2m}} \Big(\Leftrightarrow \lim_{m\to\infty} \mu\Big(Q(x) - \frac{g((n-1)^m x)}{(n-1)^{2m}}, t\Big) = 1\Big)$$

for all $x \in M$ and $t > 0$. In addition, we claim that the mapping Q satisfies (2). For this purpose, we calculate the following inequality:

$$\begin{aligned} \mu\Big(\frac{\mathcal{D}_u Q(x_1, \cdots, x)}{L}, t\Big) &\geq \prod_{1\leq i<j\leq n} \Big(\mu\Big(Q(ux_i - ux_j) - \frac{g((n-1)^m (ux_i - ux_j))}{(n-1)^{2m}}, \frac{Lt}{2^{i+j} n}\Big) \\ &\quad \circ \mu\Big(uQ(\sum_{j=1}^{n} x_j - nx_i)u^* - \frac{ug((n-1)^m (\sum_{j=1}^{n} x_j - nx_i))u^*}{(n-1)^{2m}}, \frac{Lt}{2^{i+j}}\Big)\Big) \\ &\quad \circ \mu'\Big(\varepsilon(x_1, \cdots, x_n), \Big(\frac{(n-1)^2}{\beta}\Big)^m \cdot \frac{Lt}{2^{i+j}}\Big) \end{aligned}$$

for all $x \in M, u \in \mathcal{U}(\mathcal{A}), m \in \mathbb{N}, t > 0$, where $L := \dfrac{n^3 - n^2 + 2n + 2}{2}$. This means that $\mathcal{D}_u Q(x_1, \cdots, x_n) = 0$ for all $x_1, \cdots x_n \in M, u \in \mathcal{U}(\mathcal{A})$. Hence, the mapping Q satisfies (2) and so $Q((n-1)x) = (n-1)^2 Q(x)$ for all $x \in M$. It follows that

$$\begin{aligned}
\mu\Big(Q(x) - g(x), t\Big) &\geq \mu\Big(\frac{Q((n-1)x)}{(n-1)^2} - \frac{g((n-1)^{m+1}x)}{(n-1)^{2m+2}} \\
&\quad + \sum_{k=1}^{m} \frac{(n-1)^2 g((n-1)^{k-1}x) - g((n-1)^k x)}{(n-1)^{2k}}, t\Big) \\
&\geq \Phi\Big((n-1)x, \frac{(n-1)^2 t}{2}\Big) \circ \prod_{k=1}^{m} \Phi\Big(x, \frac{(n-1)^{2k} t}{2\beta^{k-1}}\Big) \\
&\geq \Phi\Big(x, \frac{(n-1)^2}{2\beta} t\Big)
\end{aligned}$$

for all $x \in M, t > 0$.

To prove the uniqueness, let Q' be another mapping satisfying (2) and

$$\mu\Big(g(x) - Q'(x), t\Big) \geq \Phi\Big(x, \frac{(n-1)^2}{2\beta} t\Big)$$

for all $x \in M$. Thus, we have

$$\begin{aligned}
\mu\Big(\frac{1}{2}\Big(Q(x) - Q'(x)\Big), t\Big) &\geq \mu\Big(\frac{Q((n-1)^m x) - g((n-1)^m x)}{(n-1)^{2m}}, t\Big) \\
&\quad \circ \mu\Big(\frac{g((n-1)^m x) - Q'((n-1)^m x)}{(n-1)^{2m}}, t\Big) \\
&\geq \Phi\Big(x, \frac{(n-1)^{2m}}{2\beta} t\Big)
\end{aligned}$$

for all $x \in M, t > 0$. Taking the limit as $m \to \infty$, then we conclude that $Q(x) = Q'(x)$ for all $x \in M$.

Under the assumption that either f is measurable or $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in M$, the quadratic mapping Q satisfies $Q(tx) = t^2 Q(x)$ for all $x \in M$ and for all $t \in \mathbb{R}$ by the same reasoning as the proof of [25]. That is, Q is $\mathbb{R}$-quadratic. Let $P := \frac{n^4 - 2n^3 + 3n^2 - 3n + 14}{4}$. Putting $x_1 := -(n-1)^k x$ and $x_i := 0$ for all $i = 2, \cdots, n$ in (4) and dividing the resulting inequality by $(n-1)^{2k}$, we have

$$\begin{aligned}
&\mu\Big(\frac{1}{P}\Big(n(n-1)Q(-ux) - uQ((n-1)x)u^* - (n-1)uQ(-x)u^*\Big), 4t\Big) \\
&\geq \mu\Big(Q(-ux) - \frac{g(-u(n-1)^k x)}{(n-1)^{2k}}, \frac{Pt}{n(n-1)}\Big) \\
&\quad \circ \mu\Big(uQ(-(n-1)x)u^* - u\frac{g(-\lambda \cdot (n-1)^k x)}{(n-1)^{2k}} u^*, Pt\Big) \\
&\quad \circ \mu\Big(uQ(x)u^* - u\frac{g((n-1)^k x)}{(n-1)^{2k}} u^*, \frac{Pt}{n(n-1)}\Big) \\
&\quad \circ \mu\Big(\mathcal{D}_u f(-(n-1)^k x, 0, \cdots, 0), (n-1)^{2k} Pt\Big) \\
&\quad \circ \mu\Big(f(0), \frac{4(n-1)^{2k} Pt}{(n-2)(n-1)n(n+1)}\Big)
\end{aligned}$$

for all $x \in M, u \in \mathcal{U}(\mathcal{A}), t > 0$. Taking $k \to \infty$ and using the evenness of Q, we obtain that $Q(ux) = uQ(x)u^*$ for all $x \in M$ and for each $u \in \mathcal{U}(\mathcal{A})$. The last relation is also true for $u = 0$.

Now, let a be a nonzero element in $\mathcal{A}$ and K a positive integer greater than $4\|a\|$. Then, we have $\frac{\|a\|}{K} < \frac{1}{4} < 1 - \frac{2}{3}$. By [26] [Theorem 1], there exist three elements $u_1, u_2, u_3 \in \mathcal{U}(\mathcal{A})$ such that $3\frac{a}{K} = u_1 + u_2 + u_3$. Thus, we calculate in conjunction with [27] [Lemma 2.1] that

$$\begin{aligned} Q(ax) &= Q\left(\frac{K}{3}3\frac{a}{K}x\right) = \left(\frac{K}{3}\right)^2 Q(u_1x + u_2x + u_3x) \\ &= \left(\frac{K}{3}\right)^2 B(u_1x + u_2x + u_3x, u_1x + u_2x + u_3x) \\ &= \left(\frac{K}{3}\right)^2 (u_1 + u_2 + u_3)B(x,x)(u_1^* + u_2^* + u_3^*) \\ &= \left(\frac{K}{3}\right)^2 3\frac{a}{K}Q(x)3\frac{a^*}{K} = aQ(x)a^* \end{aligned}$$

for all $a \in \mathcal{A}(a \neq 0)$ and for all $x \in M$. Thus, the unique $\mathbb{R}$-quadratic mapping Q is also $\mathcal{A}$-quadratic, as desired. This completes the proof. □

Corollary 1. *Let $(\mathcal{A}, \rho)$ be a ρ-complete convex modular $*$-algebra with norm $\|\cdot\|$ and M be a left $\mathcal{A}$-module, $\mathcal{U}(\mathcal{A})$ the unitary group of $\mathcal{A}$. Assume that there exist two mappings $f \in \mathcal{A}^M$ and $\varepsilon \in \mathbb{R}^{M^n}$ such that*

$$\begin{aligned} \rho(\mathcal{D}_u f(x_1, \cdots, x_n)) &\leq \varepsilon(x_1, \cdots, x_n), \\ \varepsilon((n-1)x_1, \cdots, (n-1)x_n) &\leq \beta\varepsilon(x_1, \cdots, x_n) \end{aligned} \tag{7}$$

for all $(x_1, \cdots, x_n) \in X^n, u \in \mathcal{U}(\mathcal{A})$, where $2 \leq 2\beta < (n-1)^2$, and either f is measurable or $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in M$. Then, there exists a unique mapping $Q \in Q_{\mathcal{A}}(M, \mathcal{A})$ which satisfies Equation (2) and the inequality

$$\begin{aligned} \rho\Big(f(x) + \frac{(n-1)f(0)}{2} - Q(x)\Big) \leq \frac{12\beta}{(n-1)^4} \min_{(i,j)\in\mathcal{J}} \Big\{ \max\Big\{\varepsilon_j(-x), \frac{(n^2-(i+1)n+1)}{n}\varepsilon_i(x) \\ , \frac{(n^2-(i+1)n+1)}{n}\varepsilon_{i+1}(x)\Big\}\Big\} \end{aligned} \tag{8}$$

for all $x \in M$.

Proof. Let $X = \mathbb{R}$ with the fuzzy modular $\mu' : X \times (0, \infty) \to \mathbb{R}$ as

$$\mu'(z, t) = \frac{t}{t + |z|}$$

for all $z \in \mathbb{R}, t > 0$. In addition, define the following convex fuzzy modular μ as

$$\mu(y, t) = \frac{t}{t + \rho(y)},$$

for all $y \in M, t > 0$. As noted in Example 1, $(\mathcal{A}, \mu, \circ_M)$ is a μ-complete convex fuzzy modular $*$-algebra and $(\mathbb{R}, \mu'.\circ_M)$ is a fuzzy modular space. The result follows from the fact that (4) and (5) are equivalent to (7) and (8), respectively. □

Corollary 2. *Let $(\mathcal{A}, \|\cdot\|)$ be a Banach $*$-algebra and M be a left $\mathcal{A}$-module and $\theta > 0, p \in (0, 2 - \log_\lambda 2)$. Assume that there exists a mapping $f \in \mathcal{A}^M$ such that*

$$\|\mathcal{D}_u f(x_1, \cdots, x_n)\| \leq \theta(\|x_1\|^p + \cdots + \|x_n\|^p)$$

for all $(x_1, \cdots, x_n) \in X^n, u \in \mathcal{U}(\mathcal{A})$*, and either* f *is measurable or* $f(tx)$ *is continuous in* $t \in \mathbb{R}$ *for each fixed* $x \in M$*. Then, there exists a unique quadratic mapping* $Q \in Q_{\mathcal{A}}(M, \mathcal{A})$ *which satisfies Equation (2) and the inequality*

$$\|f(x) + \frac{(n-1)f(0)}{2} - Q(x)\| \leq \frac{12}{(n-1)^{4-p}}\varepsilon\theta\|x\|^p$$

for all $x \in M$*, where* ε *is a real number defined by*

$$\varepsilon := \begin{cases} \min\left\{\frac{n^2-(i+1)n+1}{n} \geq 1 \Big| i = 1, \cdots, n-1\right\}, & \text{if } n > 3, \\ 1, & \text{if } n = 3. \end{cases}$$

Proof. Letting $\varepsilon(x_1, \cdots, x_n) := \theta(\|x_1\|^p + \cdots + \|x_n\|^p)$, $\beta := (n-1)^p$ and applying Corollary 1, we obtain the desired result, as claimed. □

Next, we provide an alternative stability theorem of Theorem 2 equipped with Δ_{n-1}-condition in μ-complete convex fuzzy modular $*$-algebras.

Theorem 3. *Let* $(\mathcal{A}, \mu, \circ)$ *be a* μ*-complete convex fuzzy modular* $*$*-algebra with* Δ_{n-1}*-condition and norm* $\|\cdot\|$ *and* M *be a* $\mathcal{A}$*-left module,* $(X, \mu', \circ)$ *fuzzy modular space. Assume that there exist two mappings* $f \in \mathcal{A}^M$ *and* $\varepsilon \in X^{M^n}$ *such that*

$$\begin{aligned} \mu(\mathcal{D}_u f(x_1, \cdots, x_n), t) &\geq \mu'(\varepsilon(x_1, \cdots, x_n), t), \\ \mu'\left(\varepsilon\Big(\frac{x_1}{n-1}, \cdots, \frac{x_n}{n-1}\Big), t\right) &\geq \mu'(\varepsilon(x_1, \cdots, x_n), \gamma t) \end{aligned} \tag{9}$$

for all $(x_1, \cdots, x_n) \in X^n, u \in \mathcal{U}(\mathcal{A})$*, where* $(n-1)^2\gamma > 2\kappa_{n-1}^4$*, and either* f *is measurable or* $f(tx)$ *is continuous in* $t \in \mathbb{R}$ *for each fixed* $x \in M$*. Then, there exists a unique mapping* $Q \in Q_{\mathcal{A}}(M, \mathcal{A})$ *which satisfies Equation (2) and the inequality*

$$\mu(f(x) - Q(x), t) \geq \Psi\left(x, \frac{(n-1)t}{2\kappa_{n-1}}\right) \tag{10}$$

for all $x \in M, t > 0$*, where*

$$\begin{aligned} \Psi(x,t) = \max_{(i,j)\in\mathcal{J}} \Bigg\{ &\mu'\left(\varepsilon_j(-x), \frac{\gamma(n-1)^2 t}{6\kappa_{n-1}^2}\right) \circ \mu'\left(\varepsilon_i(x), \frac{\gamma(n-1)^2 nt}{6\kappa_{n-1}^2(n^2-(i+1)n+1)}\right) \\ &\circ \mu'\left(\varepsilon_{i+1}(x), \frac{\gamma(n-1)^2 nt}{6\kappa_{n-1}^2(n^2-(i+1)n+1)}\right)\Bigg\}. \end{aligned}$$

Proof. Letting $(x_1, \cdots, x_n) := (0, \cdots, 0)$ in (9) and using it, we get

$$\mu'(\varepsilon(0, \cdots, 0), t) \geq \mu'(\varepsilon(0, \cdots, 0), \gamma^m t)$$

for all $t > 0, m \in \mathbb{N}$. Thus, $\varepsilon(0, \cdots, 0) = 0$ and

$$\mu\Big(\frac{n(n-1)^2}{2} f(0), t\Big) = \mu(\mathcal{D}_u \delta(0, \cdots, 0), t) \geq \mu'(\varepsilon(0, \cdots, 0), t) = 1$$

for all $t > 0$, which implies $f(0) = 0$. From Equation (6), we get the following equality

$$\begin{aligned} & f(x) - (n-1)^2 f\Big(\frac{x}{n-1}\Big) \\ & = \mathcal{D}_1 f_j\Big(-\frac{x}{n-1}\Big) + \Big(\frac{n^2-(i+1)n+1}{n}\Big)\Big[\mathcal{D}_1 f_i\Big(\frac{x}{n-1}\Big) - \mathcal{D}_1 f_{i+1}\Big(\frac{x}{n-1}\Big)\Big] \end{aligned} \tag{11}$$

for all $(i,j) \in \mathcal{J}$. Using (11) and Δ_{n-1}-condition of μ, one gets

$$\begin{aligned} & \mu\Big(f(x) - (n-1)^{2m} f\Big(\frac{x}{(n-1)^m}\Big), t\Big) \\ & \geq \mu\Big(\sum_{k=1}^{m} \frac{(n-1)^{4k-2}}{(n-1)^{2k}}\Big(f\Big(\frac{x}{(n-1)^k}\Big) - (n-1)^2 f\Big(\frac{x}{(n-1)^k}\Big)\Big), \sum_{k=1}^{m}\frac{t}{2^k}\Big) \\ & \geq \prod_{k=1}^{m}\Big(\mu'\Big(\varepsilon_j(-x), \Big(\frac{\gamma(n-1)^2}{2\kappa_{n-1}^4}\Big)^k \cdot \frac{\kappa_{n-1}^2 t}{3}\Big) \\ & \quad \circ \mu'\Big(\varepsilon_i(x), \Big(\frac{\gamma(n-1)^2}{2\kappa_{n-1}^4}\Big)^k \cdot \frac{\kappa_{n-1}^2 nt}{3(n^2-(i+1)n+1)}\Big) \\ & \quad \circ \mu\Big(\varepsilon_{i+1}(x), \Big(\frac{\gamma(n-1)^2}{2\kappa_{n-1}^4}\Big)^k \frac{\kappa_{n-1}^2 nt}{3(n^2-(i+1)n+1)}\Big)\Big) \\ & = \mu'\Big(\varepsilon_j(-x), \frac{\gamma(n-1)^2 t}{6\kappa_{n-1}^2}\Big) \circ \mu'\Big(\varepsilon_i(x), \frac{\gamma(n-1)^2 nt}{6\kappa_{n-1}^2(n^2-(i+1)n+1)}\Big) \\ & \quad \circ \mu'\Big(\varepsilon_{i+1}(x), \frac{\gamma(n-1)^2 nt}{6\kappa_{n-1}^2(n^2-(i+1)n+1)}\Big) \end{aligned}$$

for all $x \in M, t > 0, (i,j) \in \mathcal{J}$. This relation leads to

$$\mu\Big(f(x) - (n-1)^{2m} f\Big(\frac{x}{(n-1)^m}\Big), t\Big) \geq \Psi(x,t) \tag{12}$$

for all $x \in M$ and $t > 0$. Now, replacing x by $\frac{x}{(n-1)^m}$ in (12), we have

$$\begin{aligned} & \mu\Big((n-1)^{2m} f\Big(\frac{x}{(n-1)^m}\Big) - (n-1)^{2m+2p} f\Big(\frac{x}{(n-1)^{m+p}}\Big), t\Big) \\ & \geq \mu\Big(f\Big(\frac{x}{(n-1)^m}\Big) - (n-1)^{2p} f\Big(\frac{x}{(n-1)^{m+p}}\Big), \frac{t}{\kappa_{n-1}^{2m}}\Big) \\ & \geq \Psi\Big(\frac{x}{(n-1)^m}, \frac{t}{\kappa_{n-1}^{2m}}\Big) \geq \Psi\Big(x, \Big(\frac{\gamma}{\kappa_{n-1}^2}\Big)^m t\Big) \end{aligned}$$

which converges to zero as $m \to \infty$. Thus, $\{(n-1)^{2m} f(x/(n-1)^m)\}$ is μ-Cauchy for all $x \in M$, and so it is μ-convergent in $\mathcal{A}$ since the space $\mathcal{A}$ is μ-complete. Thus, we may define a mapping $Q : M \to \mathcal{A}$ as

$$\begin{aligned} & Q(x) := \mu - \lim_{m\to\infty} (n-1)^{2m} f\Big(\frac{x}{(n-1)^m}\Big) \\ & \Big(\iff \lim_{m\to\infty} (n-1)^{2m} \mu\Big(Q(x) - f\Big(\frac{x}{(n-1)^m}\Big), t\Big) = 1\Big) \end{aligned}$$

for all $x \in M$ and all $t > 0$. Using Δ_{n-1}-condition and convexity of μ, we find the following inequality

$$\begin{aligned}\mu\Big(f(x)-Q(x),t\Big) &\geq \mu\Big(f(x)-(n-1)^{2m}f\Big(\frac{x}{(n-1)^{2m}}\Big),\frac{(n-1)t}{2\kappa_{n-1}}\Big)\\ &\quad\circ\mu\Big((n-1)^{2m}f\Big(\frac{x}{(n-1)^{2m}}\Big)-Q(x)\Big),\frac{(n-1)t}{2\kappa_{n-1}}\Big)\\ &\geq \Psi\Big(x,\frac{(n-1)t}{2\kappa_{n-1}}\Big)\end{aligned}$$

for all $x \in M, t > 0$ and for enough large $m \in \mathbb{N}$. By the similar way of the proof of Theorem 2, we get Q is $\mathcal{A}$-quadratic functional equation.

To prove the uniqueness, let T be another $\mathcal{A}$-quadratic mapping satisfying (10). Then, we get $T((n-1)^m x) = (n-1)^{2m}T(x)$ for all $x \in M$ and all $m \in \mathbb{N}$. Thus, we have

$$\begin{aligned}\mu\Big(\frac{T(x)-Q(x)}{2},t\Big) &\geq \mu\Big(T\Big(\frac{x}{(n-1)^m}\Big)-f\Big(\frac{x}{(n-1)^m}\Big),\frac{t}{\kappa_{n-1}^{2m}}\Big)\\ &\quad\circ\mu\Big(f\Big(\frac{x}{(n-1)^m}\Big)-Q\Big(\frac{x}{(n-1)^m}\Big),\frac{t}{\kappa_{n-1}^{2m}}\Big)\\ &\geq \Psi\Big(\frac{x}{(n-1)^m},\frac{(n-1)t}{\kappa_{n-1}^{2m+1}}\Big)\geq\Psi\Big(x,\frac{(n-1)\gamma^m t}{\kappa_{n-1}^{2m+1}}\Big)\end{aligned}$$

Taking the limit as $m \to \infty$, then we conclude that $T(x) = Q(x)$ for all $x \in M$. This completes the proof. □

Corollary 3. *Let $(\mathcal{A},\rho)$ be a ρ-complete convex modular $*$-algebra with Δ_{n-1}-condition and norm $\|\cdot\|$. Assume that there exist two mappings $f \in \mathcal{A}^M$ and $\varepsilon \in \mathbb{R}^{M^n}$ such that*

$$\begin{aligned}\rho(\mathcal{D}_u f(x_1,\cdots,x_n)) &\leq \varepsilon(x_1,\cdots,x_n),\\ \varepsilon\Big(\frac{x_1}{n-1},\cdots\frac{x_n}{n-1}\Big) &\leq \frac{1}{\gamma}\varepsilon(x_1,\cdots,x_n)\end{aligned}$$

for all $(x_1,\cdots,x_n) \in X^n, u \in \mathcal{U}(\mathcal{A})$, where $\gamma(n-1)^2 > 2\kappa_{n-1}^4$ and either f is measurable or $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in M$. Then, there exists a unique mapping $Q \in Q_{\mathcal{A}}(M,\mathcal{A})$ which satisfies Equation (2) and the inequality

$$\begin{aligned}\rho(f(x)-Q(x)) \leq \frac{12\kappa_{n-1}^3}{\gamma(n-1)^3}\min_{(i,j)\in\mathcal{J}}\Big\{\max\Big\{\varepsilon_j(-x),\frac{(n^2-(i+1)n+1)}{n}\varepsilon_i(x)\\ ,\frac{(n^2-(i+1)n+1)}{n}\varepsilon_{i+1}(x)\Big\}\Big\}\end{aligned}$$

for all $x \in M$.

4. Conclusions

We have studied a quadratic functional equation from the gravity of the n-distinct vectors and obtained the solution of the quadratic functional equation and investigated the stability results of a $\mathcal{A}$-quadratic mapping on μ-complete convex fuzzy modular $*$-algebras without using β-homogeneous property and lower semicontinuity. Furthermore, as corollaries, we have presented the stability results of the $\mathcal{A}$-quadratic mapping in ρ-complete convex modular $*$-algebras and Banach $*$-algebras, respectively.

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References

1. Ulam, S.M. *A Collection of the Mathematical Problems*; Interscience Publishers: New York, NY, USA, 1960.
2. Hyers, D.H. On the stability of the linear functional equation. *Proc. Natl. Acad. Sci. USA* **1941**, *27*, 222–224. [CrossRef] [PubMed]
3. Jun, K.-W.; Kim, H.-M. On the generalized A-quadratic mappings associated with the variance of a discrete-type distribution. *Nonlinear Anal. Theory Methods Appl.* **2005**, *62*, 975–987. [CrossRef]
4. Choi, G.; Jung, S.-M. The Approximation Property of a One-Dimensional, Time Independent Schrödinger Equation with a Hyperbolic Potential Well. *Mathematics* **2020**, *8*, 1351. [CrossRef]
5. Alanazi, A.M.; Muhiuddin, G.; Tamilvanan, K.; Alenze, E.N.; Ebaid, A.; Loganathan, K. Fuzzy Stability Results of Finite Variable Additive Functional Equation: Direct and Fixed Point Methods. *Mathematics* **2020**, *8*, 1050. [CrossRef]
6. Park, C.G.; Rassias, J.M.; Bodaghi, A.; Kim, S.O. Approximate homomorphisms from ternary semigroups to modular spaces. *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat.* **2019**, *113*, 2175–2188. [CrossRef]
7. Saha, P.; Samanta, T.K.; Mondal, P.; Choudhury, B.S.; De La Sen, M. Applying Fixed Point Techniques to Stability Problems in Intuitionistic Fuzzy Banach Spaces. *Mathematics* **2020**, *8*, 974. [CrossRef]
8. EL-Fassi, I. Solution and approximation of radical quintic functional equation related to quintic mapping in quasi-β-Banach spaces. *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat.* **2019**, *113*, 675–687. [CrossRef]
9. Liu, K.; Fečkan, M.; Wang, J. A Fixed-Point Approach to the Hyers–Ulam Stability of Caputo–Fabrizio Fractional Differential Equations. *Mathematics* **2020**, *8*, 647. [CrossRef]
10. Eskandani, G.Z.; Rassias, J.M. Stability of general A-cubic functional equations in modular spaces. *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat.* **2018**, *112*, 425–435. [CrossRef]
11. Kim, H.-M.; Shin, H.-Y. Approximate Lie $*$-derivations on ρ-complete convex modular algebras. *J. Appl. Anal. Comput.* **2019**, *9*, 765–776. [CrossRef]
12. Nourouzi, K. Probabilistic modular spaces. In Proceedings of the 6th International ISAAC Congress, Ankara, Turkey, 13–18 August 2007.
13. Nourouzi, K. Baire's theorem in probabilistic modular spaces. In Proceedings of the World Congress on Engineering (WCE '08), London, UK, 2–4 July 2008; Volume 2, pp. 916–917.
14. Fallahi, K.; Nourouzi, K. Probabilistic modular spaces and linear operators. *Acta Appl. Math.* **2009**, *105*, 123–140. [CrossRef]
15. Shen, Y.; Chen, W. On fuzzy modular spaces. *J. Appl. Math.* **2013**, *2013*, 576237.
16. George, A.; Veeramani, P. On some results in fuzzy metric spaces. *Fuzzy Sets Syst.* **2009**, *64*, 395–399. [CrossRef]
17. Khamsi, M.A.; Kozłowski, W.M.; Reich, S. Fixed point theory in modular function spaces. *Nonlinear Anal. Theory Methods Appl.* **1990**, *14*, 935–953. [CrossRef]
18. Wongkum, K.; Kumam, P. The stability of sextic functional equation in fuzzy modular spaces. *J. Nonlinear Sci. Appl.* **2016**, *9*, 3555–3569. [CrossRef]
19. Klement P.; Mesiar, R. Triangular norms. *Tatra Mt. Math. Publ.* **1997**, *13*, 169–193.
20. Sadeqi, I.; Amiripour, A. Fuzzy Banach algebras. In Proceedings of the First, Joint Congress on Fuzzy and Intelligent Systems, Ferdorwski University of Mashhad, Mashhad, Iran, 29–31 August 2007.
21. Dinda, B.; Samanta, T.K.; Bera, U.K. Intuitionistic fuzzy Banach algebra. *Bull. Math. Anal. Appl.* **2010**, *3*, 273–281.
22. Kim, H.-M.; Shin, H.-Y. Approximate cubic Lie derivations on ρ-complete convex modular algebras. *J. Funct. Spaces* **2018**, *2018*, 3613178.
23. Baker, J.A. A general functional equation and its stability. *Proc. Am. Math. Soc.* **2005**, *133*, 1657–1664. [CrossRef]

24. Vukman, J. Some functional equations in Banach algebras and an application. *Proc. Am. Math. Soc.* **1987**, *100*, 133–136. [CrossRef]
25. Czerwik, S. The stability of the quadratic functional equation. In *Stability of Mappings of Hyers-Ulam Type*; Rassias, T.M., Tabor, J., Eds.; Hadronic Press: Palm Harbor, FL, USA, 1994; pp. 81–91.
26. Kadison, R.V.; Pedersen, G. Means and convex combinations of unitary operators. *Math. Scand.* **1985**, *57*, 249–266. [CrossRef]
27. Jun, K.W.; Kim, H.M. On the Hyers-Ulam stability of a generalized quadratic and additive functional equations. *Bull. Korean Math. Soc.* **2005**, *42*, 133–148. [CrossRef]