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Nonnegative Inverse Elementary Divisors Problem for Lists with Nonnegative Real Parts

Hans Nina [1,*], Hector Flores Callisaya [2], H. Pickmann-Soto [3] and Jonnathan Rodriguez [1]

1 Departamento de Matemáticas, Facultad de Ciencias Básicas, Universidad de Antofagasta, Antofagasta 1240000, Chile; jonnathan.rodriguez@uantof.cl
2 Departamento de Matemáticas, Universidade Federal de Mato Grosso, Cuiabá 78060-900, Brazil; fchect@ufmt.br
3 Departamento de Matemática, Universidad de Tarapacá, Arica 1000000, Chile; hpickmanns@academicos.uta.cl
* Correspondence: hans.nina@uantof.cl

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Abstract: In this paper, sufficient conditions for the existence and construction of nonnegative matrices with prescribed elementary divisors for a list of complex numbers with nonnegative real part are obtained, and the corresponding nonnegative matrices are constructed. In addition, results of how to perturb complex eigenvalues of a nonnegative matrix while keeping its elementary divisors and its nonnegativity are derived.

Keywords: nonnegative inverse elementary divisors problem; nonnegative matrices; Jordan canonical form

MSC: 15A29; 15A18; 15A21

1. Introduction

In this paper, we consider the nonnegative inverse elementary divisors problem (NIEDP) which asks to find necessary and sufficient conditions under which the polynomials $(\lambda-\lambda_1)^{n_1}$, $(\lambda-\lambda_2)^{n_2},\ldots,(\lambda-\lambda_k)^{n_k}$, $n_1+n_2+\cdots+n_k=n$, are the elementary divisors of an $n\times n$ entrywise nonnegative matrix A (nonnegative matrix) [1,2]. The NIEDP contains the nonnegative inverse eigenvalue problem (NIEP), which asks to find necessary and sufficient conditions for a list of complex numbers $\Lambda=\{\lambda_1,\ldots,\lambda_n\}$ to be the spectrum of an $n\times n$ nonnegative matrix. Given a list of complex numbers Λ, if there exists a nonnegative matrix A with spectrum Λ, we say that Λ is realizable and that A realizes Λ. This problem has been studied by several authors [3–14]. A known fact to consider due to the Perron–Frobenius theory is that if $\Lambda=\{\lambda_1,\lambda_2,\ldots,\lambda_n\}$ is realizable then one of its elements, say λ_1 called Perron root, is real and such that $\lambda_1\geq|\lambda_j|$, $j=2,\ldots,n$.

The nonnegative inverse eigenvalue problem arises from many areas such as differential equations, functional spaces, mechanics, geophysics, engineering, economy, Markov chains, among others. When the reconstructed matrix is required with some structure, symmetric, stochastic, doubly stochastic, normal, persymmetric, Toeplitz, etc., it is called the inverse problem of structured eigenvalue. If the matrix is also asked to have a prescribed canonical Jordan form, it is called the nonnegative inverse elementary divisors problem. H. Minc was the first who raised this problem and presented some results in [1].

In [15], the authors solve completely the NIEDP for list of complex numbers $\Lambda=\{\lambda_1,\lambda_2,\ldots,\lambda_n\}$ with $\lambda_1>0$, $\operatorname{Re}\lambda_j<0$, $j=2,\ldots,n$, and $|\operatorname{Re}\lambda_j|\geq\operatorname{Im}\lambda_j$, $j=2,\ldots,n$, and they give sufficient conditions for lists Λ with $\operatorname{Re}\lambda_j<0$, $j=2,\ldots,n$. In [16], the authors give a necessary and sufficient condition for the NIEDP and lists in the left half plane, i.e., lists Λ satisfying $\lambda_1 > 0$, $\operatorname{Re}\lambda_j < 0, |\sqrt{3}\operatorname{Re}\lambda_j| \geq |\operatorname{Im}\lambda_j|$, $j = 2,\ldots,n$, and a new and better condition when $\operatorname{Re}\lambda_j < 0$, $j = 2,\ldots,n$. The NIEDP was recently considered for lists with two positive eigenvalues and certain restrictions [17]. The NIEP remains open for lists that yield in the left half plane, in the right half plane, and in the whole complex plane. So far there are only a few results with sufficient conditions [2,15–17].

In this work, we give new sufficient conditions for the NIEDP and lists of complex numbers that yield in the complex right half plane, i.e., lists $\Lambda = \{\lambda_1,\ldots,\lambda_n\}$ with $\lambda_1 > 0$ and $\operatorname{Re}\lambda_j \geq 0$, $j = 2,\ldots,n$.

It is well know that given an $n \times n$ complex matrix A there exits a nonsingular matrix S of order n such that

$$J(A) = S^{-1}AS = \begin{bmatrix} J_{n_1}(\lambda_1) & & & \\ & J_{n_2}(\lambda_2) & & \\ & & \ddots & \\ & & & J_{n_k}(\lambda_k) \end{bmatrix}, \; n_1 + n_2 + \cdots + n_k = n,$$

is the Jordan canonical form of A, hereafter JCF of A. The $n_j \times n_j$ submatrices

$$J_{n_j}(\lambda_j) = \begin{bmatrix} \lambda_j & 1 & & \\ & \lambda_j & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_j \end{bmatrix}$$

are called the Jordan blocks of $J(A)$. Then the elementary divisors of A are the polynomials $(\lambda - \lambda_j)^{n_j}$ which are the characteristic polynomials of $J_{n_j}(\lambda_j)$, $j = 1,\ldots,k$.

On the other hand, it is known when the matrix A has only real entries, the complex eigenvalues occur in conjugate pairs, and the Jordan blocks of all sizes corresponding to complex eigenvalues occur in conjugate pairs of the equal size. Furthermore, any $2k \times 2k$ Jordan matrix of the form

$$\begin{bmatrix} J_k(\lambda) & 0 \\ 0 & J_k(\overline{\lambda}) \end{bmatrix} \tag{1}$$

with $\lambda = a + ib$, $a, b \in \mathbb{R}$, is similar to a real $2k \times 2k$ block of the form

$$C_k(a,b) = \begin{bmatrix} C(a,b) & I & & \\ & C(a,b) & \ddots & \\ & & \ddots & I \\ & & & C(a,b) \end{bmatrix}, \tag{2}$$

where

$$C(a,b) = \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \text{ and } I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

see [18]. This last fact shall be crucial for the construction of the desired Jordan blocks.

A matrix $A = [a_{ij}]$ is stochastically generalized or has constant row sums if all rows sum up to the same constant, say α, i.e., $A\mathbf{e} = \alpha\mathbf{e}$ with $\mathbf{e} = (1,1,\ldots,1)^T$. In [8], Johnson showed that any realizable list is realizable by a nonnegative matrix with constant row sums equal to its Perron root. We denote by $\mathcal{CS}_\alpha$ the set of all real matrices with constant row sum equal to α and by $\mathbf{e}_k$ the vector which has a 1 as its k-th component and zeros elsewhere. Moreover, E_{ij} denote the $n \times n$ matrix with 1 in the (i,j) position and zeros elsewhere.

The paper is organized as follows: In Section 2, we recall results of perturbations due to Brauer and Rado and its consequences on the NIEDP. In Section 3, by using Brauer and Rado Theorems, sufficient conditions for the NIEDP are derived. In Section 4, we show perturbation results of the real and/or imaginary parts of complex eigenvalues from a list of numbers in the complex half right plane.

2. Preliminaries

We will make frequent use of the following matrix perturbation result by another rank one matrix due to Brauer [19].

Theorem 1. *[19] Let A be an $n \times n$ arbitrary matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$. Let $\mathbf{v} = (v_1, \ldots, v_n)^T$ be an eigenvector of A associated with the eigenvalue λ_k and let $\mathbf{q}$ be any n-dimensional vector. Then the matrix $A + \mathbf{v}\mathbf{q}^T$ has eigenvalues $\lambda_1, \ldots, \lambda_{k-1}, \lambda_k + \mathbf{v}^T\mathbf{q}, \lambda_{k+1}, \ldots, \lambda_n$.*

An extension of Brauer's result, due to Rado and introduced by Perfect in [20], is the following:

Theorem 2. *[20] Let A be an $n \times n$ arbitrary matrix with spectrum $\Lambda = \{\lambda_1, \ldots, \lambda_n\}$. Let $X = [\mathbf{x}_1 | \cdots | \mathbf{x}_r]$ be such that $rank(X) = r$ and $A\mathbf{x}_j = \lambda_j \mathbf{x}_j$, $j = 1, \ldots, r$, $r \leq n$. Let C be an $r \times n$ arbitrary matrix. Then $A + XC$ has eigenvalues $\mu_1, \ldots, \mu_r, \lambda_{r+1}, \ldots, \lambda_n$, where $\mu_1, \ldots, \mu_r$ are eigenvalues of the matrix $\Omega + CX$ with $\Omega = \mathrm{diag}(\lambda_1, \ldots, \lambda_r)$.*

The following results show under what conditions the Rado and Brauer perturbation maintain the JCF.

Lemma 1. *[15] Let A, X, Y, C and Ω be as in Theorem 2. If the matrices $B = \Omega + CX$ and VAY have no common eigenvalues, then*

$$J(A + XC) = J(B) \oplus J(VAY).$$

In particular, if $CX = 0$, A and $A + XC$ are similar.

If in Lemma 1 we consider a perturbation of rank one we have:

Lemma 2. *[2] Let $\mathbf{q} = (q_1, \ldots, q_n)^T$ an arbitrary n-dimensional vector and let $A \in \mathcal{CS}_{\lambda_1}$ with JCF $J(A) = S^{-1}AS$. Let $\lambda_1 + \mathbf{q}^T\mathbf{e} \neq \lambda_i$, $i = 2, \ldots, n$. Then $A + \mathbf{e}\mathbf{q}^T$ has JCF*

$$J(A) + (\mathbf{q}^T\mathbf{e})E_{11}.$$

In particular, if $\mathbf{q}^T\mathbf{e} = 0$, then A and $A + \mathbf{e}\mathbf{q}^T$ are similar.

3. Sufficient Conditions for The NIEDP

In [21], Laffey and Šmigoc considered the nonnegative inverse eigenvalue problem for a list $\Lambda = \{\lambda_1, \ldots, \lambda_n\}$ of complex numbers with $\overline{\Lambda} = \Lambda$, $\lambda_1 \geq |\lambda_j|$ and $\mathrm{Re}\,\lambda_j \geq 0$ for $j = 2, \ldots, n$, i.e., Λ lies in the complex half right plane. In this section, through a perturbation of rank one, we establish a new sufficient condition for the existence and construction of a nonnegative matrix with spectrum and elementary divisors prescribed for a list of numbers in the complex half right plane. We also show an extension of this result using a perturbation of rank k for $k \leq n$.

Definition 1. *Given a list of complex numbers $\Lambda = \{\lambda_1, \ldots, \lambda_n\}$ with $\overline{\Lambda} = \Lambda$, $\lambda_1 > \lambda_2 \geq \cdots \geq \lambda_\ell \geq 0$ be real and $\lambda_{\ell+1}, \ldots, \lambda_n$ be complex with $\mathrm{Im}\,\lambda_j \neq 0$ and $\mathrm{Re}\,\lambda_j \geq 0$, $j = \ell + 1, \ldots, n$, we define the negativity $N(\Lambda)$ of Λ by*

$$N(\Lambda) = \frac{1}{2} \sum_{j=\ell+1}^{n} |\,\mathrm{Im}\,\lambda_j|.$$

Theorem 3. *Let* $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ *be a list of complex numbers with* $\overline{\Lambda} = \Lambda$, $\lambda_1 \geq \max_j |\lambda_j|$, $j = 2, \ldots, n$. *Suppose that* $\lambda_1 > \lambda_2 \geq \cdots \geq \lambda_\ell \geq 0$ *are real and* $\lambda_{\ell+1}, \ldots, \lambda_n$ *are complex with* $\operatorname{Im} \lambda_j \neq 0$ *and* $\operatorname{Re} \lambda_j \geq 0$, $j = \ell + 1, \ldots, n$. *Let* $N(\Lambda) < \lambda_1 - \lambda_j$, $j = 2, \ldots, \ell$, *and*

$$m = \max_{\ell+1 \leq j \leq n} \{ m_j = \operatorname{Re} \lambda_j + \operatorname{Im} \lambda_j + N(\Lambda) \} . \tag{3}$$

If

$$\lambda_1 > m,$$

then there exists an $n \times n$ *nonnegative matrix* $A \in \mathcal{CS}_{\lambda_1}$ *with spectrum* Λ *and with prescribed elementary divisors*

$$(\lambda - \lambda_1), (\lambda - \lambda_2)^{n_2}, \ldots, (\lambda - \lambda_k)^{n_k}, \; n_2 + \cdots + n_k = n - 1.$$

Proof. Suppose that the elements of Λ are ordered in such way that $\lambda_1 > \lambda_2 \geq \cdots \geq \lambda_\ell \geq 0$ are real numbers and let $(\lambda_{\ell+1}, \ldots, \lambda_n) = (\lambda_{\ell_1}, \overline{\lambda}_{\ell_1}, \ldots, \lambda_{\ell_r}, \overline{\lambda}_{\ell_r})$ be complex with $\operatorname{Im} \lambda_j \neq 0$ and $r = \frac{n-\ell}{2}$. Then, we define

$$a_{\ell_j} = \operatorname{Re} \lambda_{\ell_j} \geq 0 \quad \text{and} \quad b_{\ell_j} = \operatorname{Im} \lambda_{\ell_j} \geq 0, \quad \text{for } j = 1, \ldots, r.$$

Now, consider the matrix of order n

$$J = \begin{bmatrix} \lambda_1 & & & & & & \\ & \lambda_2 & & & & & \\ & & \ddots & & & & \\ & & & \lambda_\ell & & & \\ & & & & C(a_{\ell_1}, b_{\ell_1}) & & \\ & & & & & \ddots & \\ & & & & & & C(a_{\ell_r}, b_{\ell_r}) \end{bmatrix}. \tag{4}$$

It is clear that the JCF of J is $D = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$. Suppose that E is an $n \times n$ upper triangular matrix with zeros in the diagonal such that $J + E$ has the required elementary divisors.

Let $S = [\mathbf{e} | \mathbf{e_2} | \ldots | \mathbf{e_n}]$ an $n \times n$ matrix. Then $B = S(J + \varepsilon E)S^{-1}$ is a matrix in $\mathcal{SC}_{\lambda_1}$ with spectrum Λ and desired JCF. Notice that for $\varepsilon > 0$ enough small all the elements on the first column of B are positive. We choose

$$\mathbf{q}^T = \left(-N(\Lambda), \underbrace{0, \ldots, 0}_{\ell-1}, \operatorname{Im} \lambda_{\ell_1}, 0, \ldots, \operatorname{Im} \lambda_{\ell_r}, 0 \right).$$

From Lemma 2, we deduce that matrix $A = B + \mathbf{e}\mathbf{q}^T$ has spectrum Λ and the required elementary divisors, since $\mathbf{q}^T \mathbf{e} = 0$.

We shall prove that A is nonnegative. By the form that we choose the vector $\mathbf{q}^T$, the entries in columns $2, \ldots, n$ of A are all nonnegative. The entries in positions $(j, 1)$, $j = 2, \ldots, \ell$ are of the form $\lambda_1 - \lambda_j - N(\Lambda)$ or they are of the form $\lambda_1 - \lambda_j - N(\Lambda) - \varepsilon$. In the positions $(j, 1)$, $j = \ell_1 + 1, \ldots, n$, the entries are either of the form $\lambda_1 - \operatorname{Re} \lambda_j - \operatorname{Im} \lambda_j - N(\Lambda)$ or $\lambda_1 - \operatorname{Re} \lambda_j + \operatorname{Im} \lambda_j - N(\Lambda)$, or they are the form $\lambda_1 - \operatorname{Re} \lambda_j - \operatorname{Im} \lambda_j - N(\Lambda) - \varepsilon$ or $\lambda_1 - \operatorname{Re} \lambda_j + \operatorname{Im} \lambda_j - N(\Lambda) - \varepsilon$.

Since $\lambda_1 > m$ and $N(\Lambda) < \lambda_1 - \lambda_j$, $j = 2, \ldots, \ell$, then the entries on the first column of A are all nonnegative for $\varepsilon > 0$ sufficiently small. Therefore, $A \in \mathcal{CS}_{\lambda_1}$ is nonnegative with spectrum Λ and by Lemma 2 it has the required elementary divisors. □

The following corollary considers a list $\Lambda = \{\lambda_1, \ldots, \lambda_n\}$ in the complex half right plane and some negative real numbers.

Corollary 1. *Let $\Lambda = \{\lambda_1, \ldots, \lambda_p, \lambda_{p+1}, \ldots, \lambda_s, \lambda_{s+1}, \ldots, \lambda_n\}$ be a list of complex numbers with $\overline{\Lambda} = \Lambda$, $\lambda_1 \geq \max_j |\lambda_j|$, $j = 2, \ldots, n$, where $\lambda_1, \ldots, \lambda_p$ are nonnegative real numbers, $\lambda_{p+1}, \ldots, \lambda_s$ are negative, $\lambda_{s+1}, \ldots, \lambda_n$ are complex with $\operatorname{Im}\lambda_j \neq 0$ and $\operatorname{Re}\lambda_j \geq 0$, $j = s+1, \ldots, n$. Let $N(\Lambda) < \lambda_1 - \lambda_j - \sum_{\ell=p+1}^{s} \lambda_\ell$, $j = 2, \ldots, p$, and*

$$m = \max_{s+1 \leq j \leq n} \left\{ m_j = \operatorname{Re}\lambda_j + \operatorname{Im}\lambda_j + \sum_{\ell=p+1}^{s} |\lambda_\ell| + N(\Lambda) \right\}. \tag{5}$$

If

$$\lambda_1 > m,$$

then there exists an $n \times n$ nonnegative matrix $A \in \mathcal{CS}_{\lambda_1}$ with spectrum Λ and with prescribed elementary divisors

$$(\lambda - \lambda_1), (\lambda - \lambda_2)^{n_2}, \ldots, (\lambda - \lambda_k)^{n_k}, \quad n_2 + \cdots + n_k = n - 1.$$

We shall apply the Theorem 2 to extend realizability conditions previously obtained as follows: we start with a block diagonal matrix where each block A_k, $k = 1, \ldots, r$, is nonnegative with Perron eigenvalue ω_k and spectrum $\Gamma = \{\omega_k, \lambda_{k2}, \ldots, \lambda_{kp_k}\}$. Then $A + XC$ will be nonnegative with spectrum $\{\lambda_1, \ldots, \lambda_n\}$ where the eigenvalues $\lambda_1, \ldots, \lambda_r$ were replaced with $\omega_1, \ldots, \omega_r$. The r columns $\mathbf{x}_j$ of X are linearly independent eigenvectors of A corresponding to the eigenvalues ω_j and C such that $\Omega + CX$, with $\Omega = \operatorname{diag}(\omega_1, \ldots, \omega_r)$, has the new eigenvalues $\lambda_1, \ldots, \lambda_r$. Thus, $A + XC$ will have the required spectrum. In [15], the authors showed that the diagonal blocks $A_1, \ldots, A_r$ can have negative entries which we may take away by setting appropriate entries in the suitable positions of the matrix C.

Lemma 3. *[15] Let A be an $n \times n$ real block diagonal matrix, where each diagonal block $A_k \in \mathcal{CS}_{\lambda_{k1}}$ (not necessarily nonnegative) has spectrum*

$$\Lambda_k = \{\lambda_{k1}, \lambda_{k2}, \ldots, \lambda_{kp_k}\}, \quad k = 1, 2, \ldots, r,$$

satisfying of Theorem 3. Let $\hat{\mathbf{q}}_k^T = (q_{k1}, \ldots, q_{kp_k})$ with $\hat{\mathbf{q}}_k^T \mathbf{e} = 0$, such that $A_k + \mathbf{e}\hat{\mathbf{q}}_k^T$ is nonnegative. Then $M = A + XC$ is nonnegative with spectrum

$$\{\mu_1, \ldots, \mu_r, \lambda_{12}, \ldots, \lambda_{1p_1}, \lambda_{22}, \ldots, \lambda_{2p_2}, \ldots, \lambda_{r2}, \ldots, \lambda_{rp_r}\},$$

where $\mu_1, \ldots, \mu_r$ are eigenvalues of $\Omega + CX$ with $\Omega = \operatorname{diag}(\lambda_{11}, \lambda_{21}, \ldots, \lambda_{r1})$.

Now we extend Theorem 3.

Theorem 4. *Let $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ be a list of complex numbers with $\Lambda = \overline{\Lambda}$, $\operatorname{Re}\lambda_j \geq 0$, $\lambda_1 \geq \max_j |\lambda_j|$, $j = 2, \ldots, n$. Let $\Lambda = \Lambda_1 \cup \cdots \cup \Lambda_{p_1+1}$ be a pairwise disjoint partition, with $\Lambda_k = \{\lambda_{k1}, \lambda_{k2}, \ldots, \lambda_{kp_k}\}$, $\lambda_{11} = \lambda_1$, $k = 1, \ldots, p_1 + 1$, where Λ_1 is realizable, p_1 is the number of elements of the list Λ_1 and some lists Λ_k can be empty. Let $\omega_2, \ldots, \omega_{p_1+1}$ be real numbers satisfying $0 \leq \omega_k \leq \lambda_1$, $k = 2, \ldots, p_1 + 1$. Suppose that*

(i) for each $k = 2, \ldots, p_1 + 1$, there exists a list $\Gamma_k = \{\omega_k, \lambda_{k1}, \ldots, \lambda_{kp_k}\}$ such that $\lambda_{k1} \geq \lambda_{k2} \geq \cdots \geq \lambda_{k\ell_k} \geq 0$ be real and $\lambda_{k(\ell_k+1)}, \ldots, \lambda_{kp_k}$ be complex with $\operatorname{Im}\lambda_{kj} \neq 0$ and $\operatorname{Re}\lambda_{kj} \geq 0$, $j = \ell_k + 1, \ldots, p_k$; with $\omega_k \geq \lambda_{k1} + m_k$, as in Theorem 3, where

$$m_k = \max_{\ell_k+1 \leq j \leq p_k} \left\{ m_{kj} = \operatorname{Re}\lambda_{kj} + \operatorname{Im}\lambda_{kj} + N(\Gamma_k) \right\},$$

(ii) there exists a $p_1 \times p_1$ nonnegative matrix B with spectrum Λ_1 and diagonal entries $\omega_2, \ldots, \omega_{p_1+1}$.

Then there exists an $n \times n$ nonnegative matrix $M \in \mathcal{CS}_{\lambda_1}$ with spectrum Λ and with prescribed elementary divisors.

Proof. Let

$$J_k = \begin{bmatrix} \omega_k & & & & & \\ & \ddots & & & & \\ & & \lambda_{ks} & & & \\ & & & C(a_{k(s+1)}, b_{k(s+1)}) & & \\ & & & & \ddots & \\ & & & & & C(a_{k(p_k-1)}, b_{k(p_k-1)}) \end{bmatrix}$$

with spectrum Γ_k and let E_k be an upper triangular matrix with zeros in the diagonal such that $J_k + E_k$ is the desired JCF. Let $S_k = \left[\mathbf{e}|\mathbf{e_2}|\ldots|\mathbf{e_{p_k}}\right]$. It is clear that the matrices J_k and S_k are of order p_k, $k = 1, \ldots, p_1 + 1$. Then $A_k = S_k(J_k + \varepsilon E_k)S_k^{-1} \in \mathcal{CS}_{\omega_k}$, $k = 2, \ldots, p_1 + 1$ and

$$A = \begin{bmatrix} A_2 & & & \\ & A_3 & & \\ & & \ddots & \\ & & & A_{p_1+1} \end{bmatrix}.$$

Notice that from Lemma 3 that for each $k = 2, \ldots, p_1 + 1$, we have that $A_k + \mathbf{e}_k\mathbf{q}_k^T$ is nonnegative for an appropriate vector $\mathbf{q}_k^T = (q_{k0}, q_{kp_1}, \ldots, q_{kp_k})$ with $\mathbf{q}_k^T\mathbf{e}_k = 0$. From Lemma 2, $A_k + \mathbf{e}_k\mathbf{q}_k^T$ has the prescribed elementary divisors. Then, we may apply Rado's perturbation as Lemma 3. Thus $A + XC$, is nonnegative with spectrum Λ, where X and $C = C_B + C_R$ are the matrices defined as in the proof of Lemma 2.1 in [15], and from Lemma 1 the matrix A has the required elementary divisors. □

Example 1. *Given the list $\Lambda = \{10, 2, 1 + 2i, 1 - 2i, 1 + 2i, 1 - 2i, 3i, -3i\}$, construct an 8×8 nonnegative matrix A with spectrum Λ and elementary divisors*

$$(\lambda - 10), (\lambda - 2), (\lambda^2 + 9), (\lambda - 1 - 2i)^2, (\lambda - 1 + 2i)^2.$$

We consider the partition $\Lambda = \Lambda_1 \cup \Lambda_2 \cup \Lambda_3 \cup \Lambda_4$ where

$$\Lambda_1 = \{10, 3i, -3i\}, \;\; \Lambda_2 = \{1 + 2i, 1 - 2i, 1 + 2i, 1 - 2i\}, \;\; \Lambda_3 = \{2\}, \;\; \Lambda_4 = \emptyset$$

$$\Gamma_2 = \{8, 1 + 2i, 1 - 2i, 1 + 2i, 1 - 2i\}, \;\; \Gamma_3 = \{2, 2\}, \;\; \Gamma_4 = \{0\}.$$

The matrix

$$B = \begin{bmatrix} 8 & 0 & 2 \\ \frac{73}{10} & 2 & \frac{3}{10} \\ 0 & 10 & 0 \end{bmatrix}$$

has spectrum Λ_1 *with diagonal estries* $8, 2, 0$. *Next, from Theorem 4 we obtain the nonnegative matrix*

$$A = \begin{bmatrix} 8 & 0 & 0 & 0 & 0 & & & \\ 4 & 1 & 2 & 1 & 0 & & & \\ 8 & -2 & 1 & 0 & 1 & & & \\ 5 & 0 & 0 & 1 & 2 & & & \\ 9 & 0 & 0 & -2 & 1 & & & \\ & & & & & 2 & 0 & \\ & & & & & 0 & 2 & \\ & & & & & & & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -4 & 2 & 0 & 2 & 0 & 0 & 0 & 2 \\ \frac{73}{10} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{7}{10} \\ 0 & 0 & 0 & 0 & 0 & 10 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 2 & 0 & 2 & 0 & 0 & 0 & 2 \\ 0 & 3 & 2 & 3 & 0 & 0 & 0 & 2 \\ 4 & 0 & 1 & 2 & 1 & 0 & 0 & 2 \\ 1 & 2 & 0 & 3 & 2 & 0 & 0 & 2 \\ 5 & 2 & 0 & 0 & 1 & 0 & 0 & 2 \\ \frac{73}{10} & 0 & 0 & 0 & 0 & 2 & 0 & \frac{7}{10} \\ \frac{73}{10} & 0 & 0 & 0 & 0 & 0 & 2 & \frac{7}{10} \\ 0 & 0 & 0 & 0 & 0 & 10 & 0 & 0 \end{bmatrix}$$

with the prescribed spectrum and elementary divisors.

4. Complex Eigenvalues Perturbation

The problem of perturbing real parts of a pair of complex eigenvalues of a realizable list, by keeping the realizability of the new list, was considered by Laffey in [22], and Guo and Guo in [23]. In [15], the authors showed that imaginary parts of a pair of complex eigenvalues from a realizable list of complex numbers $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ with $\Lambda = \overline{\Lambda}$ with $\operatorname{Re}\lambda_j \leq 0, |\operatorname{Re}\lambda_j| \geq \operatorname{Im}\lambda_j, j = 2, \ldots, n$ can also be perturbed and obtain a new list which is also realizable. In [15], the authors showed that it is also possible to perturb the imaginary parts of a pair of complex eigenvalue $a + bi, a - bi$ of a realizable list while keeping the realizability for a list in the complex left half plane. In this section, we present results of perturbation of real and/or imaginary parts of a pair of complex eigenvalues of a realizable list in the complex half right plane.

Theorem 5. *Let* $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ *be a realizable list of complex numbers with* $\operatorname{Re}\lambda_j \geq 0$, $j = 2, \ldots, n$ *and* $\lambda_1 > m$, *where* m *is defined in (3). Let* $a_p = \operatorname{Re}\lambda_p$ *and* $b_p = \operatorname{Im}\lambda_p \geq 0$, $2 \leq p \leq n-1$. *Then for all* $t \in \mathbb{R}$, *the perturbed lists*

$$\Lambda(a, t) = \{\lambda_1 + \varrho(t), \lambda_2, \ldots, \lambda_p, (a_p + t) + b_p i, (a_p + t) - b_p i, \lambda_{p+1}, \ldots, \lambda_n\},$$

where the function $\varrho(t)$ *is defined by*

$$\varrho(t) = \begin{cases} t, & t \geq 0 \\ -2t, & t < 0, \end{cases}$$

are also realizable.

Proof. Suppose that $\lambda_1, \lambda_2, \ldots, \lambda_{p-1}$ are real numbers and let $\lambda_p, \bar{\lambda}_p, \ldots, \lambda_{n-1}, \bar{\lambda}_{n-1}$ be complex numbers.

We define

$$J_{(a,t)} = \operatorname{diag}\left(\lambda_1, \ldots, \lambda_{p-1}, C_{(a,t)}(a_p, b_p), \ldots, C(a_n, b_n)\right) \tag{6}$$

with

$$C_{(a,t)}(a_p, b_p) = \begin{bmatrix} a_p + t & b_p \\ -b_p & a_p + t \end{bmatrix}.$$

Let

$$B_{(a,t)} = SJ_{(a,t)}S^{-1} \in \mathcal{CS}_{\lambda_1},$$

where $S = [\mathbf{e}|\mathbf{e}_2| \ldots |\mathbf{e}_n]$.

Since Λ is realizable and $\operatorname{Re}\lambda_j \geq 0$, $j = 2, \ldots, n$, we have $\sum\limits_{j=1}^{n} \lambda_j > 0$. For $t \geq 0$, we define the vector $q^T = (q_1, q_2, \ldots, q_n)$ with

$$\begin{aligned} q_1 &= -N(\Lambda) + \varrho(t), \\ q_j &= -\lambda_j, \ j = 2, \ldots, p-1, \\ q_p &= \operatorname{Im}\lambda_p, \quad q_{p+1} = 0, \\ q_\ell &= \operatorname{Im}\lambda_\ell, \quad q_{\ell+1} = 0, \ \ell = p+2, p+3, \ldots, n. \end{aligned}$$

Since $\lambda_1 > m$, where m is defined as in (3), $\mathbf{q}^T\mathbf{e} = \varrho(t)$ and $A = B_{(a,t)} + \mathbf{e}\mathbf{q}^T$ is a nonnegative matrix with spectrum $\Lambda(a,t)$. It is clear that if $t < 0$, choosing $q_1 = N(\Lambda)$, $q_p = \operatorname{Im}\lambda_p - t$ and $q_{p+1} = -t$, we obtain $\Lambda(a,t)$ is also realizable. □

Corollary 2. *Let $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ be a realizable list of complex numbers with $\operatorname{Re}\lambda_j \geq 0$, $j = 2, \ldots, n$, and $\lambda_1 > m$, where m is defined in (3). Let $a_p = \operatorname{Re}\lambda_p$ and $b_p = \operatorname{Im}\lambda_p$, $2 \leq p \leq n-1$. Then for all $t \in \mathbb{R}$, the perturbed list*

$$\Lambda(b,t) = \{\lambda_1 + \varrho(t), \lambda_2, \ldots, \lambda_p, a_p + (b_p + t)i, a_p - (b_p + t)i, \lambda_{p+1}, \ldots, \lambda_n\},$$

where the function $\varrho(t)$ is defined by

$$\varrho(t) = \begin{cases} 2t, & t \geq 0 \\ 0, & t < 0, \end{cases}$$

are also realizable.

The results below show how to perturb the real and/or imaginary parts of a pair of conjugate complex eigenvalues of a realizable list with the nonnegative real part, keeping the structure of the elementary divisors associated with the former eigenvalues.

Theorem 6. *Let $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ be a list of complex numbers with $\Lambda = \overline{\Lambda}, \lambda_1 \geq \max_j |\lambda_j|$, $j = 2, \ldots, n$, and $\operatorname{Re}\lambda_j \geq 0, j = 2, \ldots, n$. If Λ is realizable with prescribed elementary divisors*

$$(\lambda - \lambda_1), \ldots, (\lambda - \lambda_p)^{n_p}, (\lambda - \overline{\lambda}_p)^{n_p}, \ldots, (\lambda - \lambda_k)^{n_k}, \tag{7}$$

then for all $t > 0$ the list

$$\widetilde{\Lambda} = \{\widetilde{\lambda}_1, \lambda_2, \ldots, \lambda_{p-1}, \widetilde{\lambda}_p, \overline{\widetilde{\lambda}}_p, \lambda_{p+2}, \ldots, \lambda_n\}, \tag{8}$$

where $\widetilde{\lambda}_1 = \lambda_1 + t, \widetilde{\lambda}_p = \big(\operatorname{Re}\lambda_p + t\big) + i\operatorname{Im}\lambda_p$, is also realizable with elementary divisors

$$(\lambda - \widetilde{\lambda}_1), \ldots, (\lambda - \widetilde{\lambda}_p)^{n_p}, (\lambda - \overline{\widetilde{\lambda}}_p)^{n_p}, \ldots, (\lambda - \lambda_k)^{n_k}. \tag{9}$$

Proof. Let A be a nonnegative matrix with spectrum Λ and with the divisors elementary in (7). Since $\operatorname{Re}\lambda_j \geq 0$, $j = 2, \ldots, n$, then from Theorem 3, $\lambda_1 > m_j + N(\Lambda), j = 2, \ldots, n$. Let A_t be a nonnegative matrix with the spectrum $\widetilde{\Lambda}$ in (8) and with the elementary divisors in (9). Without loss generality suppose that $n_p = 2$ and it is enough to consider the following piece of A_t

$$\begin{bmatrix} \ddots & & & \\ & C(a_p+t,b_p) & \varepsilon I & \\ & & C(a_p+t,b_p) & \\ & & & \ddots \end{bmatrix},$$

with $a_p = \operatorname{Re}\lambda_p, b_p = \operatorname{Im}\lambda_p \geq 0$ and $\varepsilon > 0$ enough small. Then for all $t > 0$, we have from Theorem 3 $\widetilde{m_j} = m + t$. Therefore the new Perron eigenvalue is $\widetilde{\lambda}_1 = \lambda_1 + t$. $\square$

Corollary 3. *Let $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ be a list of complex numbers with $\Lambda = \overline{\Lambda}, \lambda_1 \geq \max_j |\lambda_j|,\ j = 2, \ldots, n,$ and $\operatorname{Re}\lambda_j \geq 0, j = 2, \ldots, n$. If Λ is realizable with prescribed elementary divisors*

$$(\lambda - \lambda_1), \ldots, (\lambda - \lambda_p)^{n_p}, (\lambda - \overline{\lambda}_p)^{n_p}, \ldots, (\lambda - \lambda_k)^{n_k}, \tag{10}$$

then for all $t > 0$ the list

$$\widetilde{\Lambda} = \{\widetilde{\lambda}_1, \lambda_2, \ldots, \lambda_{p-1}, \widetilde{\lambda}_p, \overline{\widetilde{\lambda}}_p, \lambda_{p+2}, \ldots, \lambda_n\}, \tag{11}$$

where $\widetilde{\lambda}_1 = \lambda_1 + \varrho(t), \widetilde{\lambda}_p = \operatorname{Re}\lambda_p + i(\operatorname{Im}\lambda_p + t)$ and the function ϱ is defined by

$$\varrho(t) = \begin{cases} 3t, & t \geq 0 \\ 2t, & t < 0, \end{cases}$$

is also realizable with elementary divisors

$$(\lambda - \widetilde{\lambda}_1), \ldots, (\lambda - \widetilde{\lambda}_p)^{n_p}, (\lambda - \overline{\widetilde{\lambda}}_p)^{n_p}, \ldots, (\lambda - \lambda_k)^{n_k}. \tag{12}$$

Remark 1. *It is clear that in the previous theorems we can perturb simultaneously two or more pairs of conjugate complex numbers, under the condition that we appropriately increase λ_1 as well.*

5. Conclusions

In this paper, we give new sufficient conditions for the nonnegative inverse elementary divisors problem for complex lists with nonnegative real parts. The results obtained provide procedures for the reconstruction of a matrix nonnegative from a list of given complex numbers and prescribed elementary divisors. We also give sufficient conditions for the perturbation of complex eigenvalues with nonnegative real parts of a nonnegative matrix keeping its elementary divisors and nonnegativity.

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