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Transitivity in Fuzzy Hyperspaces

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Abstract: Given a metric space (X,d), we deal with a classical problem in the theory of hyperspaces: how some important dynamical properties (namely, weakly mixing, transitivity and point-transitivity) between a discrete dynamical system $f:(X,d)\to(X,d)$ and its natural extension to the hyperspace are related. In this context, we consider the Zadeh's extension $\widehat{f}$ of f to $\mathcal{F}(X)$, the family of all normal fuzzy sets on X, i.e., the hyperspace $\mathcal{F}(X)$ of all upper semicontinuous fuzzy sets on X with compact supports and non-empty levels and we endow $\mathcal{F}(X)$ with different metrics: the supremum metric d_∞, the Skorokhod metric d_0, the sendograph metric d_S and the endograph metric d_E. Among other things, the following results are presented: (1) If (X,d) is a metric space, then the following conditions are equivalent: (a) (X,f) is weakly mixing, (b) $((\mathcal{F}(X),d_\infty),\widehat{f})$ is transitive, (c) $((\mathcal{F}(X),d_0),\widehat{f})$ is transitive and (d) $((\mathcal{F}(X),d_S)),\widehat{f})$ is transitive, (2) if $f:(X,d)\to(X,d)$ is a continuous function, then the following hold: (a) if $((\mathcal{F}(X),d_S),\widehat{f})$ is transitive, then $((\mathcal{F}(X),d_E),\widehat{f})$ is transitive, (b) if $((\mathcal{F}(X),d_S),\widehat{f})$ is transitive, then (X,f) is transitive; and (3) if (X,d) be a complete metric space, then the following conditions are equivalent: (a) $(X\times X,f\times f)$ is point-transitive and (b) $((\mathcal{F}(X),d_0)$ is point-transitive.

Keywords: fuzzy set; skorokhod metric; endograph metric; sendograph metric; Zadeh's extension; Transitivity; weakly mixing; point transitivity

1. Introduction

For a given metric space (X,d), let $\mathcal{F}(X)$ be the family of all normal fuzzy sets on X, i.e., all upper semicontinuous fuzzy sets on X with compact supports and non-empty levels. The hyperspace $\mathcal{F}(X)$ plays an important role in fuzzy theory. It appears, for example, in multi-point boundary value problems, topological entropy, fuzzy numbers, dynamical systems, properties of fuzzy mappings, chaos theory, etc. (see, among others, ref. [1–8]).

The aim of this paper was to study the relationship between several dynamical properties related to transitivity of a dynamical system $((X,d),f)$ and transitivity of the dynamical system $(\mathcal{F}(X),\widehat{f}))$ (where $\widehat{f}$ stands for the Zadeh's extension to $\mathcal{F}(X)$ of the function f) when the hyperspace $\mathcal{F}(X)$ is equipped with different metrics: the supremum metric d_∞, the Skorokhod metric d_0, the sendograph metric d_S and the endograph metric d_E. If the metric $d\in\{d_\infty,d_0,d_S,d_E\}$, we denote the metric space $(\mathcal{F}(X)),d)$ by $\mathcal{F}_\infty(X)$, $\mathcal{F}_0(X)$, $\mathcal{F}_S(X)$ and $\mathcal{F}_E(X)$, respectively.

It is worth noting that the space $\mathcal{F}_0(X)$ is relevant in the theory of fuzzy numbers and it is the least studied in the theory of fuzzified discrete dynamical systems. The Skorokhod topology was introduced by Skorokhod in [9] as an alternative to the topology of uniform convergence on the set $D[0,1]$ of right-continuous functions on $[0,1]$ having limits to the left at each $t\in(0,1]$. In [10],

Billingsley showed that the Skorokhod topology is metrizable, actually it proves that $D[0,1]$ endowed with the Skorokhod topology is a separable complete metric space. It plays an important role for the convergence of probability measures on $D[0,1]$, namely the convergence in distribution of stochastic processes with jumps: indeed, many central limit results and invariance principles were obtained (see [10,11]). Joo and Kim [12] introduced the Skorokhod metric in the field of fuzzy numbers which has been also studied in the context of $\mathcal{F}(\mathbb{R}^n)$ (see [13]). Given a metric space (X,d), the Skorokhod metric on $\mathcal{F}(X)$ was defined in [14]. The endograph (respectively, sendograph) metric is defined by means of the Hausdorff distance between the endographs (respectively, sendographs) of two normal fuzzy sets. The endograph metric has many applications in fuzzy theory. For example, it is used in fuzzy reference by fuzzy numbers defined on the unit interval (see [15]). It can be characterized by means of the notion of Γ-convergence (see [16] for details). To see the relationship between Γ-convergence and the sendograph metric, the interested reader can consult [17].

The paper is organized as follows. Preliminaries are given in Section 2. Section 3 is devoted to the main results of the paper. Among other things, we prove that a discrete dynamical system $((X,d),f)$ is weakly mixing if and only if the dynamical system $(\mathcal{F}_\infty(X),\widehat{f})$ is transitive if and only if the dynamical system $(\mathcal{F}_0(X),\widehat{f})$ is transitive if and only if $(\mathcal{F}_S(X),\widehat{f})$ is transitive (see Theorem 3). This theorem extends previous results of Banks [18] and Peris [19]. It is worth mentioning that the authors of [20] show the first two equivalences of Theorem 3 for compact metric spaces. In fact, they use compactness in the proof of ([20] Lemma 7). Theorem 3 also generalizes a result of [20]. We conclude the paper with a section of conclusions.

Motivation and Novelties

If $X \equiv (X,d)$ is a metric space and $f\colon X \to X$ is a continuous function, the pair (X,f) is called a discrete dynamical system. The evolution of (X,f) can be considered as the individual behavior of the phenomenon described by the system (X,f). The motivation for studying the discrete dynamical system defined by a hyperspace $\mathcal{H}(X)$ of X and a natural extension of f to $\mathcal{H}(X)$ is that, in this case, we focus our attention in the collective behavior. This includes the study of fractals sets on $\mathbb{R}^n$ (see [21]). In the case of considering Zadeh's extension $\widehat{f}$ of f to $\mathcal{F}(X)$, we compare the individual behavior with a fuzzy collective behavior. Thus, we related dynamical properties of (X,f) with fuzzy dynamical properties of $(\mathcal{F}(X),\widehat{f})$. Of course, this relationship depends on the metric we consider on $\mathcal{F}(X)$.

Thus, the topic we deal with is classical. The novelty lies in addressing the previous question for transitivity (respectively, point-transitivity) when not only $\mathcal{F}(X)$ is equipped with a metric $d \in \{d_\infty, d_0, d_S, d_E\}$, but we also compare the fuzzy collective behavior for several of the previous metrics.

2. Preliminaries

In this section, we introduce the results on fuzzy theory that we need in the sequel. A fuzzy set u on a topological space X is a function $u\colon X \to \mathbb{I}$, where $\mathbb{I}$ denotes the closed unit interval $[0,1]$. Define $u_\alpha = \{x \in X : u(x) \geq \alpha\}$ for each $\alpha \in (0,1]$. The support of u, denoted by u_0, is the set $\overline{\{x \in X : u(x) > 0\}}$. Let us note that $u_0 = \overline{\bigcup\{u_\alpha : \alpha \in (0,1]\}}$. Let $\mathcal{F}(X)$ be the family of all normal fuzzy sets on X, i.e., all upper semicontinuous fuzzy sets $u\colon X \to \mathbb{I}$ such that u_0 is compact and u_1 is non-empty.

Let (X,d) be a metric space. If $f\colon (X,d) \to (X,d)$ is a function, the Zadeh's extension of f to $\mathcal{F}(X)$ is denoted by $\widehat{f}\colon \mathcal{F}(X) \to \mathcal{F}(X)$ and is defined as follows:

$$\widehat{f}(u)(x) = \begin{cases} \sup\{u(z) : z \in f^{-1}(x)\}, & f^{-1}(x) \neq \emptyset. \\ 0, & f^{-1}(x) = \emptyset. \end{cases}$$

Two useful results on Zadeh's extension are the following:

Proposition 1 ([14]). *Let X be a Hausdorff space. If $f\colon X \to X$ is a continuous function, then $[\widehat{f}(u)]_\alpha = f(u_\alpha)$ for each $u \in \mathcal{F}(X)$ and $\alpha \in \mathbb{I}$.*

Proposition 2 ([22] (Proposition 2)). *If $f\colon (X,d) \to (X,d)$ is a continuous function, then $\left(\widehat{f}\right)^n = \widehat{f^n}$ for each $n \in \mathbb{N}$.*

In the sequel, the previous results allow us to write $\widehat{f^n}$ instead of $\left(\widehat{f}\right)^n$.

Given a non-empty subset $A \subseteq X$, we denote by $\chi_A\colon X \to \mathbb{I}$ the characteristic function of A. For the one-point set $\{x\}$, we put χ_x instead of $\chi_{\{x\}}$. If $\mathcal{K}(X)$ denotes the hyperspace of all the non-empty compact subsets of (X,d), we have the following propositions which shows that $\widehat{f}$ sends $\mathcal{K}(X)$ into itself.

Proposition 3 ([14]). *Let f be a continuous function from (X,d) into itself. Then $\widehat{f}(\chi_K) = \chi_{f(K)}$ for each $K \in \mathcal{K}(X)$.*

Next are some basic results on fuzzy metric hyperspaces. For $x \in X$ and $\epsilon > 0$, the symbol $B(x,\epsilon)$ denotes the open ball (with respect to d) with center at x and radius ϵ. The metrics we will consider on the hyperspace $\mathcal{F}(X)$ of all normal fuzzy subsets of X are related to the Hausdorff metric [23]. It is defined in the following way. If (X,d) is a metric space, let us denote by $\mathcal{C}(X)$ the set of all non-empty closed subsets of X.

For a given pair A, B of non-empty closed subsets of X, define $d(x,B) = \inf\{d(x,b) : b \in B\}$, for $x \in X$, and $H(A,B) = \sup\{d(a,B) : a \in A\}$. The Hausdorff distance d_H between A and B is defined by:

$$d_H(A,B) = \max\{H(A,B), H(B,A)\}.$$

Now we take up the metric d_∞. Consider the function $d_\infty\colon \mathcal{F}(X) \times \mathcal{F}(X) \to [0,\infty)$ defined by

$$d_\infty(u,v) = \sup_{\alpha\in\mathbb{I}}\{d_H(u_\alpha, v_\alpha)\}$$

where d_H is the Hausdorff metric on the hyperspace $\mathcal{K}(X)$. It is a well-known fact that d_∞ is a metric on $\mathcal{F}(X)$ such that $\mathcal{F}_\infty(X) \equiv (\mathcal{F}(X), d_\infty)$ is a nonseparable complete metric space. From now on, if $u \in \mathcal{F}(X)$ and $\epsilon > 0$, then the symbol $B_\infty(u,\epsilon)$ denotes the open ball (with respect to d_∞) with center at u and radius ϵ.

Next we introduce the Skorokhod metric. Denote by T the family of strictly increasing homeomorphisms from $\mathbb{I}$ onto itself. Given a metric space (X,d), we can define a metric on $\mathcal{F}(X)$ as follows:

$$d_0(u,v) = \inf\{\epsilon : \exists\, t \in T \text{ such that } \sup_{\alpha\in\mathbb{I}} |t(\alpha) - \alpha| \le \epsilon \text{ and } d_\infty(u, tv) \le \epsilon\}.$$

It is shown in [14] that d_0 is a metric on $\mathcal{F}(X)$, the so-called Skorokhod metric. For $u \in \mathcal{F}(X)$ and $\epsilon > 0$, the symbol $B_0(u,\epsilon)$ denotes the open ball, in $\mathcal{F}_0(X) \equiv (\mathcal{F}(X), d_0)$, with center at u and radius ϵ.

Clearly, $d_0(u,v) \le d_\infty(u,v)$ for each $u, v \in \mathcal{F}(X)$. Hence, the topology τ_0 induced by d_0 is weaker than the topology τ_∞ induced by d_∞, i.e., $\tau_0 \subseteq \tau_\infty$. However, for elements of $\mathcal{K}(X)$, we have the following easy proposition. The proof is left to the reader.

Proposition 4. *If $K \in \mathcal{K}(X)$, then $d_0(u,\chi_K) = d_\infty(u,\chi_K)$ for each $u \in \mathcal{F}(X)$.*

To finish the section, we introduce the sendograph and the endograph metric. For a given metric space (X,d), we define the metric $\overline{d}$ on the product $X \times \mathbb{I}$ as follows:

$$\overline{d}((x,a),(y,b)) = \max\{d(x,y), |a-b|\}.$$

Take now $u \in \mathcal{F}(X)$. The endograph of u is defined as the following set

$$end(u) = \{(x, \alpha) \in X \times \mathbb{I} : u(x) \geq \alpha\}$$

and the sendograph of u is defined as $send(u) = end(u) \cap (u_0 \times \mathbb{I})$.

The endograph metric d_E on $\mathcal{F}(X)$ is the Hausdorff distance $\overline{d}_H$ (with respect to $X \times \mathbb{I}$) between $end(u)$ and $end(v)$ for each $u, v \in \mathcal{F}(X)$, and the sendograph metric d_S on $\mathcal{F}(X)$ is the Hausdorff metric $\overline{d}_H$ (on $\mathcal{K}(X \times \mathbb{I})$) between the non-empty compact subsets $send(u)$ and $send(v)$ for every $u, v \in \mathcal{F}(X)$ (see [24]).

It is a well-known fact that $d_E \leq d_S \leq d_\infty$ (see [2]). Kloeden proved in [25] that $\mathcal{F}_S(X)$ is compact whenever X is compact. Thus, if τ_E and τ_S denote the topologies on $\mathcal{F}(X)$ induced by d_E and d_S, respectively, then we have that $\tau_E \subseteq \tau_S \subseteq \tau_\infty$. Moreover H. Huang proved in ([26] Theorem 7.1) that $\tau_S \subseteq \tau_0$.

3. Transitivity on $(\mathcal{F}(X), d_0)$

In this section, we mainly characterize the transitivity of the dynamical system $\widehat{f}\colon (\mathcal{F}(X), d_0) \to (\mathcal{F}(X), d_0)$ (see Theorem 3). Let X be a topological space and $f\colon X \to X$ a continuous function. Let us recall that a dynamical system (X, f) is transitive if for every non-empty open subsets U and V of X, there exists $n \in \mathbb{N}$ such that $f^n(U) \cap V \neq \emptyset$. We also say that (X, f) is weakly mixing if $f \times f\colon X \times X \to X \times X$ is transitive. Let us recall that $\overline{f}\colon \mathcal{K}(X) \to \mathcal{K}(X)$ is defined by $\overline{f}(K) = f(K)$ for each $K \in \mathcal{K}(X)$.

Let $f\colon X \to X$ be a continuous function on a topological space X. Banks [18] and Peris [19] showed that (X, f) is weakly mixing if and only if $(\mathcal{K}(X), \overline{f})$ is transitive. To be precise, they show the following

Theorem 1. *Let $f\colon X \to X$ be a continuous function on a topological space X. Then, the following conditions are equivalent:*

(1) *(X, f) is weakly mixing.*
(2) *$(\mathcal{K}(X), \overline{f})$ is weakly mixing.*
(3) *$(\mathcal{K}(X), \overline{f})$ is transitive.*

It is worth mentioning the following result on weakly mixing dynamical systems. Let X be a topological space and $f\colon X \to X$ a continuous function. A dynamical system (X, f) is weakly mixing of order m ($m \geq 2$) if the function

$$\underbrace{f \times \cdots \times f}_{m-times}\colon X^m \to X^m$$

is transitive. We have

Theorem 2 ([18] (Theorem 1))**.** *If $f\colon X \to X$ is continuous and weakly mixing, then f is weakly mixing of all orders.*

For a given topological space (X, τ), we need to consider the hyperspace $\mathcal{K}(X)$ endowed with the Vietoris topology τ_V. Let us remember that a base for τ_V is defined as follows:

$\langle U_1, \ldots, U_n \rangle = \{F \in \mathcal{K}(X) : F \subseteq \bigcup_{i=1}^n U_i \text{ and } F \cap U_k \neq \emptyset, \forall\, k = 1, \ldots, n\}$,

where $\{U_1, U_2, \ldots, U_n\}$ runs over all finite families of $\tau \setminus \{\emptyset\}$. It is known that for a metric space (X, d) the Vietoris topology coincides with the topology induced by the Hausdorff metric d_H on $\mathcal{K}(X)$.

Proposition 5 ([14])**.** *Let X be a Hausdorff space and $u \in \mathcal{F}(X)$. If $L\colon \mathbb{I} \to (\mathcal{K}(X), \tau_V)$ is the function defined by $L(\alpha) = u_\alpha$ for all $\alpha \in I$, then the following hold:*

(i) *L is left continuous on $(0,1]$;*
(ii) $\lim_{\lambda\to\alpha^+} L(\lambda) = \overline{\bigcup_{\beta>\alpha} u_\beta}$ *and* $\lim_{\lambda\to\alpha^+} L(\lambda) \subset u_\alpha$ *for each* $\alpha \in [0,1)$*;*
(iii) *L is right continuous at 0.*

Conversely, for any decreasing family $\{u_\alpha : \alpha \in \mathbb{I}\} \subseteq \mathcal{K}(X)$ satisfying (i)–(iii), there exists a unique $w \in \mathcal{F}(X)$ such that $w_\alpha = u_\alpha$ for every $\alpha \in \mathbb{I}$.

Let (X,d) be a metric space. For any $u \in \mathcal{F}(X)$ and $\alpha \in [0,1)$, define $u_{\alpha^+} = \lim_{\lambda\to\alpha^+} L(\lambda)$. It follows from ii) of previous proposition that L is right continuous at α if and only if $u_{\alpha^+} = u_\alpha$. The following fact is well known.

Proposition 6. *Let (X,d) be a metric space. If $A, B, C, F, G \in \mathcal{K}(X)$, then we have*

(i) *$A \subseteq B \subseteq C$ implies that $d_H(A,B) \leq d_H(A,C)$ and $d_H(B,C) \leq d_H(A,C)$.*
(ii) *if $d_H(A,F) \leq \epsilon$ and $d_H(B,G) \leq \epsilon$, then $d_H(A \cup B, F \cup G) \leq \epsilon$.*

We need the following two lemmas.The first can be proved as in [12] and the second follows easily from Proposition 6.

Lemma 1. *Suppose that (X,d) is a metric space. For any $u \in \mathcal{F}(X)$ and $\epsilon > 0$ there exist numbers $0 = \alpha_0 < \alpha_1 < \cdots < \alpha_n = 1$ such that $d_H(u_{\alpha_k^+}, u_{\alpha_{k+1}}) < \epsilon$ for $k = 0, 1, \ldots, n-1$.*

Lemma 2. *Suppose that (X,d) is a metric space. Take $u \in \mathcal{F}(X)$, $\epsilon > 0$ and a partition $0 = \alpha_0 < \alpha_1 < \cdots < \alpha_n = 1$ such that $d_H(u_{\alpha_k^+}, u_{\alpha_{k+1}}) < \epsilon$ for $k = 0, 1, \ldots, n-1$. If $0 = \beta_0 < \beta_1 < \cdots < \beta_m = 1$ is a refinement of $0 = \alpha_0 < \alpha_1 < \cdots < \alpha_n = 1$, then $d_H(u_{\beta_k^+}, u_{\beta_{k+1}}) < \epsilon$ for $k = 0, 1, \ldots, m-1$.*

We now are ready to present the main result of this paper:

Theorem 3. *Let (X,d) be a metric space. Then, the following conditions are equivalent:*

(i) *(X,f) is weakly mixing;*
(ii) *$(\mathcal{K}(X), \overline{f})$ is transitive;*
(iii) *$(\mathcal{F}_\infty(X), \widehat{f})$ is transitive;*
(iv) *$(\mathcal{F}_0(X), \widehat{f})$ is transitive;*
(v) *$(\mathcal{F}_S(X), \widehat{f})$ is transitive.*

Proof. By Theorem 1, we have that (i) implies (ii).

Let us show that (ii) implies (iii). Take $u, v \in \mathcal{F}(X)$ and $\epsilon, \delta > 0$. Put $U = B_\infty(u, \epsilon)$ and $V = B_\infty(v, \delta)$. By Lemma 1, there exist numbers $0 = \alpha_0 < \alpha_1 < \cdots < \alpha_n = 1$ such that $d_H(u_{\alpha_k^+}, u_{\alpha_{k+1}}) < \epsilon/3$ for $k = 0, 1, \ldots, n-1$. In addition, there exist numbers $0 = \beta_0 < \beta_1 < \cdots < \beta_m = 1$ such that $d_H(v_{\beta_k^+}, v_{\beta_{k+1}}) < \frac{\delta}{3}$ for $k = 0, 1, \ldots, m-1$. By Lemma 2, we can assume that $n = m$ and $\alpha_i = \beta_i$ for each $i = 0, 1, ..., n$.

We shall show that for every $k = 0, 1, \ldots, n-1$, we have the inequality

$$d_H(u_\beta, u_{\alpha_{k+1}}) < \epsilon/3 \text{ if } \beta \in (u_{\alpha_k}, u_{\alpha_{k+1}}]. \tag{1}$$

For this, notice that we have that $u_{\alpha_{k+1}} \subseteq u_\beta \subseteq u_{\alpha_k^+}$. Proposition 6 implies that

$$d_H(u_{\alpha_{k+1}}, u_\beta) \leq d_H(u_{\alpha_{k+1}}, u_{\alpha_k^+}) < \epsilon/3.$$

This shows the inequality (1).

Theorem 1 implies that $(\mathcal{K}(X), \overline{f})$ is weakly mixing. Theorem 2 tells us that $(\mathcal{K}(X), \overline{f})$ is weakly mixing of all orders. Therefore, there exist $m > 0$ and $K_1, K_2, ..., K_n, L_1, L_2, ..., L_n \in \mathcal{K}(X)$ such that for each $1 \leq i \leq n$ we have the following:

$$d_H(u_{\alpha_i}, K_i) < \epsilon/3, \tag{2}$$

$$d_H(v_{\alpha_i}, L_i) < \delta/3, \tag{3}$$

$$d_H(\overline{f}^m(K_i), L_i) < \delta/4. \tag{4}$$

Put $w_{\alpha_i} = \bigcup_{k \geq i} K_i$ for each $1 \leq i \leq n$. Proposition 6 and Inequality (2) imply that for every $1 \leq i \leq n$, we have

$$d_H(w_{\alpha_i}, u_{\alpha_i}) \leq \epsilon/3. \tag{5}$$

Let us define w_α for each $\alpha \in \mathbb{I}$ as follows:

$$w_\alpha = \begin{cases} w_{\alpha_1}, & 0 \leq \alpha \leq \alpha_1 \\ w_{\alpha_i}, & \alpha \in (\alpha_{i-1}, \alpha_i] \ \& \ i > 1 \end{cases}$$

The family $\{w_\alpha : \alpha \in \mathbb{I}\}$ satisfies conditions of Proposition 5. Hence, it determines an element $w \in \mathcal{F}(X)$. Let us show that $w \in U$. Take $\alpha \in \mathbb{I}$. Suppose that $\alpha \in [0, \alpha_1]$. Then $w_\alpha = w_{\alpha_1}$. Proposition 6 and the choice of $\alpha_0 = 0$ and α_1 imply that $d_H(u_\alpha, u_{\alpha_1}) < \epsilon/3$. The latter inequality and relation (5) give the following

$$d_H(w_\alpha, u_\alpha) \leq d_H(w_{\alpha_1}, u_{\alpha_1}) + d_H(u_{\alpha_1}, u_\alpha) < \frac{2}{3}\epsilon.$$

We now take $\alpha \in (\alpha_{i-1}, \alpha_i]$ for some $1 < i \leq n$. Inequalities (1) and (5) imply:

$$d_H(w_\alpha, u_\alpha) \leq d_H(w_{\alpha_i}, u_{\alpha_i}) + d_H(u_{\alpha_i}, u_\alpha) < \frac{2}{3}\epsilon.$$

We can conclude that $d_\infty(w, u) \leq \frac{2}{3}\epsilon < \epsilon$. Hence, $w \in U$.

Put $z_{\alpha_i} = \bigcup_{k \geq i} L_i$ for each $1 \leq i \leq n$. Let us define z_α for each $\alpha \in \mathbb{I}$ as

$$z_\alpha = \begin{cases} z_{\alpha_1}, & 0 \leq \alpha \leq \alpha_1, \\ z_{\alpha_i}, & \alpha \in (\alpha_{i-1}, \alpha_i] \ \& \ i > 1. \end{cases}$$

The family $\{z_\alpha : \alpha \in \mathbb{I}\}$ satisfies conditions of Proposition 5. Using Inequality 3, we can argue as in w to prove that $d_{\mathbb{I}}nfty(z, v) \leq \frac{2}{3}\delta$.

By Proposition 6 and Equation (4), we have that $d_H(\overline{f}^m(w_{\alpha_i}), z_{\alpha_i}) \leq \delta/4$ for every $i = 1, 2, ..., n$. Since $\overline{f}^m(w_{\alpha_i}) = f^m(w_{\alpha_i}) = [\widehat{f^m}(w)]_{\alpha_i}$, we conclude that $d_H([\widehat{f^m}(w)]_{\alpha_i}, z_{\alpha_i}) \leq \delta/4$ for each $i = 1, 2, ..., n$. Definitions of w and z imply that $d_H([\widehat{f^m}(w)]_\alpha, z_\alpha) \leq \delta/4$ for every $\alpha \in \mathbb{I}$. Therefore, $d_\infty(\widehat{f^m}(w), z) \leq \delta/4 < \delta/3$. Hence,

$$d_\infty(\widehat{f^m}(w), v) \leq d_\infty(\widehat{f^m}(w), z) + d_\infty(z, v) < \delta/3 + \frac{2}{3}\delta = \delta.$$

The latter inequality shows that $\widehat{f^m}(w) \in V$. We have thus proved that $\widehat{f^m}(w) \in \widehat{f^m}(U) \cap V$. Therefore, $(\mathcal{F}_\infty(X), \widehat{f})$ is transitive.

We have that iii) $\Rightarrow$ (iv) $\Rightarrow$ (v), since the topologies defined by levelwise, Skorokhod and sendograph metrics are related by $\tau_S(X) \subset \tau_0(X) \subset \tau_\infty(X)$.

Finally, let us prove that (v) implies (i). Suppose that $(\mathcal{F}_S(X), \widehat{f})$ is transitive. Take $K, L \in \mathcal{K}(X)$ and two positive real numbers ϵ and δ. Define $u = \chi_K$ and $v = \chi_L$, which clearly are elements of $\mathcal{F}(X)$. Recall that $d_H(p_0, q_0) \leq d_S(p, q)$ for each pair of fuzzy sets $p, q \in \mathcal{F}(X)$ (see [27]). From transitivity

of $(\mathcal{F}_S(X), \widehat{f})$, it follows the existence of $w \in B_S(u, \epsilon)$ and $n \in \mathbb{N}$ such that $\widehat{f}^n(w) \in B_S(v, \delta)$. Define $A = w_0$ and observe that $d_H(A, K) = d_H(w_0, u_0) \leq d_S(w, u) < \epsilon$ and $d_H(\overline{f^n}(A), L) = d_H(f^n(A), L) = d_H([f^n(w)]_0, v_0) \leq d_S(\widehat{f}^n(w), v) < \delta$. Therefore, $(\mathcal{K}(X), \overline{f})$ is transitive. Theorem 1 implies that (X, f) is weakly mixing. The proof is complete. □

We do not know if transitivity on $\mathcal{F}_E(X)$ implies transitivity on $\mathcal{K}(X)$. We have the next result.

Proposition 7. *Let (X, d) be a metric space and $f\colon (X, d) \to (X, d)$ a continuous function. Then the following holds:*

(i) *if $(\mathcal{F}_S(X), \widehat{f})$ is transitive, then $(\mathcal{F}_E(X), \widehat{f})$ is transitive;*
(ii) *if $(\mathcal{F}_S(X), \widehat{f})$ is transitive, then (X, f) is transitive.*

Proof. Let us show (i). Suppose that $(\mathcal{F}_S(X), \widehat{f})$ is transitive. Observe that $(\mathcal{F}_E(X), \widehat{f})$ is transitive because $\tau_E(X) \subset \tau_S(X)$.

In order to show (ii), take $x, y \in X$ and a pair of positive real numbers ϵ and δ. Define the fuzzy sets $u = \chi_x$ and $v = \chi_y$. Without losing generality, we can assume that $\epsilon < 1/2$ and $\delta < 1/2$. From transitivity of $(\mathcal{F}_E(X), \widehat{f})$, it follows the existence of $n \in \mathbb{N}$ and $w \in B_E(u, \epsilon)$ such that $\widehat{f}^n(w) \in B_E(v, \delta)$. Choose a point $z \in w_1$. It is easy to see that $d(z, x) < \epsilon$. Since $f^n(z) \in [\widehat{f}^n(w)]_1$ we have $d(f^n(z), y) \leq d_E(\widehat{f}^n(w), v) < \delta$ which finishes the proof. □

We now turn our attention to point-transitivity. A dynamical system (X, f) is point-transitive if there exists a point $x \in X$ with dense orbit, i.e., the set $\mathcal{O}_f(x) = \{x, f(x), f^2(x), ..., f^n(x), ...\}$ is dense in X.

Proposition 8. *If $f : (X, d) \to (X, d)$ is point-transitive, then $(\mathcal{F}_0(X), \widehat{f})$ is separable.*

Proof. The space (X, d) is separable, therefore so is $(\mathcal{F}_0(X), \widehat{f})$ (see [14] (Theorem 4.12)). □

Proposition 9. *If $(\mathcal{F}_0(X), \widehat{f})$ is point-transitive, then so is (X, f).*

Proof. Take $u \in \mathcal{F}(X)$ such that $\{\widehat{f^n}(u) : n \in \mathbb{N}\}$ is dense in $(\mathcal{F}(X), d_0)$. Pick $x \in u_0$. Let us show that $\{f^n(x) : n \in \mathbb{N}\}$ is dense in (X, d). Indeed, take $y \in X$ and $\epsilon > 0$. Then $\widehat{f^n}(u) \in B_0(\chi_y, \epsilon)$ for some $n \in \mathbb{N}$. So $d_0(\widehat{f^n}(u), \chi_y) < \epsilon$. Propositions 1, 2 and 4 imply

$$d(f^n(x), y) \leq d_H(f^n(u_0), \{y\}) = d_H([\widehat{f^n}(u)]_0, \{y\}) \leq$$

$$d_\infty(\widehat{f^n}(u), \chi_y) = d_0(\widehat{f^n}(u), \chi_y) < \epsilon.$$

It follows that $f^n(x) \in B(y, \epsilon)$. The proof is complete. □

It is known that point-transitivity is equivalent to transitivity for discrete dynamical systems on complete separable metric spaces without isolated points.

A space X is completely metrizable if it admits a compatible complete metric. It is well known that every completely metrizable space has Baire property. Let us recall that a space has Baire property if the intersection of a countable family of dense open sets is non-empty. According to ([8] Proposition 4.6), in every second-countable space with the Baire property, transitivity implies point-transitivity.

Theorem 4. *Let (X, d) be a complete metric space. Then the following conditions are equivalent:*

(i) *$(X \times X, f \times f)$ is point-transitive;*
(ii) *$(\mathcal{K}(X), \overline{f})$ is point-transitive;*
(iii) *$(\mathcal{F}_0(X), \widehat{f})$ is point-transitive.*

Proof. If $(X \times X, f \times f)$ is point-transitive, then $(X \times X, f \times f)$ is transitive, i.e., (X, f) is weakly mixing. By Theorem 3, $(\mathcal{K}(X), \overline{f})$ is transitive. Since (X, d) is complete separable, the metric space $(\mathcal{K}(X), d_H)$ is complete separable (see [28] Exercise 4.5.23 or [29]). Then, by [8] Proposition 4.6, $(\mathcal{K}(X), \overline{f})$ is point-transitive. Hence, (i) implies (ii).

Assume that $(\mathcal{K}(X), \overline{f})$ is point-transitive. Then $(\mathcal{K}(X), \overline{f})$ is transitive. Thus, Theorem 3 implies that $(\mathcal{F}_0(X), \widehat{f})$ is transitive. Since $(\mathcal{K}(X), \overline{f})$ is point-transitive, (X, d) is separable. By Theorem [14] Theorem 4.12, $\mathcal{F}_0(X)$ is separable. By hypothesis, (X, d) is complete so that arguing as in the proof of [12] Theorem 3.9, we can conclude that $\mathcal{F}_0(X)$ is completely metrizable. Once again, ref. [8] Proposition 4.6 implies that $(\mathcal{F}_0(X), \widehat{f})$ is point-transitive. We have just proved that (ii) $\Rightarrow$ (iii).

Finally, Proposition 9 says that (iii) $\Rightarrow$ (i). □

Example 1. *Consider* $S = \{z \in \mathbb{C} : \|z\| = 1\}$, $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ *and* $f : S \to S$ *defined by* $f(z) = ze^{i\pi\alpha}$. *It is known that* (S, f) *is transitive (point-transitive), but it is not weakly mixing. This shows that condition (i) in Theorem 3 and Theorem 4 cannot be replaced by transitivity and point-transitivity, respectively.*

4. Conclusions

We study the relationship between the individual behavior and fuzzy collective behavior for transitivity and point-transitivity. In others words, we study how transitivity and point-transitivity of a dynamical system $((X, d), f)$ and the dynamical system $(\mathcal{F}(X), \widehat{f})$, where $\widehat{f}$ is the Zadeh's extension of f and $\mathcal{F}(X)$ is the hyperspace of all normal fuzzy sets of X. We consider $\mathcal{F}(X)$ equipped with the supremum metric d_∞, the Skorokhod metric d_0, the sendograph metric d_S and the endograph metric d_E. Our main results state that transitivity of $(\mathcal{F}_\infty(X), \widehat{f})$ (respectively of $(\mathcal{F}_0(X), \widehat{f})$ and of $(\mathcal{F}_S(X), \widehat{f})$) is equivalent to the fact that $((X, d) \times (X, d), f \times f)$ is transitive. For point-transitivity, we obtain that for a complete metric space the following statements are equivalent: (a) $(X \times X, f \times f)$ is point-transitive, and (b) $(\mathcal{F}_0(X), \widehat{f})$ is point-transitive. Our results generalize previous outcomes in the theory of discrete dynamical systems.

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