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Article

New DNA Codes from Cyclic Codes over Mixed Alphabets

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Abstract: Let $R = \mathbb{F}_4 + u\mathbb{F}_4$, with $u^2 = u$ and $S = \mathbb{F}_4 + u\mathbb{F}_4 + v\mathbb{F}_4$, with $u^2 = u, v^2 = v, uv = vu = 0$. In this paper, we study $\mathbb{F}_4RS$-cyclic codes of block length (α, β, γ) and construct cyclic DNA codes from them. $\mathbb{F}_4RS$-cyclic codes can be viewed as $S[x]$-submodules of $\frac{\mathbb{F}_q[x]}{\langle x^\alpha - 1\rangle} \times \frac{R[x]}{\langle x^\beta - 1\rangle} \times \frac{S[x]}{\langle x^\gamma - 1\rangle}$. We discuss their generator polynomials as well as the structure of separable codes. Using the structure of separable codes, we study cyclic DNA codes. By using Gray maps ψ_1 from R to $\mathbb{F}_4^2$ and ψ_2 from S to $\mathbb{F}_4^3$, we give a one-to-one correspondence between DNA codons of the alphabets $\{A, T, G, C\}^2$, $\{A, T, G, C\}^3$ and the elements of R, S, respectively. Then we discuss necessary and sufficient conditions of cyclic codes over $\mathbb{F}_4$, R, S and $\mathbb{F}_4RS$ to be reversible and reverse-complement. As applications, we provide examples of new cyclic DNA codes constructed by our results.

Keywords: cyclic codes; reversible codes; reversible-complement codes; cyclic DNA codes

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1. Introduction

Linear codes were introduced in the late 1940s by Shannon and have a central role in Information Theory for recovering the corrupted messages that are sent through a noisy communication channel. Initially, linear codes were studied over finite fields, but in the early 1970s, these codes were discussed over finite rings. A great deal of attention was given to linear codes over finite rings from the 1990s because of their new role in algebraic coding theory and their rich applications. A ground-breaking work of Hammons et al. [1] showed that certain good binary nonlinear codes such as Preparata and Kerdock codes can be constructed from linear codes over $\mathbb{Z}_4$ via the Gray map. This motivated the study of linear codes over finite rings. Among algebraic linear codes, cyclic codes played an important role in coding theory, because of their easiness in practical implementations. As cyclic codes have rich algebraic structure, they can be efficiently encoded and decoded using shift registers, which explains their preferred role in engineering. These codes also have excellent error-correcting properties.

In 1997, RifÃ et al. [2] first introduced codes over mixed alphabets. After that, Brouwer et al. [3] addressed mixed binary/ternary codes and obtained the bounds for the maximum possible size of this family of codes. Since then, several scholars have focused extensively on mixed alphabets. In 2010, Borges et al. [4] proceeded to explore codes over mixed alphabets and discussed $\mathbb{Z}_2\mathbb{Z}_4$-additive codes. To get the structure of such codes, the coordinates were separated into two parts, the first part corresponds to the coordinates over $\mathbb{Z}_2$ and the second part corresponds to the coordinates over $\mathbb{Z}_4$. After that, many other generalizations of additive codes over mixed alphabets were discussed by several researchers [5,6]. In 2014, Abualrub et al. [7] studied the algebraic structure of $\mathbb{Z}_2\mathbb{Z}_4$-additive cyclic codes and determined their generator polynomials and minimal generating sets. They also studied the relationship between $\mathbb{Z}_2\mathbb{Z}_4$-additive cyclic codes and their duals. By following the approach of Abualrub et al. [7], many researchers have discussed the structure of cyclic and constacyclic codes over mixed alphabets and determined their generators as well as minimal generating sets [8–11].

Deoxyribonucleic acid (DNA) is a nucleic acid containing genetic information for the biological evolution of life. It contains information about how biological cells run and repair themselves. DNA strands can be viewed as sequences of four nucleotides: adenine (A), guanine (G), thymine (T), and cytosine (C). Two DNA strands are linked together with a rule named as Watson–Crick complement (WCC). According to WCC rule, every adenine (A) has a link with a thymine (T), and every guanine (G) has a link with cytosine (C), and vice versa. By WCC rule, we write $\overline{A} = T$, $\overline{T} = A$, $\overline{C} = G$ and $\overline{G} = C$. For example, if we have a DNA strand $x = (ATAGGC)$ then its complement $\overline{x} = (TATCCG)$.

In organisms called eukaryotes, DNA is located inside a special region of the cell known as the nucleus. As the cell is very small and organisms have many DNA molecules per cell, each DNA molecule must be tightly packaged. This packaged form of the DNA is called a chromosome. DNA code is made of chemical building blocks known as nucleotides. These building blocks are made of three parts: a sugar group, a phosphate group and one of four types of nitrogen bases. To form a DNA strand, nucleotides are linked into chains, with the phosphate and sugar groups alternating.

DNA code contains the instructions required for the growth, survival and reproduction of an organism. In order to perform these functions, DNA codes must be translated into messages that can be used to generate proteins, which are complex molecules that do much of the work in our bodies. Each DNA code that contains instructions for making a protein is called a gene. The size of a gene might vary significantly, ranging from about one thousand bases to one million bases in humans. Genes make up just about one percent of the DNA code. DNA codes outside of this one percent are involved in regulating when, how and how much protein is produced.

There is a two-step process to produce proteins from DNA code instructions. First, enzymes read the information in a DNA molecule and transcribe it into an intermediary molecule called messenger ribonucleic acid (mRNA). Second, the information presented in the mRNA molecule is converted into the language of amino acids, which are the building blocks of proteins. This language informs the cell's protein-making machinery of the exact order in which the amino acids are bound together to produce a specific protein. This is a big challenge since there are twenty types of amino acids that can be arranged in several different orders to form a wide variety of proteins.

DNA code is a genetic material and carries genetic information from cell to cell and generation to generation. DNA code is the director, which controls, regulates and determines the nature of proteins to be synthesized in a cell at a given time. The process of translation requires the transfer of genetic information from a polymer of nucleotides to a polymer of amino acids.

The changes in nucleic acid are responsible for changes in amino acids in the protein. This led to the proposition of a genetic code that could direct the sequence of amino acids during the synthesis of proteins. The group of nucleotides that specify one amino acid is known as a codon. The relationship between the sequence of amino acids in a polypeptide chain is called the genetic code. DNA code contains four kinds of nucleotides (A, T, G, C) and proteins are synthesized from different types of amino acids. In a singlet code, each base specifies one amino acid. Only four of the twenty types of amino acids could code. In doublet code, two bases specify one amino acid. A triplet code was suggested by Gamow in 1954. According to triplet code, three letters or bases specify one amino acid. Thus, 64 triplets of bases determine twenty amino acids. In triplet code, 64 codons are possible. Thus, each amino acid is coded by more than one codon.

Adleman [12] initiated DNA computing in 1994. In his work, Adleman solved an instance of an NP-complete problem by using DNA molecules. That discussion was dependent on the WCC property of DNA strands. Since then, numerous studies have built on their research and expanded DNA computing to solve other mathematical problems. For instance, Benenson et al. [13] solved the boolean satisfiability problem, Kari et al. [14] discussed the bounded post correspondence problem, an NP-complete problem, etc. Marathe et al. [15] announced four constraints, i.e., the Hamming constraint, the reverse constraint, the reverse-complement constraint, and the fixed GC-content constraint to study the DNA codes. The fixed GC-content assures that all codewords have identical thermodynamic characteristics and the rest three constraints were used to avoid the inadmissible hybridization between any two different strands.

The increasing complexity of computation and communication technology in the modern age leads us to the necessity of a new paradigm. As a result, tremendous attention is being paid to new methods such as DNA coding theory. DNA molecules' storage ability, processing of information and transmission properties stimulate both the notion of DNA coding theory and DNA cryptography. The structure of DNA strands has several applications in genetics and bioengineering. For example, biomolecular computation is used to design the DNA chips for mutational analysis. DNA strands are structured to hybridize each strand with its WCC sequence in a unique way and not to any other sequence. DNA strands concentrate on constructing giant sets of DNA codewords with the prescribed minimum Hamming distance. The structure of DNA is used as a model to construct good error-correcting codes, conversely error-correcting codes that have similar properties with DNA structure are also used to explain DNA itself.

Based on promising theoretical efficiency and preliminary implementations, DNA computing will have a lot of interest in the near future. Understanding and implementing this feature in the algebraic codes applied to communication is one of the researchers' objectives. DNA codes will be of great value since DNA computing is faster and can store more memory than silicon-based computing systems.

From the biological structure characteristic of DNA, we know that cyclic codes and reversible codes are analogous to DNA codes. As the concept of cyclic codes and reversible codes evolved, researchers showed their interest in the study of cyclic DNA codes.

To study DNA computing using the algebraic coding theory techniques, researchers studied error-correcting codes over finite fields and finite rings of cardinality 4^n by mapping the DNA nucleotides to the elements of finite fields and finite rings. After that, research on cyclic DNA codes and their generalizations evolved rapidly [16]. Abualrub et al. [17] discussed DNA codes over a field of four elements. In that article, authors have developed a theory for constructing linear and cyclic codes of the particular length over $GF(4)$ to study DNA computing. Siap et al. [18] discussed cyclic DNA codes over the finite ring $\mathbb{F}_2[u]/\langle u^2-1\rangle$. Guenda et al. [19] constructed DNA codes over the finite ring $\mathbb{F}_2+u\mathbb{F}_2(u^2=0)$ and constructed an infinite family of BCH DNA codes. Further, Liang et al. [20] also studied cyclic DNA codes over the same ring and discussed some necessary and sufficient conditions for reversible and reversible-complement codes. Yildiz et al. [21] considered cyclic DNA codes of odd length over the ring $\mathbb{F}_2[u]/\langle u^4-1\rangle$. In this article, 16 elements of this ring are matched with the set of paired DNA nucleotides and algebraic properties of cyclic DNA codes have been studied. After that, Bayram et al. [22] discussed DNA codes and their applications over the ring $\mathbb{F}_4+v\mathbb{F}_4(v^2=v)$. Zhu et al. [23] considered cyclic codes of an arbitrary length over a finite non-chain ring $\mathbb{F}_2[u,v]/\langle u^2, v^2-v, uv-vu\rangle$, where cyclic codes satisfying the reverse and reverse-complement constraints were discussed. Oztas et al. [24] constructed a new family of polynomials over a finite field $GF(16)$, which generates reversible codes over this field. More of studies of cyclic DNA codes over different rings can be found in [25–30].

In 2019, Diao et al. [31] discussed additive cyclic codes over mixed alphabets and their application in constructing quantum code. Recently, Dinh et al. [32] studied cyclic codes over mixed alphabets and the construction of LCD codes and quantum codes. However, according to our knowledge, there has not been any study of DNA codes over mixed alphabets. So motivated by the idea of the construction of LCD codes and quantum codes over mixed alphabets, in this paper, we study the construction of cyclic DNA codes over single alphabets. Then, we extend this study to mixed alphabets.

This paper is organized as follows: In Section 2, some definitions are studied and the structure of $\mathbb{F}_4RS$-cyclic codes is discussed. In Section 3, decomposed linear code structure over R and S is explored and an extended Gray map from $\mathbb{F}_4^{\alpha} \times R^{\beta} \times S^{\gamma}$ to $\mathbb{F}_4^{\alpha+2\beta+3\gamma}$ is described. In Section 4, the algebraic structure of $\mathbb{F}_4RS$-cyclic codes are discussed and their generator polynomials are determined. Further, we discuss the structure of the separable codes. In Section 5, as an application of our study, we discuss cyclic DNA codes. Section 5 is divided into four subsections: In Section 5.1, we present necessary and sufficient conditions for cyclic codes to be reversible and reversible-complement over $\mathbb{F}_4$. In Section 5.2, we define a one-to-one correspondence between the elements of the ring R and DNA codons $\{A, T, G, C\}^2$ and then necessary and sufficient conditions for cyclic codes to be reversible and reversible-complement over R are discussed. To illustrate our results, we present some examples and construct cyclic DNA codes. In Section 5.3, A one-to-one correspondence is also defined between the elements of the ring S and DNA codons $\{A, T, G, C\}^3$ and discuss similar results like the previous subsection. To support the results obtained in this subsection, we present some examples and construct cyclic DNA codes. In Section 5.4, we extend the results discussed in Sections 5.1–5.3, and discuss necessary and sufficient conditions for cyclic codes to be reversible and reversible-complement over $\mathbb{F}_4RS$. To illustrate the results discussed in this subsection, we present some examples. We conclude this paper in Section 6.

2. Preliminaries

Let $\mathfrak{R}$ be a finite commutative ring. A code $\mathfrak{C}$ of length $\mathfrak{n}$ over $\mathfrak{R}$ is defined as a non-empty subset of $\mathfrak{R}^{\mathfrak{n}}$. A code $\mathfrak{C}$ is called a linear code, if $\mathfrak{C}$ forms an $\mathfrak{R}$-submodule of $\mathfrak{R}^{\mathfrak{n}}$. A linear code $\mathfrak{C}$ of length $\mathfrak{n}$ over $\mathfrak{R}$ is called a cyclic code if $\mathbf{y} = (y_0, y_1, \dots, y_{\mathfrak{n}-1}) \in \mathfrak{C}$, then its cyclic shift $\tau(\mathbf{y}) := (y_{\mathfrak{n}-1}, y_0, \dots, y_{\mathfrak{n}-2}) \in \mathfrak{C}$. The elements of a code $\mathfrak{C}$ are called codewords.

We denote $\mathfrak{R}_{\mathfrak{n}} = \frac{\mathfrak{R}[x]}{\langle x^{\mathfrak{n}}-1\rangle}$. We define a map $\psi : \mathfrak{R}^{\mathfrak{n}} \longrightarrow \mathfrak{R}_{\mathfrak{n}}$ such that $\mathbf{w} = (w_0, w_1, \dots, w_{\mathfrak{n}-1}) \mapsto w_0 + w_1x + \cdots + w_{\mathfrak{n}-1}x^{\mathfrak{n}-1}$. It can be easily seen that the map ψ is a $\mathfrak{R}$-module isomorphism. By this identification, we can identify every codeword $\mathbf{w} = (w_0, w_1, \dots, w_{\mathfrak{n}-1}) \in \mathfrak{R}^{\mathfrak{n}}$ with a polynomial $w(x) = w_0 + w_1x + \cdots + w_{\mathfrak{n}-1}x^{\mathfrak{n}-1}$ in $\mathfrak{R}_{\mathfrak{n}}$. By this polynomial identification, we see that a linear code $\mathfrak{C}$ of length $\mathfrak{n}$ over $\mathfrak{R}$ is a cyclic code if and only if its corresponding polynomial representation forms an ideal of the ring $\frac{\mathfrak{R}[x]}{\langle x^{\mathfrak{n}}-1\rangle}$.

Let $\mathbf{y} = (y_0, y_1, \dots, y_{\mathfrak{n}-1}) \in \mathfrak{C}$, then the Hamming weight $w_H(\mathbf{y})$ of $\mathbf{y}$ is defined as the number of non-zero coordinates of $\mathbf{y}$. The minimum Hamming weight of a linear code $\mathfrak{C}$ is denoted by $w_H(\mathfrak{C})$, and defined as $w_H(\mathfrak{C})$=min$\{w_H(\mathbf{y}) \mid \mathbf{y} \in \mathfrak{C}, \mathbf{y} \neq 0\}$. Let $\mathbf{y} = (y_0, y_1, \dots, y_{\mathfrak{n}-1})$ and $\mathbf{y}' = (y'_0, y'_1, \dots, y'_{\mathfrak{n}-1}) \in R^{\mathfrak{n}}$, then the Hamming distance between $\mathbf{y}$ and $\mathbf{y}'$ is defined as $d_H(\mathbf{y}, \mathbf{y}') = |\{i \mid y_i \neq y'_i\}|$ such that, $d_H(\mathbf{y}, \mathbf{y}') = w_H(\mathbf{y} - \mathbf{y}')$. The minimum Hamming distance of a linear code $\mathfrak{C}$ is defined as $d_H(\mathfrak{C})$=min$\{d_H(\mathbf{y}, \mathbf{y}') \mid \mathbf{y} \neq \mathbf{y}'\}$.

We define the Euclidean inner product between $\mathbf{y}$ and $\mathbf{y}'$ in $\mathfrak{R}^{\mathfrak{n}}$ as $\mathbf{y} \cdot \mathbf{y}' = y_0y'_0 + y_1y'_1 + \cdots + y_{\mathfrak{n}-1}y'_{\mathfrak{n}-1}$. The dual code of $\mathfrak{C}$ is defined as $\mathfrak{C}^{\perp} = \{\mathbf{y} \in \mathfrak{R}^{\mathfrak{n}} \mid \mathbf{y} \cdot \mathbf{y}' = 0, \forall\, \mathbf{y}' \in \mathfrak{C}\}$.

Now we extend the discussion of cyclic codes over single alphabets to the mixed alphabets.

Throughout this paper, we denote by $\mathbb{F}_4$ the finite field of order 4 given by $\mathbb{F}_4 = \{0, 1, w, w^2 = 1 + w\}$, where $1 + w + w^2 = 0$ and $R = \mathbb{F}_4 + u\mathbb{F}_4$, with $u^2 = u$ and $S = \mathbb{F}_4 + u\mathbb{F}_4 + v\mathbb{F}_4$, with $u^2 = u, v^2 = v, uv = vu = 0$. Consider $S_{\alpha,\beta,\gamma} = \frac{\mathbb{F}_4[x]}{\langle x^{\alpha}-1\rangle} \times \frac{R[x]}{\langle x^{\beta}-1\rangle} \times \frac{S[x]}{\langle x^{\gamma}-1\rangle}$. We define the set

$$\mathbb{F}_4RS = \{(m_1, m_2, m_3) \mid m_1 \in \mathbb{F}_4,\ m_2 \in R,\ m_3 \in S\}.$$

The set $\mathbb{F}_4RS$ forms a ring under the componentwise addition and multiplication. Consider an element $d = a + ub + vc \in S$, we define $\rho_1 : S \to \mathbb{F}_4$ such that $\rho_1(d) = a$ and $\rho_2 : S \to R$ such that $\rho_2(d) = a + ub$. We can see that both ρ_1 and ρ_2 are ring homomorphisms. For any $d \in S$, we define the S-scalar multiplication on $\mathbb{F}_4RS$ as follows.

$$\bullet : S \times \mathbb{F}_4RS \to \mathbb{F}_4RS$$

such that

$$d \bullet (m_1, m_2, m_3) = (\rho_1(d)m_1, \rho_2(d)m_2, dm_3).$$

This multiplication can be extend componentwise on $\mathbb{F}_4^\alpha \times R^\beta \times S^\gamma$ as $\bullet : S \times (\mathbb{F}_4^\alpha \times R^\beta \times S^\gamma) \to \mathbb{F}_4^\alpha \times R^\beta \times S^\gamma$ such that

$$d \bullet \mathbf{c} = (\rho_1(d)a_0, \rho_1(d)a_1, \dots, \rho_1(d)a_{\alpha-1}, \rho_2(d)b_0, \rho_2(d)b_1, \dots, \rho_2(d)b_{\beta-1}, dc_0, dc_1, \dots, dc_{\gamma-1}),$$

for any $d \in S$ and $\mathbf{c} = (a_0, a_1, \dots, a_{\alpha-1}, b_0, b_1, \dots, b_{\beta-1}, c_0, c_1, \dots, c_{\gamma-1}) \in \mathbb{F}_q^\alpha \times R^\beta \times S^\gamma$. By this S-scalar multiplication, we can see that $\mathbb{F}_q^\alpha \times R^\beta \times S^\gamma$ forms an S-module.

Now we present the definition of linear codes and constacyclic codes over mixed alphabets.

Definition 1. *A non-empty subset C of $\mathbb{F}_4^\alpha \times R^\beta \times S^\gamma$ is called a $\mathbb{F}_4RS$-linear code of block length (α, β, γ) if C is an S-submodule of $\mathbb{F}_4^\alpha \times R^\beta \times S^\gamma$.*

Let $\mathbf{c} = (a_0, a_1, \dots, a_{\alpha-1}, b_0, b_1, \dots, b_{\beta-1}, c_0, c_1, \dots, c_{\gamma-1})$ and $\mathbf{c}' = (a'_0, a'_1, \dots, a'_{\alpha-1}, b'_0, b'_1, \dots, b'_{\beta-1}, c'_0, c'_1, \dots, c'_{\gamma-1})$ be any two elements of $\mathbb{F}_4^\alpha \times R^\beta \times S^\gamma$. Then the inner product is defined as

$$\mathbf{c} \cdot \mathbf{c}' = u \sum_{i=0}^{\alpha-1} a_i a'_i + v \sum_{j=0}^{\beta-1} b_j b'_j + \sum_{k=0}^{\gamma-1} c_k c'_k \in S.$$

We define the dual of a $\mathbb{F}_4RS$-linear code as follows.

Definition 2. *If C is a $\mathbb{F}_4RS$-linear code of block length (α, β, γ), then its dual code $C^\perp$ is defined as*

$$C^\perp = \{\mathbf{c} \in \mathbb{F}_4^\alpha \times R^\beta \times S^\gamma \mid \mathbf{c} \cdot \mathbf{c}' = 0, \forall\, \mathbf{c}' \in C\}.$$

C is called self-dual if $C^\perp = C$ and self-orthogonal if $C \subseteq C^\perp$.

Definition 3. *A $\mathbb{F}_4RS$-linear code C of block length (α, β, γ) is called a $\mathbb{F}_4RS$-cyclic code if for any $\mathbf{c} = (a_0, a_1, \dots, a_{\alpha-1}, b_0, b_1, \dots, b_{\beta-1}, c_0, c_1, \dots, c_{\gamma-1}) \in C$, its cyclic shift $\rho(\mathbf{c}) := (a_{\alpha-1}, a_0, a_1, \dots, a_{\alpha-2}, b_{\beta-1}, b_0, b_1, \dots, b_{\beta-2}, c_{\gamma-1}, c_0, c_1, \dots, c_{\gamma-2}) \in C$.*

Now we present a relation between $\mathbb{F}_4RS$-cyclic codes and their duals.

Lemma 1. *Let C be a $\mathbb{F}_4RS$-cyclic code of block length (α, β, γ). Then its dual $C^\perp$ is also a $\mathbb{F}_4RS$-cyclic code.*

Proof. The proof is similar to the [32] (Proposition 3). □

Consider an element $\mathbf{m}' = (r'_0, r'_1, \dots, r'_{\alpha-1}, s'_0, s'_1, \dots, s'_{\beta-1}, t'_0, t'_1, \dots, t'_{\gamma-1}) \in \mathbb{F}_q^\alpha \times R^\beta \times S^\gamma$. This element can be identified with an element of $S_{\alpha,\beta,\gamma}$ as

$$m'(x) = (r'_0 + r'_1 x + \dots + r'_{\alpha-1} x^{\alpha-1}, s'_0 + s'_1 x + \dots + s'_{\beta-1} x^{\beta-1}, t'_0 + t'_1 x + \dots + t'_{\gamma-1} x^{\gamma-1}).$$

For convenience, we denote $m'(x) = (r'(x), s'(x), t'(x))$. This identification gives us a one-to-one correspondence between the elements of $\mathbb{F}_4^\alpha \times R^\beta \times S^\gamma$ and $S_{\alpha,\beta,\gamma}$. The multiplication of any element $e(x) = e_0 + e_1 x + \dots + e_{t-1} x^{t-1} \in S[x]$ with the element $(r'(x), s'(x), t'(x)) \in S_{\alpha,\beta,\gamma}$ is defined as follows

$$e(x) \star (r'(x), s'(x), t'(x)) = (\rho_1(e(x)) r'(x), \rho_2(e(x)) s'(x), e(x) t'(x)),$$

where $\rho_1(e(x)) = \rho_1(e_0) + \rho_1(e_1)x + \cdots + \rho_1(e_{t-1})x^{t-1}$ and $\rho_2(e(x)) = \rho_2(e_0) + \rho_2(e_1)x + \cdots + \rho_2(e_{t-1})x^{t-1}$. It can be seen that $S_{\alpha,\beta,\gamma}$ forms an $S[x]$-module with respect to the usual addition and multiplication $\star$.

For any $m'(x) = (r'_0 + r'_1x + \cdots + r'_{\alpha-1}x^{\alpha-1}, s'_0 + s'_1x + \cdots + s'_{\beta-1}x^{\beta-1}, t'_0 + t'_1x + \cdots + t'_{\gamma-1}x^{\gamma-1}) \in S_{\alpha,\beta,\gamma}$, we get $x \star m'(x) = (r'_{\alpha-1} + r'_0x + \cdots + r'_{\alpha-2}x^{\alpha-1}, s'_{\beta-1} + s'_0x + \cdots + s'_{\beta-2}x^{\beta-1}, t'_{\gamma-1} + t'_0x + \cdots + t'_{\gamma-2}x^{\gamma-1})$. Therefore, $x \star m'(x)$ corresponds to the element $(r'_{\alpha-1}, r'_0, \ldots, r'_{\alpha-2}, s'_{\beta-1}, s'_0, \ldots, s'_{\beta-2}, t'_{\gamma-1}, t'_0, \ldots, t'_{\gamma-2}) \in \mathbb{F}_4^\alpha \times R^\beta \times S^\gamma$. This implies that $x \star m'(x)$ is a cyclic shift of the corresponding vector of $m'(x)$. This argument takes us to the result below.

Theorem 1. *A linear code C is called a $\mathbb{F}_4RS$-cyclic code of block length (α, β, γ) if and only if C is an $S[x]$-submodule of $S_{\alpha,\beta,\gamma}$.*

Proof. The proof is similar to the [32] (Proposition 4). □

3. Linear Codes over R and S

In this section, we present a idempotent decomposition of the rings R and S, then we discuss the form of the linear codes from this decomposition. Furthermore, we define a Gray map on $\mathbb{F}_4^\alpha \times R^\beta \times S^\gamma$ and study some basic properties of this Gray map.

Let D_1, D_2 be any two codes such that D_1 and D_2 intersect trivially. We denote $D_1 \oplus D_2 = \{d_1 + d_2 \mid d_i \in D_i,\ i = 1, 2\}$.

Any arbitrary element $r = r_1 + ur_2 \in R$ can be written in the form $r = r_1 + ur_2 = \xi_1\hat{r}_1 + \xi_2\hat{r}_2$, where $r_1, \hat{r}_1, r_2, \hat{r}_2 \in \mathbb{F}_4$ such that $\hat{r}_1 = r_1,\ \hat{r}_2 = r_1 + r_2$ and

$$\xi_1 = 1 - u, \quad \xi_2 = u.$$

We see that $\xi_i^2 = \xi_i,\ \xi_i\xi_j = 0$ and $\xi_1 + \xi_2 = 1$, for $i, j = 1, 2;\ i \neq j$. Therefore, we get $R = \xi_1R \oplus \xi_2R$, and we note that any element $r \in R$ can be uniquely written as $r = \xi_1a + \xi_2b$, where $a, b \in \mathbb{F}_4$.

Now a Gray map on R is defined as follows.

$$\psi_1 : R \to \mathbb{F}_4^2$$

given by

$$\psi_1(r) = (a, b).$$

This map can be extended from R^β to $\mathbb{F}_4^{2\beta}$ as

$$\psi_1 : R^\beta \to \mathbb{F}_4^{2\beta},$$

given by

$$(a_0, a_1, \ldots, a_{\beta-1}) \mapsto (a_{0,1}, a_{1,1}, \ldots, a_{\beta-1,1}, b_{0,2}, b_{1,2}, \ldots, b_{\beta-1,2}),$$

where $\mathbf{r} = (a_0, a_1, \ldots, a_{\beta-1}) \in R^\beta$ and $a_j = \xi_1a_{j,1} + \xi_2b_{j,2}$ for $j = 0, 1, \ldots, \beta - 1$. For any $a_j = \xi_1a_{j,1} + \xi_2b_{j,2} \in R$, we define the Lee weight of a_j as $w_L(a_j) = w_H(\phi_1(a_j))$, where w_H denotes the Hamming weight over $\mathbb{F}_4$. Further, the Lee distance between any two elements $\mathbf{r} = (a_0, a_1, \ldots, a_{\beta-1})$ and $\mathbf{r}' = (a'_0, a'_1, \ldots, a'_{\beta-1}) \in R^\beta$ is defined as $d_L(\mathbf{r}, \mathbf{r}') = w_L(\mathbf{r} - \mathbf{r}') = w_H(\phi_1(\mathbf{r} - \mathbf{r}'))$. It can be easily seen that the Gray map ψ_1 is a $\mathbb{F}_4$-linear distance preserving map from R^β (Lee distance) to $\mathbb{F}_4^{2\beta}$ (Hamming distance).

Suppose we have a linear code C_β of length β over R. Then we define

$$C_{\beta,1} = \{\mathbf{a} \in \mathbb{F}_4^\beta \mid \xi_1\mathbf{a} + \xi_2\mathbf{b} \in C_\beta \text{ for some } \mathbf{b} \in \mathbb{F}_4^\beta\},$$
$$C_{\beta,2} = \{\mathbf{b} \in \mathbb{F}_4^\beta \mid \xi_1\mathbf{a} + \xi_2\mathbf{b} \in C_\beta \text{ for some } \mathbf{a} \in \mathbb{F}_4^\beta\}.$$

Thus, $C_{\beta,1}$ and $C_{\beta,2}$ are linear codes of length β over $\mathbb{F}_4$. Hence, we get that C_β can be uniquely written as $C_\beta = \xi_1 C_{\beta,1} \oplus \xi_2 C_{\beta,2}$ and $|C_\beta| = |C_{\beta,1}||C_{\beta,2}|$.

In the above discussion, we have studied the structure of linear codes above R. Now we discuss linear codes over S in a similar fashion.

Any element $t = s_1 + us_2 + vs_3 \in S$ can be written in the form $t = s_1 + us_2 + vs_3 = \eta_1\hat{s}_1 + \eta_2\hat{s}_2 + \eta_3\hat{s}_3$, where $s_1, \hat{s}_1, s_2, \hat{s}_2, s_3, \hat{s}_3 \in \mathbb{F}_4$ such that $\hat{s}_1 = s_1,\ \hat{s}_2 = s_1 + s_2,\ \hat{s}_3 = s_1 + s_3$ and

$$\eta_1 = 1 - u - v, \quad \eta_2 = u, \quad \eta_3 = v.$$

We see that $\eta_i^2 = \eta_i, \quad \eta_i\eta_j = 0$ and $\sum_i^3 \eta_i = 1$, for $i, j = 1,2,3;\ i \neq j$. Therefore, $S = \eta_1 S \oplus \eta_2 S \oplus \eta_3 S$, and we note that any element $t \in S$ can be uniquely written as $t = \eta_1 t_1 + \eta_2 t_2 + \eta_3 t_3$, where $t_1, t_2, t_3 \in \mathbb{F}_4$.

Similar as above, we define a Gray map on S as follows.

$$\psi_2 : S \to \mathbb{F}_4^3$$

given by

$$\psi_2(t) = (t_1, t_2, t_3).$$

This map can be extended from S^γ to $\mathbb{F}_4^{3\gamma}$ as

$$\psi_2 : S^\gamma \to \mathbb{F}_4^{3\gamma},$$

given by

$$(t_0, t_1, \ldots, t_{\gamma-1}) \mapsto (t_{0,1}, t_{1,1}, \ldots, t_{\gamma-1,1}, t_{0,2}, t_{1,2}, \ldots, t_{\gamma-1,2}, t_{0,3}, t_{1,3}, \ldots, t_{\gamma-1,3}),$$

where $\mathbf{t} = (t_0, t_1, \ldots, t_{\gamma-1}) \in S^\gamma$ and $t_i = \eta_1 t_{i,1} + \eta_2 t_{i,2} + \eta_3 t_{i,3}$ for $i = 0, 1, \ldots, \gamma - 1$. For any $t_i = \eta_1 t_{i,1} + \eta_2 t_{i,2} + \eta_3 t_{i,3} \in S$, the Lee weight of t_i is defined as $w_L(t_i) = w_H(\phi_2(t_i))$. Further, we define the Lee distance between any two elements $\mathbf{t} = (t_0, t_1, \ldots, t_{\gamma-1})$ and $\mathbf{t}' \in S^\gamma$ as $d_L(\mathbf{t}, \mathbf{t}') = w_L(\mathbf{t} - \mathbf{t}') = w_H(\phi_2(\mathbf{t} - \mathbf{t}'))$. It can be easily seen that the Gray map ψ_2 is a $\mathbb{F}_4$-linear distance preserving map from S^γ (Lee distance) to $\mathbb{F}_4^{3\gamma}$ (Hamming distance).

Suppose we have a linear code C_γ of length γ over S. Then we define

$$C_{\gamma,1} = \{\mathbf{t}_1 \in \mathbb{F}_4^\gamma \mid \eta_1\mathbf{t}_1 + \eta_2\mathbf{t}_2 + \eta_3\mathbf{t}_3 \in C_\gamma \text{ for some } \mathbf{t}_2, \mathbf{t}_3 \in \mathbb{F}_4^\gamma\},$$
$$C_{\gamma,2} = \{\mathbf{t}_2 \in \mathbb{F}_4^\gamma \mid \eta_1\mathbf{t}_1 + \eta_2\mathbf{t}_2 + \eta_3\mathbf{t}_3 \in C_\gamma \text{ for some } \mathbf{t}_1, \mathbf{t}_3 \in \mathbb{F}_4^\gamma\},$$
$$C_{\gamma,3} = \{\mathbf{t}_3 \in \mathbb{F}_4^\gamma \mid \eta_1\mathbf{t}_1 + \eta_2\mathbf{t}_2 + \eta_3\mathbf{t}_3 \in C_\gamma \text{ for some } \mathbf{t}_1, \mathbf{t}_2 \in \mathbb{F}_4^\gamma\}.$$

Similar as above, $C_{\gamma,1}, C_{\gamma,2}$ and $C_{\gamma,3}$ are linear codes of length γ over $\mathbb{F}_4$. Hence, we get that C_γ can be uniquely written as $C_\gamma = \eta_1 C_{\gamma,1} \oplus \eta_2 C_{\gamma,2} \oplus \eta_3 C_{\gamma,3}$ and $|C_\gamma| = |C_{\gamma,1}||C_{\gamma,2}||C_{\gamma,3}|$.

We have studied the structure of linear codes over R and S in our above discussion. We also have defined Gray maps ψ_1 and ψ_2 on R^β and S^γ, respectively. Using these maps, we now define a Gray map on $\mathbb{F}_4^\alpha \times R^\beta \times S^\gamma$.

Any element $(m, r, t) \in \mathbb{F}_4 RS$ can be expressed as $(m, r, t) = (m, \xi_1 a + \xi_2 b, \eta_1 t_1 + \eta_2 t_2 + \eta_3 t_3)$. We define a Gray map from $\mathbb{F}_4 RS$ to $\mathbb{F}_q^6$ as

$$\Psi : \mathbb{F}_4 RS \to \mathbb{F}_4^6,$$

given by

$$\Psi(m,r,t) = (m, \psi_1(r), \psi_2(t)),$$

such that

$$(m,r,t) \mapsto (m,a,b,t_1,t_2,t_3).$$

We can see that the map Ψ is a $\mathbb{F}_4$-linear map and this can be extended on $\mathbb{F}_4^\alpha \times R^\beta \times S^\gamma$ as follows.

$$\Psi : \mathbb{F}_4^\alpha \times R^\beta \times S^\gamma \longrightarrow \mathbb{F}_4^{\alpha+2\beta+3\gamma}$$

given by

$$\begin{aligned}
&(m_0, m_1, \dots, m_{\alpha-1}, a_0, a_1, \dots, a_{\beta-1}, t_0, t_1, \dots, t_{\gamma-1}) \\
&\mapsto (m_0, m_1, \dots, m_{\alpha-1}, a_{0,1}, a_{1,1}, \dots, a_{\beta-1,1}, \\
&\quad b_{0,2}, b_{1,2}, \dots, b_{\beta-1,2}, t_{0,1}, t_{1,1}, \dots, t_{\gamma-1,1}, \\
&\quad t_{0,2}, t_{1,2}, \dots, t_{\gamma-1,2}, t_{0,3}, t_{1,3}, \dots, t_{\gamma-1,3}),
\end{aligned}$$

where $(m_0, m_1, \dots, m_{\alpha-1}) \in \mathbb{F}_4^\alpha$, $(a_0, a_1, \dots, a_{\beta-1}) \in R^\beta$, $(t_0, t_1, \dots, t_{\gamma-1}) \in S^\gamma$, and $a_j = \xi_1 a_{j,1} + \xi_2 b_{j,2} \in R$ and $t_i = \eta_1 t_{i,1} + \eta_2 t_{i,2} + \eta_3 t_{i,3} \in S$ for $j = 0, 1, \dots, \beta-1$ and $i = 0, 1, \dots, \gamma-1$.

By the same argument given in [33], the Lee weight of any element $(\mathbf{m}, \mathbf{r}, \mathbf{t}) \in \mathbb{F}_4^\alpha \times R^\beta \times S^\gamma$ is defined as $w_L(\mathbf{m}, \mathbf{r}, \mathbf{t}) = w_H(\mathbf{m}) + w_L(\mathbf{r}) + w_L(\mathbf{t})$, where w_H denotes the Hamming weight over $\mathbb{F}_4$ and w_L denotes the Lee weight. Further, we define the Lee distance between any two elements $\mathbf{e}_1$ and $\mathbf{e}_2 \in \mathbb{F}_4^\alpha \times R^\beta \times S^\gamma$ as $d_L(\mathbf{e}_1, \mathbf{e}_2) = w_L(\mathbf{e}_1 - \mathbf{e}_2) = w_H(\Psi(\mathbf{e}_1 - \mathbf{e}_2))$.

Proposition 1. *Let Ψ be the Gray map defined above. Then*

1. *Ψ is a $\mathbb{F}_4$-linear map which preserves distance from $\mathbb{F}_4^\alpha \times R^\beta \times S^\gamma$ (Lee distance) to $\mathbb{F}_4^{\alpha+2\beta+3\gamma}$ (Hamming distance).*
2. *If C is a $\mathbb{F}_4RS$-linear code of block length (α, β, γ) with $|C| = q^k$, then $\Psi(C)$ is a $[\alpha + 2\beta + 3\gamma, k, d_H]$ linear code over $\mathbb{F}_4$, where $d_L = d_H$.*

Proof. (1). Consider $\mathbf{s}_1 = (\mathbf{m}, \mathbf{r}, \mathbf{t}),\ \mathbf{s}_2 = (\mathbf{m}', \mathbf{r}', \mathbf{t}') \in \mathbb{F}_4^\alpha \times R^\beta \times S^\gamma$, where

$$\mathbf{m} = (m_0, m_1, \dots, m_{\alpha-1}),\ \mathbf{m}' = (m'_0, m'_1, \dots, m'_{\alpha-1}) \in \mathbb{F}_q^\alpha,$$

$$\mathbf{r} = \xi_1 \mathbf{a} + \xi_2 \mathbf{b},\ \mathbf{r}' = \xi_1 \mathbf{a}' + \xi_2 \mathbf{b}' \in R^\beta,$$

where

$$\begin{aligned}
\mathbf{a} &= (a_{0,1}, a_{1,1}, \dots, a_{\beta-1,1}), \\
\mathbf{b} &= (b_{0,2}, b_{1,2}, \dots, b_{\beta-1,2}), \\
\mathbf{a}' &= (a'_{0,1}, a'_{1,1}, \dots, a'_{\beta-1,1}), \\
\mathbf{b}' &= (b'_{0,2}, b'_{1,2}, \dots, b'_{\beta-1,2}) \in \mathbb{F}_4^\beta,
\end{aligned}$$

and

$$\mathbf{t} = \eta_1 \mathbf{t}_1 + \eta_2 \mathbf{t}_2 + \eta_3 \mathbf{t}_3,\ \mathbf{t}' = \eta_1 \mathbf{t}'_1 + \eta_2 \mathbf{t}'_2 + \eta_3 \mathbf{t}'_3 \in S^\gamma,$$

where

$$\begin{aligned}
\mathbf{t}_i &= (t_{i,0}, t_{i,1}, \dots, t_{i,\gamma-1}), \\
\mathbf{t}'_i &= (t'_{i,0}, t'_{i,1}, \dots, t'_{i,\gamma-1}) \in \mathbb{F}_4^\gamma, \text{ for } i = 1, 2, 3.
\end{aligned}$$

Then

$$\begin{aligned}
&\Psi(\mathbf{s}_1+\mathbf{s}_2)\\
&=(\mathbf{m}+\mathbf{m}',\mathbf{a}+\mathbf{a}',\mathbf{b}+\mathbf{b}',\mathbf{t}_1+\mathbf{t}_1',\mathbf{t}_2+\mathbf{t}_2',\mathbf{t}_3+\mathbf{t}_3')\\
&=(\mathbf{m},\mathbf{a},\mathbf{b},\mathbf{t}_1,\mathbf{t}_2,\mathbf{t}_3)+(\mathbf{m}',\mathbf{a}',\mathbf{b}',\mathbf{t}_1',\mathbf{t}_2',\mathbf{t}_3')\\
&=\Psi(\mathbf{s}_1)+\Psi(\mathbf{s}_2),
\end{aligned}$$

and $\Psi(\lambda_1\mathbf{s}_1) = (\lambda_1\mathbf{m},\lambda_1\mathbf{a},\lambda_1\mathbf{b},\lambda_1\mathbf{t}_1,\lambda_1\mathbf{t}_2,\lambda_1\mathbf{t}_3) = \lambda_1\Psi(\mathbf{s}_1)$, where $\lambda_1 \in \mathbb{F}_4$. Hence, Ψ is a $\mathbb{F}_4$-linear map.

As Ψ is a $\mathbb{F}_4$-linear map, then we have $d_L(\mathbf{s}_1,\mathbf{s}_2) = w_L(\mathbf{s}_1-\mathbf{s}_2) = w_H(\Psi(\mathbf{s}_1-\mathbf{s}_2)) = d_H(\Psi(\mathbf{s}_1),\Psi(\mathbf{s}_2))$. Hence, we get that the map Ψ is a distance preserving map.

(2). As Ψ is a $\mathbb{F}_4$-linear distance preserving and bijective map, then we get that $\Psi(C)$ is a $[\alpha+2\beta+3\gamma,k,d_H]$ linear code over $\mathbb{F}_4$. □

4. The Structure of $\mathbb{F}_4RS$-Cyclic Codes

In this section, we discuss the generator polynomials of $\mathbb{F}_4RS$-cyclic codes and separable codes of block length (α,β,γ). Before determining the generator polynomials of $\mathbb{F}_4RS$-cyclic codes, we first see the algebraic structure and the generator polynomials of cyclic codes over R and S, respectively.

Theorem 2. *[34] (Theorem 12.9) Let C_α be a cyclic code of length α over $\mathbb{F}_4$. Then there exists a unique monic polynomial $f(x)\in\mathbb{F}_4[x]/\langle x^\alpha-1\rangle$ such that $C_\alpha=\langle f(x)\rangle$ and $f(x)$ divides $(x^\alpha-1)$. Furthermore, C_α has 4^{k_1} codewords, where $k_1=\alpha-\deg(f(x))$, and the set $\{f(x),xf(x),\cdots,x^{k_1-1}f(x)\}$ forms a basis of C_α.*

Now we present the generator polynomials of cyclic codes over R. These polynomials have been studied by Bayram et al. [22].

Theorem 3. *Let $C_\beta=\xi_1C_{\beta,1}\oplus\xi_2C_{\beta,2}$ be a linear code of length β over R. Then*

1. *C_β is a cyclic code of length β if and only if $C_{\beta,1}$ and $C_{\beta,2}$ are cyclic codes of length β over $\mathbb{F}_4$.*
2. *If C_β is a cyclic code of length β over R, then its dual $C_\beta^\perp=\xi_1C_{\beta,1}^\perp\oplus\xi_2C_{\beta,2}^\perp$ is also a cyclic code over R.*
3. *If C_β is a cyclic code of length β over R, then $C_\beta=\langle a(x)\rangle$, where $a(x)=\xi_1a_1(x)+\xi_2a_2(x)$ with $a(x)\mid(x^\beta-1)$ and $C_{\beta,1}=\langle a_1(x)\rangle$ and $C_{\beta,2}=\langle a_2(x)\rangle$. Moreover, $|C_\beta|=4^{2\beta-\sum_{i=1}^{2}\deg(a_i(x))}$.*

Similar to the above theorem, we get the following result for cyclic codes over S.

Theorem 4. *Let $C_\gamma=\eta_1C_{\gamma,1}\oplus\eta_2C_{\gamma,2}\oplus\eta_3C_{\gamma,3}$ be a linear code of length γ over S. Then*

1. *C_γ is a cyclic code of length γ if and only if $C_{\gamma,1}$, $C_{\gamma,2}$ and $C_{\gamma,3}$ are cyclic codes of length γ over $\mathbb{F}_4$.*
2. *If C_γ is a cyclic code of length γ over S, then its dual $C_\gamma^\perp=\eta_1C_{\gamma,1}^\perp\oplus\eta_2C_{\gamma,2}^\perp\oplus C_{\gamma,3}^\perp$ is also a cyclic code over S.*
3. *If C_γ is a cyclic code of length γ over S, then $C_\gamma=\langle t(x)\rangle$, where $t(x)=\eta_1t_1(x)+\eta_2t_2(x)+\eta_3t_3(x)$ with $t(x)\mid(x^\gamma-1)$ and $C_{\gamma,1}=\langle t_1(x)\rangle$, $C_{\gamma,2}=\langle t_2(x)\rangle$ and $C_{\gamma,3}=\langle t_3(x)\rangle$. Moreover, $|C_\gamma|=4^{3\gamma-\sum_{i=1}^{3}\deg(t_i(x))}$.*

Proof. The proof follows from Theorems 10 and 12 presented in [32]. □

In Theorems 3 and 4, we have studied the generator polynomials of cyclic codes of length β and γ over R and S, respectively. By using these polynomials, the generator polynomials of the $\mathbb{F}_4RS$-cyclic codes are now determined as follows.

Theorem 5. *Let C be a $\mathbb{F}_4RS$-cyclic code of block length (α, β, γ). Then*

$$C = \langle (f(x), 0, 0), (l(x), a(x), 0), (l_1(x), l_2(x), t(x)) \rangle,$$

where $f(x) \mid (x^\alpha - 1)$, $a(x) = \xi_1 a_1(x) + \xi_2 a_2(x)$ with $a(x) \mid (x^\beta - 1)$, $t(x) = \eta_1 t_1(x) + \eta_2 t_2(x) + \eta_3 t_3(x)$ with $t(x) \mid (x^\gamma - 1)$ and $l(x), l_1(x) \in \mathbb{F}_4[x]$, $l_2(x) \in R[x]$.

Proof. Since C and $\frac{S[x]}{\langle x^\gamma - 1\rangle}$ are $S[x]$-submodules of $S_{\alpha,\beta,\gamma}$, we define a map $\Phi : C \to \frac{S[x]}{\langle x^\gamma - 1\rangle}$ given by $\Phi(r_1(x), r_2(x), r_3(x)) = r_3(x)$. We can see that Φ is a homomorphism between two $S[x]$-modules. Notice that $\Phi(C)$ is an ideal of $\frac{S[x]}{\langle x^\gamma - 1\rangle}$, i.e., $\Phi(C)$ is a cyclic code of length γ over S. Therefore, by Theorem 4.3, we have $\Phi(C) = \langle t(x) \rangle$, where $t(x) = (\eta_1 t_1(x) + \eta_2 t_2(x) + \eta_3 t_3(x))$. Further, we have $Ker(\Phi) = \{(r_1(x), r_2(x), 0) \in S_{\alpha,\beta,\gamma} \mid (r_1(x), r_2(x), 0) \in C\}$. Define $I = \{(r_1(x), r_2(x)) \in R_{\alpha,\beta} \mid (r_1(x), r_2(x), 0) \in Ker(\Phi)\}$, where $R_{\alpha,\beta} = \frac{\mathbb{F}_4[x]}{\langle x^\alpha - 1\rangle} \times \frac{R[x]}{\langle x^\beta - 1\rangle}$. We can see that I is an $R[x]$-submodule of $R_{\alpha,\beta}$. Therefore, from [31] (Lemma 3.3), I has the generator polynomials of the form $\langle (f(x), 0), (l(x), a(x)) \rangle$, where $f(x) \mid (x^\alpha - 1)$, $a(x) = \xi_1 a_1(x) + \xi_2 a_2(x)$ with $a(x) \mid (x^\beta - 1)$ and $l(x) \in \mathbb{F}_4[x]$. For any $(r_1(x), r_2(x), 0) \in Ker(\Phi)$, we get $(r_1(x), r_2(x)) \in I$. Then there exist some polynomials $m_1(x) \in \mathbb{F}_4[x]$ and $m_2(x) \in R[x]$ such that

$$(r_1(x), r_2(x)) = m_1(x) \star (f(x), 0) + m_2(x) \star (l(x), a(x)).$$

Hence,

$$(r_1(x), r_2(x), 0) = m_1(x) \star (f(x), 0, 0) + m_2(x) \star (l(x), a(x), 0),$$

which implies

$$Ker(\Phi) = \langle (f(x), 0, 0), (l(x), a(x), 0) \rangle.$$

Therefore, by the first isomorphism theorem of modules, we get

$$C/Ker(\Phi) \cong \Phi(C) = \langle t(x) \rangle.$$

Suppose $(l_1(x), l_2(x), t(x)) \in C$ with $\Phi((l_1(x), l_2(x), t(x)) = t(x)$. Hence, from the above discussion, we get that any $\mathbb{F}_4RS$-cyclic code can be generated as an $S[x]$-submodule of $S_{\alpha,\beta,\gamma}$ by the elements of the form $(f(x), 0, 0)$, $(l(x), a(x), 0)$ and $(l_1(x), l_2(x), t(x))$, where $l(x), l_1(x) \in \mathbb{F}_4[x]$, $l_2(x) \in R[x]$ and $f(x) \mid (x^\alpha - 1)$, $a(x) \mid (x^\beta - 1)$, $t(x) \mid (x^\gamma - 1)$. This completes the proof. □

The polynomials $l(x), l_1(x)$ and $l_2(x)$ obtained in Theorem 5, have some conditions on their degrees. These conditions are as follows.

Lemma 2. *Let $C = \langle (f(x), 0, 0), (l(x), a(x), 0), (l_1(x), l_2(x), t(x)) \rangle$ be a $\mathbb{F}_4RS$-cyclic code. Then we may assume* $\deg(l(x)) < \deg(f(x))$, $\deg(l_1(x)) < \deg(f(x))$, $\deg(l_2(x)) < \deg(a(x))$.

Proof. Let $\deg(l_1(x)) \geq \deg(f(x))$ and $\deg(l_1(x)) - \deg(f(x)) = i$. Consider

$$D = \langle (f(x), 0, 0), (l(x), a(x), 0), (l_1(x) + x^i f(x), l_2(x), t(x)) \rangle.$$

Notice that

$$(l_1(x) + x^i f(x), l_2(x), t(x)) = x^i \star (f(x), 0, 0) + (l_1(x), l_2(x), t(x)),$$

which implies that $D \subseteq C$.

On the other hand, we have

$$(l_1(x), l_2(x), t(x)) = (l_1(x) + x^i f(x), l_2(x), t(x)) - x^i \star (f(x), 0, 0),$$

which implies $C \subseteq D$. Thus, we get $C = D$. Therefore, the degree of $l_1(x)$ can be reduced in C. Hence, $\deg(l_1(x)) < \deg(f(x))$. Other parts can be proven by using a similar method. □

In the above discussion, we have determined the generator polynomials of $\mathbb{F}_4RS$-cyclic codes of block length (α, β, γ). Now we discuss the structure of the separable codes and their generator polynomials.

A $\mathbb{F}_4RS$-linear code C of block length (α, β, γ) is called a separable code if $C = C'_\alpha \times C'_\beta \times C'_\gamma$, while considering C'_α, C'_β and C'_γ as punctured codes of C by deleting the coordinates outside the α, β and γ components, respectively.

Using the result obtained in Theorem 5, we determine the generator polynomials of separable $\mathbb{F}_4RS$-cyclic codes of block length (α, β, γ).

Lemma 3. *Let $C = \langle (f(x), 0, 0), (l(x), a(x), 0), (l_1(x), l_2(x), t(x)) \rangle$ be a $\mathbb{F}_4RS$-cyclic code of block length (α, β, γ). Then*

$$\begin{aligned} C'_\alpha &= \langle \gcd(f(x), l(x), l_1(x)) \rangle, \\ C'_\beta &= \langle \gcd(a(x), l_2(x)) \rangle, \\ C'_\gamma &= \langle t(x) \rangle. \end{aligned}$$

Proof. Consider $p'(x) \in C'_\alpha$, then there exist two polynomials $q'(x) \in R[x]/\langle x^\beta - 1 \rangle$ and $r'(x) \in S[x]/\langle x^\gamma - 1 \rangle$ such that $(p'(x), q'(x), r'(x)) \in C$. It follows that there exist some polynomials $\lambda_1(x), \lambda_2(x)$ and $\lambda_3(x) \in S[x]$ such that

$$(p'(x), q'(x), r'(x)) = \lambda_1(x) \star (f(x), 0, 0) + \lambda_2(x) \star (l(x), a(x), 0) + \lambda_3(x) \star (l_1(x), l_2(x), t(x)).$$

This implies

$$p'(x) = \rho_1(\lambda_1(x)) f(x) + \rho_1(\lambda_2(x)) l(x) + \rho_1(\lambda_3(x)) l_1(x).$$

Hence, we get $\gcd(f(x), l(x), l_1(x)) \mid p'(x)$. Therefore, $p'(x) \in \langle \gcd(g(x), l(x), l_1(x)) \rangle$, which implies $C'_\alpha \subseteq \langle \gcd(f(x), l(x), l_1(x)) \rangle$.

On the other hand, for some polynomials $\lambda'_1(x), \lambda'_2(x), \lambda'_3(x) \in \mathbb{F}_4[x]$, we get

$$\gcd(f(x), l(x), l_1(x)) = \lambda'_1(x) f(x) + \lambda'_2(x) l(x) + \lambda'_3(x) l_1(x).$$

Then

$$\begin{aligned} &(\gcd(f(x), l(x), l_1(x)), \lambda'_2(x) a(x) + \lambda'_3(x) l_2(x), \lambda'_3(x) t(x)) \\ &= \lambda'_1(x) \star (f(x), 0, 0) + \lambda'_2(x) \star (l(x), a(x), 0) + \lambda'_3(x) \star (l_1(x), l_2(x), t(x)) \subseteq C, \end{aligned}$$

which implies $\langle \gcd(f(x), l(x), l_1(x)) \rangle \subseteq C'_\alpha$. Thus, we get $C'_\alpha = \langle \gcd(f(x), l(x), l_1(x)) \rangle$. Similarly, we can see $C'_\beta = \langle \gcd(a(x), l_2(x)) \rangle$ and $C'_\gamma = \langle t(x) \rangle$. □

Lemma 4. *Let $C = \langle (f(x), 0, 0), (l(x), a(x), 0), (l_1(x), l_2(x), t(x)) \rangle$ be a $\mathbb{F}_4RS$-cyclic code. Then $f(x) \mid l(x)$ if and only if $l(x) = 0$, and $f(x) \mid l_1(x)$ if and only if $l_1(x) = 0$.*

Proof. Suppose $l(x) = 0$, then obviously $f(x) \mid l(x)$.

Conversely, suppose $f(x) \mid l(x)$, then $l(x) = \lambda(x)f(x)$ for some $\lambda(x) \in \mathbb{F}_4[x]$. Let

$$D = \langle (f(x),0,0),(0,a(x),0),(l_1(x),l_2(x),t(x))\rangle.$$

On the one hand, notice that

$$(0,a(x),0) = (l(x),a(x),0) - \lambda(x) \star (f(x),0,0) \in C,$$

therefore $D \subseteq C$. On the other hand,

$$(l(x),a(x),0) = \lambda(x) \star (f(x),0,0) + (0,a(x),0) \in D,$$

which implies, $C \subseteq D$. Thus, we get $C = D$. Hence, we infer that $l(x) = 0$. Similarly, we can prove $g(x) \mid l_1(x)$ if and only if $l_1(x) = 0$. □

Similar to the above result, we get the following.

Lemma 5. *Let $C = \langle (f(x),0,0),(l(x),a(x),0),(l_1(x),l_2(x),t(x))\rangle$ be a $\mathbb{F}_4RS$-cyclic code. Then $a(x) \mid l_2(x)$ if and only if $l_2(x) = 0$.*

By Lemmas 3–5, we get the following result for a $\mathbb{F}_4RS$-cyclic code to be a separable code.

Theorem 6. *Let $C = \langle (f(x),0,0),(l(x),a(x),0), (l_1(x),l_2(x),t(x))\rangle$ be a $\mathbb{F}_4RS$-cyclic code. Then the following assertions are equivalent:*

1. *C is a separable code;*
2. *$f(x) \mid l(x)$, $g(x) \mid l_1(x)$ and $a(x) \mid l_2(x)$;*
3. *$C'_\alpha = \langle f(x)\rangle$, $C'_\beta = \langle a(x)\rangle$ and $C'_\gamma = \langle t(x)\rangle$;*
4. *$C = \langle (f(x),0,0),(0,a(x),0),(0,0,t(x))\rangle$.*

Proof. Proof holds directly from Lemmas 3–5. □

From the results obtained in the above theorem for a separable code and results discussed in Theorems 2–4, we get the followings:

$$\begin{aligned} C'_\alpha &= \langle f(x)\rangle = C_\alpha, \\ C'_\beta &= \langle a(x)\rangle = C_\beta, \\ C'_\gamma &= \langle t(x)\rangle = C_\gamma. \end{aligned}$$

The next result is obtained from these observations.

Theorem 7. *Let $C = C_\alpha \times C_\beta \times C_\gamma$ be a $\mathbb{F}_4RS$-linear code of block length (α,β,γ). Then C is a separable $\mathbb{F}_4RS$-cyclic code of block length (α,β,γ) if and only if C_α, C_β and C_γ are cyclic codes of length α, β and γ over $\mathbb{F}_4$, R and S, respectively.*

Proof. The proof follows from [32] (Theorem 24). □

From our above discussion, we conclude the following result.

Theorem 8. *Let $C = C_\alpha \times C_\beta \times C_\gamma$ be a separable $\mathbb{F}_4RS$-cyclic code of block length (α,β,γ), where $C_\alpha = \langle f(x)\rangle$, $C_\beta = \langle a(x)\rangle$ and $C_\gamma = \langle t(x)\rangle$. Then $C = \langle f(x)\rangle \times \langle a(x)\rangle \times \langle t(x)\rangle$.*

To illustrate the above results, we present an example.

Example 1. *Let* $S_{\alpha,\beta,\gamma} = \frac{\mathbb{F}_4[x]}{\langle x^3-1\rangle} \times \frac{R[x]}{\langle x^6-1\rangle} \times \frac{S[x]}{\langle x^9-1\rangle}$. *Then*

$$x^3 - 1 = (x+1)(x+w)(x+w^2) \in \mathbb{F}_4[x].$$

Let $f(x) = (x+w)(x+w^2)$. *Then* $C_\alpha = \langle f(x)\rangle$ *is a cyclic code of length* 3 *over* $\mathbb{F}_4$ *and* $|C_\alpha| = 4$.

$$x^6 - 1 = (x+1)^2(x+w)^2(x+w^2)^2 \in \mathbb{F}_4[x].$$

Let $a_1(x) = a_2(x) = (x+w)(x+w^2)^2$. *Then* $C_{\beta,1} = C_{\beta,2} = \langle a_1(x)\rangle$ *are cyclic codes of length* 6 *over* $\mathbb{F}_4$. *Thus,* C_β *is a cyclic code of length* 6 *over* R, *with cardinality* $4^{12-6} = 4^6$. *Hence,* $C_\beta = \langle \xi_1 a_1(x) + \xi_2 a_2(x)\rangle = \langle a(x)\rangle$.

$$x^9 - 1 = (x+1)(x+w)(x+w^2)(x^3+w)(x^3+w^2) \in \mathbb{F}_4[x].$$

Let $t_1(x) = (x+w)(x+w^2)$ *and* $t_2(x) = t_3(x) = (x+w^2)(x^3+w)(x^3+w^2)$. *Then* $C_{\gamma,1} = \langle t_1(x)\rangle$, $C_{\gamma,i} = \langle t_i(x)\rangle$; $i = 2,3$, *are cyclic codes of length* 9 *over* $\mathbb{F}_4$. *Thus,* $C_\gamma = \eta_1 C_{\gamma,1} \oplus \eta_2 C_{\gamma,2} \oplus \eta_3 C_{\gamma,3}$ *is a cyclic code of length* 9 *over* S *with cardinality* $4^{27-16} = 4^{11}$. *Hence,* $C_\gamma = \langle \eta_1 t_1(x) + \eta_2 t_2(x) + \eta_3 t_3(x)\rangle = \langle t(x)\rangle$, *where*

$$\begin{aligned} t(x) &= (1-u-v)(x+w)(x+w^2) + u(x+w^2)(x^3+w)(x^3+w^2) + v(x+w^2)(x^3+w)(x^3+w^2) \\ &= (u+v)x^7 + (u+v)w^2x^6 + (u+v)x^4 + (u+v)w^2x^3 + (1+u+v)x^2 + x + (u+v)w + 1. \end{aligned}$$

Hence, $C = \langle (f(x),0,0),(0,a(x),0),(0,0,t(x))\rangle = \langle f(x)\rangle \times \langle a(x)\rangle \times \langle t(x)\rangle$ *is a separable* $\mathbb{F}_4RS$*-cyclic code of block length* $(3,6,9)$. *Furthermore,* $|C| = 4^{18}$.

5. Applications in Cyclic DNA Codes

In the above discussion, we have studied $\mathbb{F}_4RS$-cyclic codes. We have further determined the structure of separable $\mathbb{F}_4RS$-cyclic codes and obtained their generator polynomials. Now in the rest of this paper, we discuss the application of this family of codes.

We proceed to the discussion of cyclic DNA codes over $\mathbb{F}_4$, R, S and $\mathbb{F}_4RS$. For this discussion, we need some basic definitions that are defined next. Throughout this section, we assume α, β and γ are odd positive integers.

Let $\mathfrak{R}$ be a finite commutative ring and $\mathfrak{C}$ be a linear code of length $\mathfrak{n}$ over $\mathfrak{R}$. Consider $\mathbf{z} = (z_0, z_1, \ldots, z_{\mathfrak{n}-1}) \in \mathfrak{R}^{\mathfrak{n}}$. The reverse of $\mathbf{z}$ is defined as $\mathbf{z}^r = (z_{\mathfrak{n}-1}, z_{\mathfrak{n}-2}, \ldots, z_0)$, the complement of $\mathbf{z}$ is defined as $\mathbf{z}^c = (\overline{z_0}, \overline{z_1}, \ldots, \overline{z_{\mathfrak{n}-1}})$ and the reverse-complement of $\mathbf{z}$ is defined as $\mathbf{z}^{rc} = (\overline{z_{\mathfrak{n}-1}}, \overline{z_{\mathfrak{n}-2}}, \ldots, \overline{z_0})$.

Definition 4. *Let* $\mathfrak{C}$ *be a linear code of length* $\mathfrak{n}$ *over* $\mathfrak{R}$. *Then* $\mathfrak{C}$ *is called* **reversible** *if for any* $\mathbf{z} \in \mathfrak{C}$, $\mathbf{z}^r \in \mathfrak{C}$, **complement** *if for any* $\mathbf{z} \in \mathfrak{C}$, $\mathbf{z}^c \in \mathfrak{C}$, **reversible-complement** *if for any* $\mathbf{z} \in \mathfrak{C}$, $\mathbf{z}^{rc} \in \mathfrak{C}$.

Definition 5. *[21] (p. 1172) Let* $\mathfrak{C}$ *be a linear code of length* $\mathfrak{n}$ *over* $\mathfrak{R}$. *Then* $\mathfrak{C}$ *is said to be a cyclic DNA code if*

1. $\mathfrak{C}$ *is a cyclic code, and*
2. *for any* $\mathbf{z} \in \mathfrak{C}$, $\mathbf{z} \neq \mathbf{z}^{rc}$, $\mathbf{z}^{rc} \in \mathfrak{C}$.

For any polynomial $k(x) = k_0 + k_1x + k_2x^2 + \cdots + k_sx^s \in \mathfrak{R}[x]$, with $k_s \neq 0$ the reciprocal polynomial of $k(x)$ is defined as

$$k^*(x) = x^s k(1/x) = k_s + k_{s-1}x + \cdots + k_0x^s.$$

It can be easily seen that if $k_0 \neq 0$, then $\deg(k^*(x)) = \deg(k(x))$ otherwise $\deg(k^*(x)) \leq \deg(k(x))$. Further, if $k^*(x) = k(x)$, then $k(x)$ is called self-reciprocal.

5.1. Reversible and Reversible-Complement Codes over $\mathbb{F}_4$

In 2006, Abualrub et al. [17] studied cyclic DNA codes over $\mathbb{F}_4$, where $\mathbb{F}_4 = \{0, 1, w, w^2 = w+1\}$ with $1 + w + w^2 = 0$. They have taken a bijection map between the elements of $\mathbb{F}_4$ and DNA alphabets $S_{D_4} = \{A, T, C, G\}$ such that $0 \mapsto A$, $1 \mapsto T$, $w \mapsto C$, $w^2 = w + 1 \mapsto G$. We will use the same bijection map throughout this section for our study of cyclic DNA codes over R and S.

Lemma 6. *[35] (Theorem 1) Let $C_\alpha = \langle f(x) \rangle$ be a cyclic code of length α over $\mathbb{F}_4$. Then C_α is reversible if and only if $f(x)$ is self-reciprocal.*

We adopt the next lemma from [17], this lemma will be useful in determining reversible-complements codes over $\mathbb{F}_4$.

Lemma 7. *[17] (Lemma 8) Let $C_\alpha = \langle f(x) \rangle$ be a cyclic code of length α over $\mathbb{F}_4$. Then C_α is complement if and only if $f(x)$ is not divisible by $x - 1$.*

By Lemmas 6 and 7, we obtain the following result.

Theorem 9. *Let $C_\alpha = \langle f(x) \rangle$ be a cyclic code of length α over $\mathbb{F}_4$. Then C_α is reversible-complement if and only if $f(x)$ is self-reciprocal and $f(x)$ is not divisible by $x - 1$.*

5.2. Reversible and Reversible-Complement Codes over R

This subsection is dedicated to the study of cyclic codes over R, which satisfy the reversible constraint and reversible-complement constraint. Bayram et al. [22] considered the same ring $R = \mathbb{F}_4 + u\mathbb{F}_4$, $u^2 = u$ and discussed the reversible constraint and reversible-complement constraint. Here, we also have considered the same ring but we use a different approach to prove these constraints than Bayram et al. [22].

In the above subsection, we have defined a bijection map between $\mathbb{F}_4$ and S_{D_4}. This map can be extended naturally from R to $S_{D_4}^2$ by considering the Gray images of elements of R from Gray map ψ_1 defined in Section 3. For example, the Gray image of $1 + wu \in R$ is $(1, 1 + w) \in \mathbb{F}_4^2$. Since $1 \mapsto T$, $1 + w \mapsto G$ over $\mathbb{F}_4$, then $(1, 1 + w)$ can be identified with $TG \in S_{D_4}^2$. By this identification, we get a one-to-one correspondence between the elements of R and $S_{D_4}^2$. The correspondence from R to $S_{D_4}^2$ is denoted by γ_1 and defined in Table 1.

Table 1. Correspondence between R and $S_{D_4}^2$.

Elements $e \in R$	**Gray Images**	**DNA Codons $\gamma_1(e)$**
0	$(0,0)$	AA
u	$(0,1)$	AT
uw	$(0,w)$	AC
$u(1+w)$	$(0,1+w)$	AG
1	$(1,1)$	TT
$1+u$	$(1,0)$	TA
$1+u(1+w)$	$(1,w)$	TC
$1+uw$	$(1,1+w)$	TG
w	(w,w)	CC
$w+uw$	$(w,0)$	CA
$w+u(1+w)$	$(w,1)$	CT
$w+u$	$(w,1+w)$	CG
$1+w+u(1+w)$	$(1+w,0)$	GA
$1+w+u$	$(1+w,w)$	GC
$1+w+uw$	$(1+w,1)$	GT
$1+w$	$(1+w,1+w)$	GG

By WCC, we have $\overline{A} = T$, $\overline{T} = A$, $\overline{C} = G$ and $\overline{G} = C$. We can extend this notation to the elements of $S_{D_4}^2$ such as $\overline{AT} = TA, \overline{AA} = TT, \cdots, \overline{GC} = CG$.

Definition 6. *Let C_β be a linear code of length β over R and $\mathbf{r} = (a_0, a_1, \ldots, a_{\beta-1}) \in C_\beta$. We define*

$$\Phi_1(\mathbf{r}) : C_\beta \longrightarrow S_{D_4}^{2\beta},$$

given by

$$(a_0, a_1, \ldots, a_{\beta-1}) \mapsto (\gamma_1(a_0)\gamma_1(a_1)\ldots\gamma_1(a_{\beta-1})),$$

by using Table 1.

For example, $(a_0, a_1, a_2) = (1, u, w + u(1 + w))$ is mapped to $(\gamma_1(1)\gamma_1(u)\gamma_1(w + u(1 + w))) = (TTATCT)$.

Now we present some relation between cyclic codes and reversible codes over R.

Theorem 10. *Let $C_\beta = \xi_1 C_{\beta,1} \oplus \xi_2 C_{\beta,2}$ be a cyclic code of length β over R. Then C_β is reversible over R if and only if $C_{\beta,1}$ and $C_{\beta,2}$ are reversible over $\mathbb{F}_4$.*

Proof. Suppose C_β is reversible over R and $\mathbf{r} = (a_0, a_1, \ldots, a_{\beta-1}) \in C_\beta$, where $a_j = \xi_1 a_{j,1} + \xi_2 b_{j,2}$, for $a_{j,1}, b_{j,2} \in \mathbb{F}_4$ and $j = 0, 1, \ldots, \beta - 1$. Then $\mathbf{r} = \xi_1 \mathbf{a} + \xi_2 \mathbf{b}$, where $\mathbf{a} = (a_{0,1}, a_{1,1} \ldots, a_{\beta-1,1}) \in C_{\beta,1}$ and $\mathbf{b} = (b_{0,2}, b_{1,2}, \ldots, b_{\beta-1,2}) \in C_{\beta,2}$. Notice that

$$\begin{aligned}
\mathbf{r}^r &= (a_{\beta-1}, a_{\beta-2}, \ldots, a_1, a_0) \\
&= (\xi_1 a_{\beta-1,1} + \xi_2 b_{\beta-1,2}, \xi_1 a_{\beta-2,1} + \xi_2 b_{\beta-2,2}, \ldots, \xi_1 a_{1,1} + \xi_2 b_{1,2}, \xi_1 a_{0,1} + \xi_2 b_{0,2}) \\
&= \xi_1 (a_{\beta-1,1}, a_{\beta-2,1}, \ldots, a_{1,1}, a_{0,1}) + \xi_2 (b_{\beta-1,2}, b_{\beta-2,2}, \ldots, b_{1,2}, b_{0,2}) \\
&= \xi_1 \mathbf{a}^r + \xi_2 \mathbf{b}^r.
\end{aligned}$$

Since C_β is reversible over R, then $\mathbf{r}^r = \xi_1 \mathbf{a}^r + \xi_2 \mathbf{b}^r \in C_\beta$, and $C_\beta = \xi_1 C_{\beta,1} \oplus \xi_2 C_{\beta,2}$. Which implies $\mathbf{a}^r \in C_{\beta,1}$ and $\mathbf{b}^r \in C_{\beta,2}$. Thus, $C_{\beta,1}$ and $C_{\beta,2}$ are reversible over $\mathbb{F}_4$.

Conversely, suppose that $C_{\beta,1}$ and $C_{\beta,2}$ are reversible over $\mathbb{F}_4$. By considering the above notations, we have $\mathbf{a}^r \in C_{\beta,1}$ and $\mathbf{b}^r \in C_{\beta,2}$. Since $\mathbf{r}^r = \xi_1 \mathbf{a}^r + \xi_2 \mathbf{b}^r \in C_\beta$, then C_β is reversible over R. □

To illustrate the above result, we present an example.

Example 2. *Let $R = \mathbb{F}_4 + u\mathbb{F}_4$, where $u^2 = u$ and $\mathbb{F}_4 = \{0, 1, w, w^2 = w + 1\}$.*

$$x^5 - 1 = (x + 1)(x^2 + wx + 1)(x^2 + w^2 x + 1) \in \mathbb{F}_4[x].$$

Consider $a_1(x) = x^2 + wx + 1$ and $a_2(x) = x^2 + w^2 x + 1$. Then $C_{\beta,1} = \langle a_1(x) \rangle$ and $C_{\beta,2} = \langle a_2(x) \rangle$ are cyclic codes of length 5 over $\mathbb{F}_4$. By Theorem 3, $C_\beta = \langle \xi_1 a_1(x) + \xi_2 a_2(x) \rangle$ is a cyclic code of length 5 over R. As $a_1(x)$ and $a_2(x)$ are self-reciprocal polynomials, so by Lemma 6, $C_{\beta,1}$ and $C_{\beta,2}$ are reversible over $\mathbb{F}_4$. Hence, by Theorem 10, C_β is reversible over R.

Example 3. *Let $R = \mathbb{F}_4 + u\mathbb{F}_4$, where $u^2 = u$ and $\mathbb{F}_4 = \{0, 1, w, w^2 = w + 1\}$.*

$$x^{13} - 1 = (x + 1)(x^6 + wx^5 + w^2 x^3 + wx + 1)(x^6 + w^2 x^5 + wx^3 + w^2 x + 1) \in \mathbb{F}_4[x].$$

Consider $a_1(x) = x^6 + wx^5 + w^2 x^3 + wx + 1$ and $a_2(x) = x^6 + w^2 x^5 + wx^3 + w^2 x + 1$. Then $C_{\beta,1} = \langle a_1(x) \rangle$ and $C_{\beta,2} = \langle a_2(x) \rangle$ are cyclic codes of length 13 over $\mathbb{F}_4$. By Theorem 3, $C_\beta = \langle \xi_1 a_1(x) + \xi_2 a_2(x) \rangle$ is a cyclic code of length 5 over R. As $a_1(x)$ and $a_2(x)$ are self-reciprocal polynomials, so by Lemma 6, $C_{\beta,1}$ and $C_{\beta,2}$ are reversible over $\mathbb{F}_4$. Hence, by Theorem 10, C_β is reversible over R.

From Table 1, we obtain next useful lemma. This lemma is used in determining reversible-complement property on cyclic codes over R.

Lemma 8. *For any $a \in R$, we have $\overline{a} + \overline{0} = a$.*

Theorem 11. *Let $C_\beta = \xi_1 C_{\beta,1} \oplus \xi_2 C_{\beta,2}$ be a cyclic code of length β over R. Then C_β is reversible-complement over R if and only if $(\overline{0}, \overline{0}, \dots, \overline{0}) \in C_\beta$ and C_β is reversible over R.*

Proof. Suppose C_β is reversible-complement over R and $\mathbf{r} = (a_0, a_1, \dots, a_{\beta-1}) \in C_\beta$. Then $\mathbf{r}^{rc} = (\overline{a_{\beta-1}}, \overline{a_{\beta-2}}, \dots, \overline{a_0}) \in C_\beta$. Since C_β is a linear code, then we have $(0, 0, \dots, 0) \in C_\beta$, which implies $(\overline{0}, \overline{0}, \dots, \overline{0}) \in C_\beta$. By using Lemma 8, we get

$$\begin{aligned} \mathbf{r}^r &= (\overline{a_{\beta-1}}, \overline{a_{\beta-2}}, \dots, \overline{a_0}) + (\overline{0}, \overline{0}, \dots, \overline{0}) \\ &= (a_{\beta-1}, a_{\beta-2}, \dots, a_0). \end{aligned}$$

Since C_β is linear as well as reversible-complement, then we get $\mathbf{r}^r \in C_\beta$. Hence, C_β is reversible over R.

Conversely, suppose $(\overline{0}, \overline{0}, \dots, \overline{0}) \in C_\beta$ and C_β is reversible. Then for any $\mathbf{r} = (a_0, a_1, \dots, a_{\beta-1}) \in C_\beta$, we get $\mathbf{r}^r = (a_{\beta-1}, a_{\beta-2}, \dots, a_0) \in C_\beta$. Again by Lemma 8, and linearity of C_β, we get

$$\begin{aligned} \mathbf{r}^{rc} &= (a_{\beta-1}, a_{\beta-2}, \dots, a_0) + (\overline{0}, \overline{0}, \dots, \overline{0}) \\ &= (\overline{a_{\beta-1}}, \overline{a_{\beta-2}}, \dots, \overline{a_0}) \in C_\beta. \end{aligned}$$

Hence, C_β is reversible-complement over R. □

To illustrate the above results, we present some examples.

Example 4. *Let $R = \mathbb{F}_4 + u\mathbb{F}_4$, where $u^2 = u$.*

$$x^3 - 1 = (x+1)(x+w)(x+w^2) \in \mathbb{F}_4[x].$$

Let $a_1(x) = a_2(x) = (x+w)(x+w^2)$. Then $C_{\beta,i} = \langle a_i(x) \rangle$ are cyclic codes of length 3 over $\mathbb{F}_4$, for $i = 1, 2$. By Theorem 3, $C_\beta = \langle \xi_1 a_1(x) + \xi_2 a_2(x) \rangle$ is a cyclic code of length 3 over R. As $a_1(x)$ and $a_2(x)$ are self-reciprocal polynomials, hence by Lemma 6, $C_{\beta,i}$ are reversible over $\mathbb{F}_4$, for $i = 1, 2$. Therefore, by Theorem 10, C_β is reversible over R. Further, C_β has $4^{6-4} = 16$ codewords. The corresponding DNA codewords obtained by using Table 1, are listed below.

Notice that C_β is reversible over R and $(TTTTTT) = (\overline{0}, \overline{0}, \overline{0}) \in C_\beta$. Thus, by Theorem 11, C_β is reversible-complement over R. Moreover, by Definition 5, we conclude that C_β is a cyclic DNA code. The image of C_β under the map ψ_1 is a DNA code of length 6, size 16 and minimum Hamming distance 3. The DNA code given in Table 2 is different from the DNA codes of the same length constructed by Zhu et al. [23] and Siap et al. [18].

Table 2. DNA code of length 6 obtained from C_β.

AGAGAG	*TGTGTG*	*CGCGCG*	*GCGCGC*
AAAAAA	*TCTCTC*	*CCCCCC*	*GTGTGT*
ACACAC	*TTTTTT*	*CTCTCT*	*GGGGGG*
ATATAT	*TATATA*	*CACACA*	*GAGAGA*

Example 5. *Let $R = \mathbb{F}_4 + u\mathbb{F}_4$, where $u^2 = u$.*

$$x^5 - 1 = (x+1)(x^2 + wx + 1)(x^2 + w^2 x + 1) \in \mathbb{F}_4[x].$$

Let $a_1(x) = a_2(x) = (x^2 + wx + 1)(x^2 + w^2x + 1)$. Then $C_{\beta,i} = \langle a_i(x)\rangle$ are cyclic codes of length 5 over $\mathbb{F}_4$, for $i = 1, 2$. By Theorem 3, $C_\beta = \langle \xi_1 a_1(x) + \xi_2 a_2(x)\rangle$ is a cyclic code of length 5 over R. As $a_1(x)$ and $a_2(x)$ are self-reciprocal polynomials, hence by Lemma 6, $C_{\beta,i}$ are reversible over $\mathbb{F}_4$, for $i = 1, 2$. Therefore, by Theorem 10, C_β is reversible over R. Further, C_β has $4^{10-8} = 16$ codewords. The corresponding DNA codewords obtained by using Table 3, are listed below.

Table 3. DNA code of length 10 obtained from C_β.

AAAAAAAAAA	*TTTTTTTTTT*	*CCCCCCCCCC*	*GGGGGGGGGG*
ATATATATAT	*TATATATATA*	*CACACACACA*	*GAGAGAGAGA*
ACACACACAC	*TCTCTCTCTC*	*CTCTCTCTCT*	*GTGTGTGTGT*
AGAGAGAGAG	*TGTGTGTGTG*	*CGCGCGCGCG*	*GCGCGCGCGC*

Notice that C_β is reversible over R and $(TTTTTTTTTT) = (\bar{0}, \bar{0}, \bar{0}, \bar{0}, \bar{0}) \in C_\beta$. Thus, by Theorem 11, C_β is reversible-complement over R. Moreover, by Definition 5, we conclude that C_β is a cyclic DNA code. The image of C_β under the map ψ_1 is a DNA code of length 10, size 16 and minimum Hamming distance 5. These codewords are given in Table 3.

5.3. Reversible and Reversible-Complement Codes over S

This subsection is dedicated to the study of cyclic codes over S, which satisfy the reversible constraint and reversible-complement constraint. We can see that the ring S has 64 elements. In literature, many researchers [26,27,29] have discussed reversible constraint and reversible-complement constraint over the ring of order 64. In this subsection, we use a different approach to study these constraints over the ring of order 64.

In 2020, Liu et al. [29] defined a one-to-one correspondence between the elements of the ring of order 64 and DNA alphabets set $S_{D_4}^3$ by using the Gray images of the elements of the ring. We also define a similar kind of one-to-one correspondence between the elements of S and $S_{D_4}^3$ for our study.

The above defined bijection map between $\mathbb{F}_4$ and S_{D_4} can be extended naturally from S to $S_{D_4}^3$ by considering the Gray images of the elements of S from Gray map ψ_2 defined in Section 3. For example, the Gray image of $1 + wu + wv \in S$ is $(1, 1 + w, 1 + w) \in \mathbb{F}_4^3$. Since $1 \mapsto T$, $1 + w \mapsto G$ over $\mathbb{F}_4$, then $(1, 1 + w, 1 + w)$ can be identified with $TGG \in S_{D_4}^3$. By this identification, we get a one-to-one correspondence between the elements of S and $S_{D_4}^3$. The correspondence from S to $S_{D_4}^3$ is denoted by γ_2 and defined in Table 4.

Table 4. Correspondence between S and $S_{D_4}^3$.

Elements $e_1 \in S$	**Gray Images**	**DNA Codons $\gamma_2(e_1)$**
0	$(0,0,0)$	AAA
u	$(0,1,0)$	ATA
uw	$(0,w,0)$	ACA
$u(1+w)$	$(0,1+w,0)$	AGA
v	$(0,0,1)$	AAT
vw	$(0,0,w)$	AAC
$v(1+w)$	$(0,0,1+w)$	AAG
$u+v$	$(0,1,1)$	ATT
$u+vw$	$(0,1,w)$	ATC
$u+v(1+w)$	$(0,1,1+w)$	ATG
$uw+v$	$(0,w,1)$	ACT
$uw+vw$	$(0,w,w)$	ACC
$uw+v(1+w)$	$(0,w,1+w)$	ACG
$u(1+w)+v$	$(0,1+w,1)$	AGT
$u(1+w)+vw$	$(0,1+w,w)$	AGC
$u(1+w)+v(1+w)$	$(0,1+w,1+w)$	AGG

Table 4. *Cont.*

Elements $e_1 \in S$	Gray Images	DNA Codons $\gamma_2(e_1)$
1	$(1,1,1)$	TTT
$1+u$	$(1,0,1)$	TAT
$1+u(1+w)$	$(1,w,1)$	TCT
$1+uw$	$(1,1+w,1)$	TGT
$1+v$	$(1,1,0)$	TTA
$1+vw$	$(1,1,1+w)$	TTG
$1+v(1+w)$	$(1,1,w)$	TTC
$1+u+v$	$(1,0,0)$	TAA
$1+u+vw$	$(1,0,1+w)$	TAG
$1+u+v(1+w)$	$(1,0,w)$	TAC
$1+uw+v$	$(1,1+w,0)$	TGA
$1+uw+vw$	$(1,1+w,1+w)$	TGG
$1+uw+v(1+w)$	$(1,1+w,w)$	TGC
$1+u(1+w)+v$	$(1,w,0)$	TCA
$1+u(1+w)+vw$	$(1,w,1+w)$	TCG
$1+u(1+w)+v(1+w)$	$(1,w,w)$	TCC
w	(w,w,w)	CCC
$w+uw$	$(w,0,w)$	CAC
$w+u(1+w)$	$(w,1,w)$	CTC
$w+u$	$(w,1+w,w)$	CGC
$w+v$	$(w,w,1+w)$	CCG
$w+vw$	$(w,w,0)$	CCA
$w+v(1+w)$	$(w,w,1)$	CCT
$w+u+v$	$(w,1+w,1+w)$	CGG
$w+u+vw$	$(w,1+w,0)$	CGA
$w+u+v(1+w)$	$(w,1+w,1)$	CGT
$w+uw+v$	$(w,0,1+w)$	CAG
$w+uw+vw$	$(w,0,0)$	CAA
$w+uw+v(1+w)$	$(w,0,1)$	CAT
$w+u(1+w)+v$	$(w,1,1+w)$	CTG
$w+u(1+w)+vw$	$(w,1,0)$	CTA
$w+u(1+w)+v(1+w)$	$(w,1,1)$	CTT
$1+w$	$(1+w,1+w,1+w)$	GGG
$1+w+u(1+w)$	$(1+w,0,1+w)$	GAG
$1+w+uw$	$(1+w,1,1+w)$	GTG
$1+w+u$	$(1+w,w,1+w)$	GCG
$1+w+v$	$(1+w,1+w,w)$	GGC
$1+w+vw$	$(1+w,1+w,1)$	GGT
$1+w+v(1+w)$	$(1+w,1+w,0)$	GGA
$1+w+u+v$	$(1+w,w,w)$	GCC
$1+w+u+vw$	$(1+w,w,1)$	GCT
$1+w+u+v(1+w)$	$(1+w,w,0)$	GCA
$1+w+uw+v$	$(1+w,1,w)$	GTC
$1+w+uw+vw$	$(1+w,1,1)$	GTT
$1+w+uw+v(1+w)$	$(1+w,1,0)$	GTA
$1+w+u(1+w)+v$	$(1+w,0,w)$	GAC
$1+w+u(1+w)+vw$	$(1+w,0,1)$	GAT
$1+w+u(1+w)+v(1+w)$	$(1+w,0,0)$	GAA

By WCC, we have $\overline{A}=T$, $\overline{T}=A$, $\overline{C}=G$ and $\overline{G}=C$. We can extend this notation to the elements of $S_{D_4}^3$ such as $\overline{AAA}=TTT, \overline{ATA}=TAT, \cdots, \overline{GAA}=CTT$.

Definition 7. *Let C_γ be a linear code of length γ over S and $\mathbf{t}=(t_0,t_1,\ldots,t_{\gamma-1})\in C_\gamma$. We define*

$$\Phi_2(\mathbf{t}) : C_\gamma \longrightarrow S_{D_4}^{3\gamma},$$

given by

$$(t_0, t_1, \ldots, t_{\gamma-1}) \mapsto (\gamma_2(t_0)\gamma_2(t_1)\ldots\gamma_2(t_{\gamma-1})),$$

by using Table 4.

For example, $(t_0, t_1, t_2) = (1, uw, v)$ is mapped to $(\gamma_2(1)\gamma_2(uw)\gamma_2(v)) = (TTTACAAAT)$.

All the results in this subsection have similar proofs to the results discussed in the above subsection, so we omit their proofs.

Theorem 12. *Let $C_\gamma = \eta_1 C_{\gamma,1} \oplus \eta_2 C_{\gamma,2} \oplus \eta_3 C_{\gamma,3}$ be a cyclic code of length γ over S. Then C_γ is reversible over S if and only if $C_{\gamma,1}$, $C_{\gamma,2}$ and $C_{\gamma,3}$ are reversible over $\mathbb{F}_4$.*

To illustrate the above result, we present an example.

Example 6. *Let $S = \mathbb{F}_4 + u\mathbb{F}_4 + v\mathbb{F}_4$, and $\mathbb{F}_4 = \{0, 1, w, w^2 = w + 1\}$.*

$$x^9 - 1 = (x+1)(x+w)(x+w^2)(x^3+w)(x^3+w^2) \in \mathbb{F}_4[x].$$

Let $t_1(x) = (x+w)(x+w^2)$ and $t_2(x) = t_3(x) = (x^3+w)(x^3+w^2)$. Then $C_{\gamma,1} = \langle t_1(x)\rangle$, $C_{\gamma,2} = \langle t_2(x)\rangle$, and $C_{\gamma,3} = \langle t_3(x)\rangle$ are cyclic codes of length 9 over $\mathbb{F}_4$. By Theorem 4, $C_\gamma = \langle \eta_1 t_1(x) + \eta_2 t_2(x) + \eta_3 t_3(x)\rangle$ is a cyclic code of length 9 over S. As $t_1(x)$, $t_2(x)$ and $t_3(x)$ are self-reciprocal polynomials, so by Lemma 6, $C_{\gamma,1}$, $C_{\gamma,2}$ and $C_{\gamma,3}$ are reversible over $\mathbb{F}_4$. Hence, by Theorem 12, C_γ is reversible over S.

Example 7. *Let $S = \mathbb{F}_4 + u\mathbb{F}_4 + v\mathbb{F}_4$, and $\mathbb{F}_4 = \{0, 1, w, w^2 = w + 1\}$.*

$$x^{13} - 1 = (x+1)(x^6 + wx^5 + w^2x^3 + wx + 1)(x^6 + w^2x^5 + wx^3 + w^2x + 1) \in \mathbb{F}_4[x].$$

Let $t_1(x) = x^6 + wx^5 + w^2x^3 + wx + 1$ and $t_2(x) = t_3(x) = x^7 + wx^6 + w^2x^5 + wx^4 + wx^3 + w^2x^2 + wx + 1$. Then $C_{\gamma,1} = \langle t_1(x)\rangle$, $C_{\gamma,2} = \langle t_2(x)\rangle$, and $C_{\gamma,3} = \langle t_3(x)\rangle$ are cyclic codes of length 13 over $\mathbb{F}_4$. By Theorem 4, $C_\gamma = \langle \eta_1 t_1(x) + \eta_2 t_2(x) + \eta_3 t_3(x)\rangle$ is a cyclic code of length 9 over S. As $t_1(x)$, $t_2(x)$ and $t_3(x)$ are self-reciprocal polynomials, so by Lemma 6, $C_{\gamma,1}$, $C_{\gamma,2}$ and $C_{\gamma,3}$ are reversible over $\mathbb{F}_4$. Hence, by Theorem 12, C_γ is reversible over S.

From Table 4, we get the next useful lemma similar to Lemma 8. This result is used in determining reversible-complement property of cyclic codes over S.

Lemma 9. *For any $b \in S$, we have $\overline{b} + \overline{0} = b$.*

Theorem 13. *Let $C_\gamma = \eta_1 C_{\gamma,1} \oplus \eta_2 C_{\gamma,2} \oplus \eta_3 C_{\gamma,3}$ be a cyclic code of length γ over S. Then C_γ is reversible-complement over S if and only if $(\overline{0}, \overline{0}, \ldots, \overline{0}) \in C_\gamma$ and C_γ is reversible over S.*

To illustrate the above result, we present an example.

Example 8. *Let $S = \mathbb{F}_4 + u\mathbb{F}_4 + v\mathbb{F}_4$, and $\mathbb{F}_4 = \{0, 1, w, w^2 = w + 1\}$.*

$$x^3 - 1 = (x+w^2)(x+w)(x+1) \in \mathbb{F}_4[x].$$

Let $t_1(x) = t_2(x) = t_3(x) = (x+w^2)(x+w)$. Then $C_{\gamma,j} = \langle t_j(x)\rangle$ are cyclic codes of length 3 over $\mathbb{F}_4$, for $j = 1,2,3$. By Theorem 4, $C_\gamma = \langle \eta_1 t_1(x) + \eta_2 t_2(x) + \eta_3 t_3(x)\rangle$ is a cyclic code of length 3 over S. As $t_1(x)$, $t_2(x)$ and $t_3(x)$ are self-reciprocal polynomials, hence by Lemma 6, $C_{\gamma,j}$ are reversible over $\mathbb{F}_4$, for $j = 1,2,3$. Therefore, by Theorem 12, C_γ is reversible over S. Further, C_γ has $4^{9-6} = 64$ codewords. The corresponding DNA codewords obtained by using Table 4, are listed below.

Notice that C_γ is reversible over S and $(TTTTTTTTT) = (\bar{0}, \bar{0}, \bar{0}) \in C_\gamma$. Thus, by Theorem 13, C_γ is reversible-complement over S. Moreover, by Definition 5, we conclude that C_γ is a cyclic DNA code. The image of C_γ under the map ψ_2 is a DNA code of length 9, size 64 and minimum Hamming distance 3. The DNA code given in Table 5 is different from the DNA code of the same length and size constructed by Dinh et al. [27].

Table 5. DNA code of length 9 obtained from C_γ.

AAAAAAAAA	TTTTTTTTT	CCCCCCCCC	GGGGGGGGG
ATAATAATA	TATTATTAT	CACCACCAC	GAGGAGGAG
ACAACAACA	TCTTCTTCT	CTCCTCCTC	GTGGTGGTG
AGAAGAAGA	TGTTGTTGT	CGCCGCCGC	GCGGCGGCG
AATAATAAT	TTATTATTA	CCGCCGCCG	GGCGGCGGC
AACAACAAC	TTGTTGTTG	CCACCACCA	GGTGGTGGT
AAGAAGAAG	TTCTTCTTC	CCTCCTCCT	GGAGGAGGA
ATTATTATT	TAATAATAA	CGGCGGCGG	GCCGCCGCC
ATCATCATC	TAGTAGTAG	CGACGACGA	GCAGCAGCA
ATGATGATG	TACTACTAC	CGTCGTCGT	GCTGCTGCT
ACTACTACT	TGATGATGA	CAGCAGCAG	GTCGTCGTC
ACCACCACC	TGCTGCTGC	CAACAACAA	GTAGTAGTA
ACGACGACG	TGGTGGTGG	CATCATCAT	GTTGTTGTT
AGTAGTAGT	TCATCATCA	CTGCTGCTG	GACGACGAC
AGCAGCAGC	TCGTCGTCG	CTACTACTA	GATGATGAT
AGGAGGAGG	TCCTCCTCC	CTTCTTCTT	GAAGAAGAA

Example 9. *Let $S = \mathbb{F}_4 + u\mathbb{F}_4 + v\mathbb{F}_4$, and $\mathbb{F}_4 = \{0, 1, w, w^2 = w + 1\}$.*

$$x^7 - 1 = (x+1)(x^3 + x + 1)(x^3 + x^2 + 1) \in \mathbb{F}_4[x].$$

Let $t_1(x) = t_2(x) = t_3(x) = (x^3 + x + 1)(x^3 + x^2 + 1)$. Then $C_{\gamma,j} = \langle t_j(x) \rangle$ are cyclic codes of length 7 over $\mathbb{F}_4$, for $j = 1, 2, 3$. By Theorem 4, $C_\gamma = \langle \eta_1 t_1(x) + \eta_2 t_2(x) + \eta_3 t_3(x) \rangle$ is a cyclic code of length 7 over S. As $t_1(x)$, $t_2(x)$ and $t_3(x)$ are self-reciprocal polynomials, hence by Lemma 6, $C_{\gamma,j}$ are reversible over $\mathbb{F}_4$, for $j = 1, 2, 3$. Therefore, by Theorem 12, C_γ is reversible over S. Further, C_γ has $4^{21-18} = 64$ codewords. The corresponding DNA codewords obtained by using Table 6, are listed below.

Table 6. DNA code of length 21 obtained from C_γ.

AAAAAAAAAAAAAAAAAAAAA	TTTTTTTTTTTTTTTTTTTTT
CCCCCCCCCCCCCCCCCCCCC	GGGGGGGGGGGGGGGGGGGGG
ATAATAATAATAATAATAATA	TATTATTATTATTATTATTAT
CACCACCACCACCACCACCAC	GAGGAGGAGGAGGAGGAGGAG
ACAACAACAACAACAACAACA	TCTTCTTCTTCTTCTTCTTCT
CTCCTCCTCCTCCTCCTCCTC	GTGGTGGTGGTGGTGGTGGTG
AGAAGAAGAAGAAGAAGAAGA	TGTTGTTGTTGTTGTTGTTGT
CGCCGCCGCCGCCGCCGCCGC	GCGGCGGCGGCGGCGGCGGCG
AATAATAATAATAATAATAAT	TTATTATTATTATTATTATTA
CCGCCGCCGCCGCCGCCGCCG	GGCGGCGGCGGCGGCGGCGGC
AACAACAACAACAACAACAAC	TTGTTGTTGTTGTTGTTGTTG
CCACCACCACCACCACCACCA	GGTGGTGGTGGTGGTGGTGGT
AAGAAGAAGAAGAAGAAGAAG	TTCTTCTTCTTCTTCTTCTTC
CCTCCTCCTCCTCCTCCTCCT	GGAGGAGGAGGAGGAGGAGGA
ATTATTATTATTATTATTATT	TAATAATAATAATAATAATAA
CGGCGGCGGCGGCGGCGGCGG	GCCGCCGCCGCCGCCGCCGCC
ATCATCATCATCATCATCATC	TAGTAGTAGTAGTAGTAGTAG
CGACGACGACGACGACGACGA	GCAGCAGCAGCAGCAGCAGCA
ATGATGATGATGATGATGATG	TACTACTACTACTACTACTAC
CGTCGTCGTCGTCGTCGTCGT	GCTGCTGCTGCTGCTGCTGCT

Table 6. *Cont.*

$ACTACTACTACTACTACTACT$	$TGATGATGATGATGATGATGA$
$CAGCAGCAGCAGCAGCAGCAG$	$GTCGTCGTCGTCGTCGTCGTC$
$ACCACCACCACCACCACCACC$	$TGCTGCTGCTGCTGCTGCTGC$
$CAACAACAACAACAACAACAA$	$GTAGTAGTAGTAGTAGTAGTA$
$ACGACGACGACGACGACGACG$	$TGGTGGTGGTGGTGGTGGTGG$
$CATCATCATCATCATCATCAT$	$GTTGTTGTTGTTGTTGTTGTT$
$AGTAGTAGTAGTAGTAGTAGT$	$TCATCATCATCATCATCATCA$
$CTGCTGCTGCTGCTGCTGCTG$	$GACGACGACGACGACGACGAC$
$AGCAGCAGCAGCAGCAGCAGC$	$TCGTCGTCGTCGTCGTCGTCG$
$CTACTACTACTACTACTACTA$	$GATGATGATGATGATGATGAT$
$AGGAGGAGGAGGAGGAGGAGG$	$TCCTCCTCCTCCTCCTCCTCC$
$CTTCTTCTTCTTCTTCTTCTT$	$GAAGAAGAAGAAGAAGAAGAA$

Notice that C_γ is reversible over S and $(TTTTTTTTTTTTTTTTTTTTT) = (\bar{0},\bar{0},\bar{0},\bar{0},\bar{0},\bar{0},\bar{0}) \in C_\gamma$. Thus, by Theorem 13, C_γ is reversible-complement over S. Moreover, by Definition 5, we conclude that C_γ is a cyclic DNA code. The image of C_γ under the map ψ_2 is a DNA code of length 21*, size* 64 *and minimum Hamming distance* 7*. These codewords are given in Table 6.*

5.4. Reversible and Reversible-Complement Codes over $\mathbb{F}_4RS$

In this subsection, we now study reversible constraint and reversible-complement constraint of $\mathbb{F}_4RS$-cyclic codes.

Let $\mathbf{s}_1 = (m_0, m_1, \ldots, m_{\alpha-1}, a_0, a_1, \ldots, a_{\beta-1}, t_0, t_1, \ldots, t_{\gamma-1}) \in \mathbb{F}_4^\alpha \times R^\beta \times S^\gamma$. Then, the reverse of $\mathbf{s}_1$ is defined as $\mathbf{s}_1^r = (m_{\alpha-1}, m_{\alpha-2}, \ldots, m_0, a_{\beta-1}, a_{\beta-2}, \ldots, a_0, t_{\gamma-1}, t_{\gamma-2}, \ldots, t_0)$, the complement of $\mathbf{s}_1$ is defined as $\mathbf{s}_1^c = (\overline{m_0}, \overline{m_1}, \ldots, \overline{m_{\alpha-1}}, \overline{a_0}, \overline{a_1}, \ldots, \overline{a_{\beta-1}}, \overline{t_0}, \overline{t_1}, \ldots, \overline{t_{\gamma-1}})$ and the reverse-complement of $\mathbf{s}_1$ is defined as $\mathbf{s}_1^{rc} = (\overline{m_{\alpha-1}}, \overline{m_{\alpha-2}}, \ldots, \overline{m_0}, \overline{a_{\beta-1}}, \overline{a_{\beta-2}}, \ldots, \overline{a_0}, \overline{t_{\gamma-1}}, \overline{t_{\gamma-2}}, \ldots, \overline{t_0})$.

Definition 8. *Let C be a $\mathbb{F}_4RS$-linear code of block length (α, β, γ). Then, C is said to be* **reversible** *if for any $\mathbf{s}_1 \in C$, $\mathbf{s}_1^r \in C$,* **complement** *if for any $\mathbf{s}_1 \in C$, $\mathbf{s}_1^c \in C$ and* **reversible-complement** *if for any $\mathbf{s}_1 \in C$, $\mathbf{s}_1^{rc} \in C$.*

Using the results obtained in the above subsections, we now discuss the reversible and reversible-complement constraints of separable $\mathbb{F}_4RS$-cyclic codes.

Theorem 14. *Suppose $C = C_\alpha \times C_\beta \times C_\gamma$ is a separable $\mathbb{F}_4RS$-cyclic code of block length (α, β, γ), where C_α, C_β and C_γ are cyclic codes of length α, β and γ over $\mathbb{F}_4$, R and S, respectively. Then C is reversible if and only if C_α, C_β and C_γ are reversible over $\mathbb{F}_4$, R and S, respectively.*

Proof. Suppose $C = C_\alpha \times C_\beta \times C_\gamma$ is reversible and $\mathbf{s}_1 = (m_0, m_1, \ldots, m_{\alpha-1}, a_0, a_1, \ldots, a_{\beta-1}, t_0, t_1, \ldots, t_{\gamma-1}) \in C$, where $(m_0, m_1, \ldots, m_{\alpha-1}) \in C_\alpha$, $(a_0, a_1, \ldots, a_{\beta-1}) \in C_\beta$, and $(t_0, t_1, \ldots, t_{\gamma-1}) \in C_\gamma$. Since C is reversible, then we have $\mathbf{s}_1^r = (m_{\alpha-1}, m_{\alpha-2}, \ldots, m_1, m_0, a_{\beta-1}, a_{\beta-2}, \ldots, a_1, a_0, t_{\gamma-1}, t_{\gamma-2}, \ldots, t_1, t_0) \in C$, which implies $(m_{\alpha-1}, m_{\alpha-2}, \ldots, m_1, m_0) \in C_\alpha$, $(a_{\beta-1}, a_{\beta-2}, \ldots, a_1, a_0) \in C_\beta$ and $(t_{\gamma-1}, t_{\gamma-2}, \ldots, t_1, t_0) \in C_\gamma$. Thus, C_α, C_β and C_γ are reversible over $\mathbb{F}_4$, R and S, respectively.

Conversely, let $\mathbf{s}_1 = (m_0, m_1, \ldots, m_{\alpha-1}, a_0, a_1, \ldots, a_{\beta-1}, t_0, t_1, \cdots, t_{\gamma-1}) \in C$, where $(m_0, m_1, \ldots, m_{\alpha-1}) \in C_\alpha$, $(a_0, a_1, \ldots, a_{\beta-1}) \in C_\beta$ and $(t_0, t_1, \ldots, t_{\gamma-1}) \in C_\gamma$. Suppose C_α, C_β and C_γ are reversible over $\mathbb{F}_4$, R and S, respectively. Then $(m_{\alpha-1}, m_{\alpha-2}, \ldots, m_1, m_0) \in C_\alpha$, $(a_{\beta-1}, a_{\beta-2}, \ldots, a_1, a_0) \in C_\beta$ and $(t_{\gamma-1}, t_{\gamma-2}, \ldots, t_1, t_0) \in C_\gamma$. Thus, $\mathbf{s}_1^r = (m_{\alpha-1}, m_{\alpha-2}, \ldots, m_1, m_0, a_{\beta-1}, a_{\beta-2}, \ldots, a_1, a_0, t_{\gamma-1}, t_{\gamma-2}, \ldots, t_1, t_0) \in C$. Therefore, C is reversible. $\square$

Example 10. *Let $C = C_\alpha \times C_\beta \times C_\gamma$ be a separable $\mathbb{F}_4RS$-cyclic code of block length $(7, 5, 9)$.*

$$x^7 - 1 = (x+1)(x^3 + x + 1)(x^3 + x^2 + 1) \in \mathbb{F}_4[x].$$

Let $f(x) = (x^3 + x + 1)(x^3 + x^2 + 1)$. Then by Theorem 2, $C_\alpha = \langle f(x) \rangle$ is a cyclic code of length 7 over $\mathbb{F}_4$. As $f(x)$ is a self-reciprocal polynomial, hence by Lemma 6, C_α is reversible over $\mathbb{F}_4$. Further, we consider the same C_β as given in Example 2, which is reversible over R and the same C_γ as given in Example 6, which is reversible over S. Therefore, by Theorem 14, we get $C = C_\alpha \times C_\beta \times C_\gamma$ is reversible.

In the next result, we discuss the necessary and sufficient conditions for a $\mathbb{F}_4RS$-cyclic code to be reversible-complement.

Theorem 15. *Suppose $C = C_\alpha \times C_\beta \times C_\gamma$ is a separable $\mathbb{F}_4RS$-cyclic code of block length (α, β, γ), where C_α, C_β and C_γ are cyclic codes of length α, β and γ over $\mathbb{F}_4$, R and S, respectively. Then C is reversible-complement if and only if C_α, C_β and C_γ are reversible-complement over $\mathbb{F}_4$, R and S, respectively.*

Proof. Suppose $C = C_\alpha \times C_\beta \times C_\gamma$ is reversible-complement and $\mathbf{s}_1 = (m_0, m_1, \ldots, m_{\alpha-1}, a_0, a_1, \ldots, a_{\beta-1}, t_0, t_1, \ldots, t_{\gamma-1}) \in C$, where $(m_0, m_1, \ldots, m_{\alpha-1}) \in C_\alpha$, $(a_0, a_1, \ldots, a_{\beta-1}) \in C_\beta$ and $(t_0, t_1, \ldots, t_{\gamma-1}) \in C_\gamma$. Since C is reversible-complement, then we have $\mathbf{s}_1^{rc} = (\overline{m_{\alpha-1}}, \overline{m_{\alpha-2}}, \ldots, \overline{m_1}, \overline{m_0}, \overline{a_{\beta-1}}, \overline{a_{\beta-2}}, \ldots, \overline{a_1}, \overline{a_0}, \overline{t_{\gamma-1}}, \overline{t_{\gamma-2}}, \ldots, \overline{t_1}, \overline{t_0}) \in C$, which implies $(\overline{m_{\alpha-1}}, \overline{m_{\alpha-2}}, \ldots, \overline{m_1}, \overline{m_0}) \in C_\alpha$, $(\overline{a_{\beta-1}}, \overline{a_{\beta-2}}, \ldots, \overline{a_1}, \overline{a_0}) \in C_\beta$ and $(\overline{t_{\gamma-1}}, \overline{t_{\gamma-2}}, \ldots, \overline{t_1}, \overline{t_0}) \in C_\gamma$. Thus, C_α, C_β and C_γ are reversible-complement over $\mathbb{F}_4$, R and S, respectively.

Conversely, let $\mathbf{s}_1 = (m_0, m_1, \ldots, m_{\alpha-1}, a_0, a_1, \ldots, a_{\beta-1}, t_0, t_1, \ldots, t_{\gamma-1}) \in C$, where $(m_0, m_1, \ldots, m_{\alpha-1}) \in C_\alpha$, $(a_0, a_1, \ldots, a_{\beta-1}) \in C_\beta$ and $(t_0, t_1, \ldots, t_{\gamma-1}) \in C_\gamma$. Suppose C_α, C_β and C_γ are reversible-complement over $\mathbb{F}_4$, R and S, respectively. Then $(\overline{m_{\alpha-1}}, \overline{m_{\alpha-2}}, \ldots, \overline{m_1}, \overline{m_0}) \in C_\alpha$, $(\overline{a_{\beta-1}}, \overline{a_{\beta-2}}, \ldots, \overline{a_0}) \in C_\beta$ and $(\overline{t_{\gamma-1}}, \overline{t_{\gamma-2}}, \ldots, \overline{t_1}, \overline{t_0}) \in C_\gamma$. Thus, $\mathbf{s}_1^{rc} = (\overline{m_{\alpha-1}}, \overline{m_{\alpha-2}}, \ldots, \overline{m_1}, \overline{m_0}, \overline{a_{\beta-1}}, \overline{a_{\beta-2}}, \ldots, \overline{a_1}, \overline{a_0}, \overline{t_{\gamma-1}}, \overline{t_{\gamma-2}}, \ldots, \overline{t_1}, \overline{t_0}) \in C$. Hence, C is reversible-complement. $\square$

Example 11. *Let $C = C_\alpha \times C_\beta \times C_\gamma$ be a separable $\mathbb{F}_4RS$-cyclic code of block length $(3, 5, 3)$.*

$$x^3 - 1 = (x+1)(x+w^2)(x+w) \in \mathbb{F}_4[x].$$

Let $f(x) = (x+w)(x+w^2)$. Then by Theorem 2, $C_\alpha = \langle f(x) \rangle$ is a cyclic code of length 3 over $\mathbb{F}_4$. As $f(x)$ is self-reciprocal polynomial, not divisible by $(x-1)$, hence by Theorem 9, C_α is reversible-complement over $\mathbb{F}_4$. Further, we consider the same C_β as given in Example 5, which is reversible-complement over R and the same C_γ as given in Example 8, which is reversible-complement over S. Hence, by Theorem 15, we get $C = C_\alpha \times C_\beta \times C_\gamma$ is reversible-complement. Therefore, by Definition 5, C is a cyclic DNA code. The image of C under the map Ψ is a DNA code of length 22, size 3072 and minimum Hamming distance 3.

Example 12. *Let $C = C_\alpha \times C_\beta \times C_\gamma$ be a separable $\mathbb{F}_4RS$-cyclic code of block length $(5, 3, 7)$.*

$$x^5 - 1 = (x+1)(x^2 + wx + 1)(x^2 + w^x + 1) \in \mathbb{F}_4[x].$$

Let $f(x) = (x^2 + w^2x + 1)$. Then by Theorem 2, $C_\alpha = \langle f(x) \rangle$ is a cyclic code of length 5 over $\mathbb{F}_4$. As $f(x)$ is self-reciprocal polynomial, not divisible by $(x-1)$, hence by Theorem 9, C_α is reversible-complement over $\mathbb{F}_4$. Further, we consider the same C_β as given in Example 4, which is reversible-complement over R and the same C_γ as given in Example 9, which is reversible-complement over S. Hence, by Theorem 15, we get $C = C_\alpha \times C_\beta \times C_\gamma$ is reversible-complement. Therefore, by Definition 5, C is a cyclic DNA code. The image of C under the map Ψ is a DNA code of length 32, size 65536 and minimum Hamming distance 3.

6. Conclusions and Future Direction

This paper considers the rings $R = \mathbb{F}_4 + u\mathbb{F}_4$, with $u^2 = u$ and $S = \mathbb{F}_4 + u\mathbb{F}_4 + v\mathbb{F}_4$, with $u^2 = u$, $v^2 = v$, $uv = vu = 0$ as alphabets to construct cyclic DNA codes. We first study the decomposed structure of both the rings and then discuss linear codes over these rings. We have further studied the algebraic structure and generator polynomials of over $\mathbb{F}_4$, R and S. Using these polynomials, we determine the generator polynomials of $\mathbb{F}_4RS$-cyclic codes of block length (α, β, γ). Moreover, we determine the generators of separable $\mathbb{F}_4RS$-cyclic codes. We study cyclic DNA codes over the ring $\mathbb{F}_4$, R, S and $\mathbb{F}_4RS$ from the structure of separable codes. We define a map from R to a set of DNA alphabets of order 16 and then study reverse constraint and reverse-complement constraint of cyclic codes over R. A similar kind of map from S to a set of DNA alphabets of order 64 has been defined. The reverse constraint and reverse-complement constraint of cyclic codes over S are studied. By using the above-obtained constraints, reverse constraint and reverse-complement constraint of $\mathbb{F}_4RS$-cyclic codes have been discussed. To support our results, we present several examples and construct cyclic DNA codes. In the modern age of e-business and e-commerce, the security of confidentiality, integrity and availability of stored and distributed data is essential. The rising technical complexity is leading us to the need for a new paradigm. As a consequence, unorthodox approaches to coding theory have evolved from the recent past, and considerable attention is being paid to DNA coding theory. This method will be a good source for constructing cyclic DNA codes over mixed alphabets. In Section 5, the applications have only been discussed by using the structure of separable codes. In the future, we will try to work on such applications over non-separable codes. We will also try to study skew cyclic DNA codes over single and mixed alphabets. More generally, it would be interesting to study how cyclic DNA codes involve in the evolution of genetic codes and possible role they may play.

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