mathematics

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Aggregation of L-Probabilistic Quasi-Uniformities

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Received: 30 September 2020; Accepted: 2 November 2020; Published: 6 November 2020

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Abstract: The problem of aggregating fuzzy structures, mainly fuzzy binary relations, has deserved a lot of attention in the last years due to its application in several fields. Here, we face the problem of studying which properties must satisfy a function in order to merge an arbitrary family of (bases of) L-probabilistic quasi-uniformities into a single one. These fuzzy structures are special filters of fuzzy binary relations. Hence we first make a complete study of functions between partially-ordered sets that preserve some special sets, such as filters. Afterwards, a complete characterization of those functions aggregating bases of L-probabilistic quasi-uniformities is obtained. In particular, attention is paid to the case $L = \{0,1\}$, which allows one to obtain results for functions which aggregate crisp quasi-uniformities. Moreover, we provide some examples of our results including one showing that Lowen's functor ι which transforms a probabilistic quasi-uniformity into a crisp quasi-uniformity can be constructed using this aggregation procedure.

Keywords: filters; L-probabilistic quasi-uniformity; aggregation

MSC: 4A40; 54E15; 06A06

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1. Introduction

Aggregation can be considered as a method for merging a list of numbers in a single representative one. This process of aggregation appears in areas where decision-making is important in probability, computer science, economics, etc. There are several aggregation functions that are commonly known, that is, functions making this aggregation process, like the arithmetic mean. However the growing interest on this topic has lead aggregation functions to become a very active area of research (see [1–3]). Thus an aggregation function is usually a function $F : [0,1]^n \to [0,1]$ such that

- $F(0,0,\ldots,0) = 0$ and $F(1,1,\ldots,1) = 1$;
- $F(\boldsymbol{x}) \leq F(\boldsymbol{y})$ whenever $\boldsymbol{x}, \boldsymbol{y} \in [0,1]^n$ and $\boldsymbol{x}_i \leq \boldsymbol{y}_i$ for all $i \in \{1,\ldots,n\}$.

Therefore, generally aggregation functions only aggregate a finite amount of numbers of the unit interval, are isotone, and satisfy some boundary conditions ([3] (Definition 1.1)), ([2] (Definition 1.5)). Nevertheless, some extensions of these basic aggregation functions have been considered in the literature. On the one hand, functions aggregating inputs for any fixed number of arguments, called *extended aggregation functions*, are often used as well as *infinitary aggregation functions* ([3] (Definition A.1.)), [4] which allow one to aggregate infinitely but countably many inputs. On the other hand, aggregation functions based on bounded partially ordered sets rather than on $([0,1], \leq)$ have also been studied [5,6].

In mathematics, this process of aggregation does not always consider numbers but other mathematical structures. For example, finite unions and finite intersections of subsets of a nonempty set X fit into this aggregation procedure. In this way, let us identify a subset A of X with its characteristic function $\chi_A : X \to \{0,1\}$. Consider the aggregation function $F : [0,1]^n \to [0,1]$ given by $F(x_1, \ldots, x_n) = \max\{x_1, \ldots, x_n\}$. Given a nonempty family $\{A_1, \ldots, A_n\}$ of subsets of X, consider $\chi_{A_1} \triangle \ldots \triangle \chi_{A_n} : X \to [0,1]^n$, the diagonal of the mappings $\{\chi_{A_i}\}_{i=1}^n$ [7] given by:

$$(\chi_{A_1} \triangle \ldots \triangle \chi_{A_n})(x) = (\chi_{A_1}(x), \ldots, \chi_{A_n}(x))$$

for all $x \in X$. Then the composition $F \circ (\chi_{A_1} \triangle \ldots \triangle \chi_{A_n})$ is the characteristic function of the set $\cup_{i=1}^n A_i$. If F is taken as the minimum, then the characteristic function of the intersection $\cap_{i=1}^n A_i$ is obtained.

Furthermore, the construction of some mathematical structures on the cartesian product of sets can be considered an aggregation process: Product topology, supremum topology, product of metric spaces, etc. Maybe the first aggregation process established in a precursor modern terminology is that which merges several metrics into a single one [8,9] developed by Doběs and his collaborators. The corresponding problem for pseudometrics and quasi-metrics was developed by Pradera and Trillas [10] and Mayor and Valero [11]. In the fuzzy context, the problem of characterizing functions that aggregate fuzzy binary relations satisfying certain properties has received a lot of attention [12–17] particularly, when indistinguishability operators [18] are considered. In this last case, the characterizations make use of the concept of $*$-triangular triplet [15] where $*$ is a t-norm (cf. Definition 11). Recently, the aggregation of relaxed indistinguishability operators [19] and partial indistinguishability operators [20] have also been studied. Closely related with the problem of aggregating fuzzy binary relations, in [21] the functions which aggregate fuzzy (quasi-)metrics were characterized. Moreover, the aggregation of other fuzzy structures have been developed: Fuzzy subgroups [22]; soft topological spaces [23]; intuitionistic fuzzy sets [24]; hesitant probabilistic fuzzy elements [25]; and probabilistic dual hesitant fuzzy sets [26]; etc.

This paper is devoted to examining the aggregation of another important fuzzy structure: The L-probabilistic quasi-uniformities. In this way, the main goal of this paper is to characterize those functions, which allow the merging of an arbitrary family of L-probabilistic quasi-uniformities into a single one (see Definitions 6, 7 and 9). Probabilistic uniformities were first considered by Höhle and Katsaras [27,28] as a fuzzy counterpart of uniformities. Lowen introduced in [29] the t-norm $\wedge$, a special type of probabilistic uniformities, now called Lowen uniformities or Lowen-Höhle uniformities [30], which were also studied by Höhle [31] for an arbitrary t-norm. Asymmetric versions of these concepts have already been considered in [32].

The probabilistic quasi-uniformities can be considered a fuzzy counterpart of classical quasi-uniformities. A quasi-uniformity on a nonempty set X is a filter of reflexive binary relations on $X \times X$ satisfying a certain triangular property [33]. On its part, probabilistic quasi-uniformities are filters of fuzzy binary relations verifying a triangular property with respect to a t-norm. Therefore, if we intend to characterize those functions merging a family of probabilistic quasi-uniformities into a single one, it is natural to guess that the aggregation of fuzzy binary relations will have an important role. Nevertheless, more ingredients must be added in order to treat the filter structure and the triangular property.

In this way, in Section 2 we deal with functions between partially ordered sets that preserve sets that satisfy some order properties involved in the definition of a probabilistic quasi-uniformity: Upper sets, filtered sets, filters, and principal filters. We will show that these functions coincide with those functions that preserve families of fuzzy sets satisfying the same property except in the case of filtered sets.

In Section 3, we address the study of those functions which allow the fusion of a family of (bases of) probabilistic quasi-uniformities into a single one. In this way, we completely characterize the functions that merge a collection, not necessarily finite, of bases of probabilistic quasi-uniformities into another one. In the case where we consider probabilistic quasi-uniformities instead of bases, we obtain some necessary and sufficient conditions for a function to aggregate them. We also provide some examples of our results. In particular, we show that the construction of a crisp quasi-uniformity starting from a probabilistic quasi-uniformity by means of the Lowen's functor ι [29] can be performed with our results.

2. Functions Preserving Special Subsets of Partially Ordered Sets

As shown in the next section, probabilistic quasi-uniformities are special subsets of a partially ordered set (see Definition 6). In fact, they are a kind of filter of fuzzy sets. Since we are interested in characterizing those functions which merge a family of probabilistic quasi-uniformities into a single one, it is natural to study functions between partially ordered sets that preserve filters of fuzzy sets (see Definition 2). We do it in the following, but we first recall some concepts related with order theory (see for example [34]).

Definition 1 ([34])**.** *A preordered set is a pair $(L, \preceq)$ where L is a nonempty set and $\preceq$ is a reflexive and transitive binary relation on L, called a preorder.*

If a preorder $\preceq$ on L is also antisymmetric then it is called a partial order and $(L, \preceq)$ is said to be a partially ordered set. In the case that $\preceq$ also verifies that $a \preceq b$ or $b \preceq a$ for any $a, b \in L$ then $\preceq$ is called a total order and $(L, \preceq)$ is said to be totally ordered.

If $(L, \preceq)$ is a preordered set:

- *A subset A of L is said to be filtered if every finite subset of A has a lower bound in A;*
- *A subset A of L is said to be upper if $\uparrow A = A$ where $\uparrow A = \{x \in L : a \preceq x \text{ for some } a \in A\}$. If $A = \{a\}$ we will write $\uparrow a$ instead of $\uparrow \{a\}$;*
- *A subset A of L is said to be a filter if it is a filtered upper set;*
- *A filter is principal if it has a minimum element. Therefore, every principal filter is of the form $\uparrow m$, where m is its minimum element.*

Example 1. *Let us consider the partially ordered set $([0, 1], \leq)$. Since $\leq$ is a total order then every subset of $[0, 1]$ is filtered. Moreover the upper sets of $[0, 1]$ coincide with the filters and are of the form $(a, 1]$ or $[a, 1]$ where $a \in [0, 1]$. The principal filters are those sets of the form $[a, 1]$ for any $a \in [0, 1]$.*

Let $(L, \preceq)$ be a partially ordered set and X be a nonempty set. An L-fuzzy set on X is a function $\mu : X \to L$ [35]. We will denote by L^X the family of all L-fuzzy sets on X. Moreover, we can establish a pointwise partial order on L^X given by

$$\mu \preceq \eta \text{ whenever } \mu(x) \preceq \eta(x) \text{ for all } x \in X.$$

In order to not overload the notation we will use the same notation for the partial order on L and the partial order on L^X. Furthermore, we will always consider that a family of L-fuzzy sets on X is endowed with the partial order inherited from $(L^X, \preceq)$.

For each $a \in L$, we will denote by a_X the constant L-fuzzy set on X given by $a_X(x) = a$ for all $x \in X$.

Definition 2. *Let $(L, \preceq_L)$, $(S, \preceq_S)$ be two partially ordered sets and let P be any of the properties of Definition 1. A function $f : (L, \preceq_L) \to (S, \preceq_S)$ is said to:*

- *Preserve P sets if whenever $A \subseteq L$ satisfies property P then $f(A)$ satisfies property P;*
- *Preserve P families of fuzzy sets if given an arbitrary nonempty set X and a family $\{\mu_i : i \in I\}$ of L-fuzzy sets on X satisfying property P then the family $\{f \circ \mu_i : i \in I\}$ of S-fuzzy sets on X also satisfies property P.*

Although in the previous definition we introduced two different concepts, we will see in the following that these two notions coincide except when the property P is to be filtered. We begin characterizing those functions, preserving filtered families of fuzzy sets.

Proposition 1. *Let $(L, \preceq_L)$, $(S, \preceq_S)$ be two partially ordered sets. A function $f : (L, \preceq_L) \to (S, \preceq_S)$ preserves filtered families of fuzzy sets if and only if f is isotone.*

Proof. We first suppose that f preserves filtered families of fuzzy sets. If it is not isotone we can find $a, b \in L$ with $a \preceq_L b$ but $f(a) \not\preceq_S f(b)$. Let $X = \{x_1, x_2\}$ be a set with two different elements and consider the L-fuzzy sets μ, η, ξ on X given by:

$$\mu(x_i) = \begin{cases} a & \text{if } i = 1 \\ b & \text{if } i = 2 \end{cases}, \qquad \eta(x_i) = \begin{cases} b & \text{if } i = 1 \\ a & \text{if } i = 2 \end{cases}, \qquad \xi(x_i) = a, \quad i = 1, 2.$$

It is obvious that $\{\mu, \eta, \xi\}$ is filtered on $(L^X, \preceq_L)$. Nevertheless $\{f \circ \mu, f \circ \eta, f \circ \xi\}$ is not filtered on $(S^X, \preceq_S)$, and is a contradiction.

Conversely, let $\{\mu_i : i \in I\}$ be a filtered family of L-fuzzy sets on a nonempty set X. Let us show that $\{f \circ \mu_i : i \in I\}$ is filtered on $(S^X, \preceq_S)$. Let $i, j \in I$. Then we can find $k \in I$ such that $\mu_k \preceq_L \mu_i$ and $\mu_k \preceq_L \mu_j$. Since f is isotone then $f \circ \mu_k \preceq_S f \circ \mu_i$ and $f \circ \mu_k \preceq_S f \circ \mu_j$ so $\{f \circ \mu_i : i \in I\}$ is filtered. $\square$

Remark 1. *If a function $f : (L, \preceq_L) \to (S, \preceq_S)$ preserves filtered families of fuzzy sets then it is isotone by Proposition 1. It is easy to check that this implies that f preserves filtered sets. Nevertheless, the converse is not true in general. For example, let us consider the partially ordered set $([0,1], \leq)$. It is obvious that every subset of $[0,1]$ is filtered since $\leq$ is a total order. Then every function $f : ([0,1], \leq) \to ([0,1], \leq)$ preserves filtered sets. Nevertheless, if f is not isotone, by the above proposition, it does not preserve filtered families of fuzzy sets.*

We next characterize those functions which preserve filtered sets by means of the following concept.

Definition 3. *Let $f : (L \preceq_L) \to (S, \preceq_S)$ be a function between two partially ordered sets. We say that $f(L)$ is almost totally ordered if:*

- *Whenever $a \preceq_L b$ then $f(a) \preceq_S f(b)$ or $f(b) \preceq_S f(a)$;*
- *$f(\uparrow a) \setminus \uparrow f(a)$ is totally ordered for every $a \in L$.*

Notice that if $(S, \preceq_S)$ is totally ordered then $f(L)$ is almost totally ordered for every function $f : (L \preceq_L) \to (S, \preceq_S)$. It is also clear that isotonicity of f implies that $f(L)$ is almost totally ordered (notice that in this case $f(\uparrow a) \setminus \uparrow f(a) = \varnothing$ for every $a \in L$) but the converse is not true (see Remark 1).

Proposition 2. *Let $(L, \preceq_L)$, $(S, \preceq_S)$ be two partially ordered sets. A function $f : (L, \preceq_L) \to (S, \preceq_S)$ preserves filtered sets if and only if $f(L)$ is almost totally ordered.*

Proof. We first suppose that f preserves filtered sets. If $a \preceq_L b$ then $\{a, b\}$ is filtered so $f(\{a, b\})$ is also filtered. Hence $f(a) \preceq_S f(b)$ or $f(b) \preceq_S f(a)$. In order to prove the second condition of being almost totally ordered, let $a \in L$ and $f(b), f(c) \in f(\uparrow a) \setminus \uparrow f(a)$. Since $f(b), f(c) \in f(\uparrow a)$ then $a \preceq_L b$ and $a \preceq_L c$. Hence $\{a, b, c\}$ is filtered so, by hypothesis, $\{f(a), f(b), f(c)\}$ also is. Moreover, since it is a finite set, it has a minimum. Since $f(b), f(c) \notin \uparrow f(a)$ then $f(a)$ is not the minimum. So either $f(b)$ or $f(c)$ is the minimum. Hence $f(b)$ and $f(c)$ are comparable.

Conversely, let A be a filtered subset of $(L, \preceq_L)$ and fix $a, b \in A$. Then we can find $c \in A$ such that $c \preceq_L a$ and $c \preceq_L b$. Since $f(L)$ is almost totally ordered then:

$$f(c) \preceq_S f(a) \text{ or } f(a) \preceq_S f(c)$$

and

$$f(c) \preceq_S f(b) \text{ or } f(b) \preceq_S f(c).$$

So we have the following four cases:

If $f(c) \preceq_S f(a)$ and $f(c) \preceq_S f(b)$, it is obvious that $\{f(a), f(b), f(c)\}$ is filtered.
If $f(c) \preceq_S f(a)$ and $f(b) \preceq_S f(c)$, it is obvious that $\{f(a), f(b), f(c)\}$ is filtered.
If $f(a) \preceq_S f(c)$ and $f(c) \preceq_S f(b)$, it is obvious that $\{f(a), f(b), f(c)\}$ is filtered.

If $f(a) \preceq_S f(c)$ and $f(b) \preceq_S f(c)$ and $f(a) \neq f(c), f(b) \neq f(c)$ (otherwise we immediately conclude that $\{f(a), f(b), f(c)\}$ is filtered), we have that $f(a), f(b) \in f(\uparrow c) \setminus \uparrow f(c)$ which is totally ordered. Hence $f(a) \preceq_S f(b)$ or $f(b) \preceq_S f(a)$. In any case we obtain that $\{f(a), f(b), f(c)\}$ is filtered.

Therefore, $f(A)$ is filtered. $\square$

However, the functions which preserve upper sets are the functions which preserve upper families of fuzzy sets as will be shown.

Proposition 3. *Let $(L, \preceq_L)$, $(S, \preceq_S)$ be two partially ordered sets. Let $f : (L \preceq_L) \to (S, \preceq_S)$ be a function. The following statements are equivalent:*

(i) *f preserves upper sets;*
(ii) *$\uparrow f(a) \subseteq f(\uparrow a)$ for all $a \in L$;*
(iii) *f preserves upper families of fuzzy sets.*

Proof. $(i) \Rightarrow (ii)$ This is obvious since $\uparrow a$ is an upper set and $f(a) \in f(\uparrow a)$.

$(ii) \Rightarrow (iii)$ Let $\{\mu_i : i \in I\}$ be an upper family of L-fuzzy sets on a nonempty set X. Let $\xi \in \uparrow \{f \circ \mu_i : i \in I\}$ so we can find $j \in I$ such that $f \circ \mu_j \preceq_S \xi$. For each $x \in X$, $\xi(x) \in \uparrow f(\mu_j(x)) \subseteq f(\uparrow \mu_j(x))$ so there exists $\eta_j(x) \in \uparrow \mu_j(x)$ such that $\xi(x) = f(\eta_j(x))$. Since x was arbitrary, we can consider η_j as an L-fuzzy set on X. As $\mu_j \preceq_L \eta_j$ then $\eta_j \in \uparrow \{\mu_i : i \in I\} = \{\mu_i : i \in I\}$ so $\xi \in \{f \circ \mu_i : i \in I\}$ as desired.

$(iii) \Rightarrow (i)$ Let $A \subseteq L$ be an upper set and let $a \in A$ and $s \in \uparrow f(a)$. Consider a nonempty set X and $a_X \in L^X$ given by $a_X(x) = a$ for all $x \in X$. Then $\uparrow a_X$ is an upper set on $(L^X, \preceq_L)$ so $\{f \circ \eta : \eta \in \uparrow a_X\}$ must also be an upper set on $(S^X, \preceq_S)$. Since the constant S-fuzzy set on X s_X belongs to $\uparrow \{f \circ \eta : \eta \in \uparrow a_X\} = \{f \circ \eta : \eta \in \uparrow a_X\}$, we can find $\mu \in \uparrow a_X$ such that $s_X = f \circ \mu$. Given $x \in X$ then $s = s_X(x) = f(\mu(x))$ and $a_X(x) = a \preceq_L \mu(x)$. Since $\mu(x) \in \uparrow a = A$, we conclude that $s \in f(A)$. Consequently, $f(A)$ is upper. $\square$

Remark 2. *Recall that given a partially ordered set $(L, \preceq_L)$, its Alexandroff topology $\tau_{\preceq_L}$ has as open sets for all the upper subsets of L. Then the above result asserts that f preserves upper families of fuzzy sets if and only if $f : (L, \tau_{\preceq_L}) \to (S, \tau_{\preceq_S})$ is open, which is equivalent to $\uparrow f(a) \subseteq f(\uparrow a)$ for all $a \in L$.*

We also recall the well-known fact that $f : (L, \preceq_L) \to (S, \preceq_S)$ is isotone if and only if $f : (L, \tau_{\preceq_L}) \to (S, \tau_{\preceq_S})$ is continuous. Hence, we have that $f : (L, \preceq_L) \to (S, \preceq_S)$ preserves upper families of fuzzy sets and filtered families of fuzzy sets if and only if $f : (L, \tau_{\preceq_L}) \to (S, \tau_{\preceq_S})$ is open and continuous, which is equivalent to $\uparrow f(a) = f(\uparrow a)$ for all $a \in L$.

We next study those functions f which preserve families of fuzzy sets that are filters. To simplify the terminology, we will say that f preserves filters of fuzzy sets. One could guess that the characterization of these functions can be obtained combining Propositions 1 and 3 but this is no longer true (see Theorem 1 and Remark 4). We provide first the following result which gives an equivalence between functions preserving principal filters and functions preserving principal filters of fuzzy sets.

Proposition 4. *Let $(L, \preceq_L), (S, \preceq_S)$ be two partially ordered sets. A function $f : (L, \preceq_L) \to (S, \preceq_S)$ preserves principal filters of fuzzy sets if and only if f preserves principal filters.*

Proof. Suppose that f preserves principal filters of fuzzy sets. Let $\uparrow l$ be a principal filter on L. Let us consider a nonempty set X and the constant L-fuzzy set $l_X \in L^X$ given by $l_X(x) = l$ for all $x \in X$. Then $\uparrow l_X$ is a principal filter of L-fuzzy sets on X so by assumption $\{f \circ \eta : \eta \in \uparrow l_X\}$ is also a principal filter of S-fuzzy sets on X. Therefore, there exists $\xi \in \uparrow l_X$ such that $\uparrow f \circ \xi = \{f \circ \eta : \eta \in \uparrow l_X\}$.

Let us check that $f \circ \xi$ is constant. Otherwise, we can find two different elements $x_1, x_2 \in X$ such that $(f \circ \xi)(x_1) \neq (f \circ \xi)(x_2)$. Therefore $\xi(x_1) \neq \xi(x_2)$. Since $\xi \in \uparrow l_X$ then $(\xi(x_1))_X, (\xi(x_2))_X \in \uparrow l_X$ so $f \circ (\xi(x_1))_X, f \circ (\xi(x_2))_X \in \{f \circ \eta : \eta \in \uparrow l_X\} = \uparrow f \circ \xi$. In particular:

$$(f \circ \xi)(x_1) \preceq_S (f \circ (\xi(x_2))_X)(x_1) = (f \circ \xi)(x_2)$$
$$(f \circ \xi)(x_2) \preceq_S (f \circ (\xi(x_1))_X)(x_2) = (f \circ \xi)(x_1)$$

so $(f \circ \xi)(x_1) = (f \circ \xi)(x_2)$, a contradiction. Consequently, $f \circ \xi = s_X$ for some $s \in S$. Finally, we prove that $f(\uparrow l) = \uparrow s$. In fact, let $a \in \uparrow l$. Then $a_X \in \uparrow l_X$ so $f \circ a_X \in \{f \circ \eta : \eta \in \uparrow l_X\} = \uparrow s_X$, that is, $s_X \preceq_S f \circ a_X$ so $f(a) \in \uparrow s$.

On the other hand, let $b \in \uparrow s$. Then $b_X \in \uparrow s_X = \{f \circ \eta : \eta \in \uparrow l_X\}$ so $b_X = f \circ \mu$ for some $\mu \in \uparrow l_X$. Then, given $x \in X$ we have that $l_X(x) = l \preceq_L \mu(x)$, that is, $\mu(x) \in \uparrow l$. Since $f(\mu(x)) = b_X(x) = b$ we have that $b \in f(\uparrow l)$.

Conversely, let X be a nonempty set and let $\uparrow \mu$ be a principal filter on $(L^X, \preceq_L)$. For each $x \in X$, $\uparrow \mu(x)$ is a principal filter on L so by hypothesis $f(\uparrow \mu(x))$ is a principal filter on S that we will denote by $\uparrow \xi(x)$. Since $x \in X$ is arbitrary, we can construct an S-fuzzy set ξ on X. We next prove that $\{f \circ \eta : \eta \in \uparrow \mu\} = \uparrow \xi$, which will finish the proof. Given $\eta \in \uparrow \mu$ then for each $x \in X$ we have that $\mu(x) \preceq_L \eta(x)$ so by construction $\xi(x) \preceq_S f \circ \eta(x)$. Hence $\xi \preceq_S f \circ \eta$, that is, $f \circ \eta \in \uparrow \xi$.

On the other hand, if $\zeta \in \uparrow \xi$ then $\xi(x) \preceq_S \zeta(x)$ for all $x \in X$. Then, for each $x \in X$, $\zeta(x) \in \uparrow \xi(x) = f(\uparrow \mu(x))$ so there exists $\rho(x) \in \uparrow \mu(x)$ with $\zeta(x) = f(\rho(x))$. By arbitrariness of x, we have obtained an L-fuzzy set ρ on X such that $\rho \in \uparrow \mu$ and $\zeta = f \circ \rho$ so $\zeta \in \{f \circ \eta : \eta \in \uparrow \mu\}$. □

Remark 3. *We have to mention that $f : (L, \preceq_L) \to (S, \preceq_S)$ preserves principal filters, equivalent to: for every $a \in L$ then $f(\uparrow a) = \uparrow f(b)$ for some $b \in \uparrow a$. Nevertheless, this does not mean that $f(\uparrow a) = \uparrow f(a)$ (see Remark 2). For example, you can consider the function $f : ([0,1], \leq) \to ([-1,1], \leq)$ given by:*

$$f(x) = \begin{cases} \sin\left(\frac{1}{1-x}\right) & \text{if } 0 \leq x < 1 \\ 1 & \text{if } x = 1 \end{cases}.$$

It is obvious that $f(\uparrow a) = [-1,1]$ for every $a \in [0,1[$ and obviously we can find $b \in \uparrow a$ such that $\uparrow f(b) = [-1,1]$. Therefore, f preserves principal filters. Nevertheless, $f\left(\uparrow \frac{\pi-1}{\pi}\right) = [-1,1]$ but $\uparrow f\left(\frac{\pi-1}{\pi}\right) = \uparrow \sin(\pi) = [0,1]$.

Theorem 1. *Let $(L, \preceq_L)$ and $(S, \preceq_S)$ be two partially ordered sets. Let $f : (L, \preceq_L) \to (S, \preceq_S)$ be a function. The following statements are equivalent:*

(i) f preserves filters of fuzzy sets;
(ii) f preserves filters;
(iii) for any $a \in L$, $\uparrow f(a) \subseteq f(\uparrow a)$ and $f(\uparrow a)$ is filtered.

Proof. $(i) \Rightarrow (ii)$ Suppose that f preserves filters of fuzzy sets. Let F be a filter on $(L, \preceq_L)$ and let $b \in \uparrow f(F)$. So there exists $a \in F$ with $f(a) \preceq_S b$. Let X be a nonempty set. Since $\uparrow a_X$ is a filter on $(L^X, \preceq_L)$, the family $\{f \circ \mu : \mu \in \uparrow a_X\}$ is also a filter on $(S^X, \preceq_S)$. Since $f \circ a_X \preceq_S b_X$ we can find $\eta \in \uparrow a_X$ such that $b_X = f \circ \eta$. For a fixed $x \in X$, we have that $a_X(x) = a \preceq_L \eta(x)$ so $\eta(x) \in F$. Moreover, $b = b_X(x) = f(\eta(x))$ so $b \in f(F)$. Consequently, $f(F)$ is upper. On the other hand, let $s, t \in F$. Then there exists $u \in F$ such that $u \preceq_L s$ and $u \preceq_L t$. As above, we have that $\{f \circ \mu : \mu \in \uparrow u_X\}$ is a filter on $(S, \preceq_S)$. Then there exists $v \in \uparrow u_X$ with $f \circ v \preceq_S f \circ s_X$ and $f \circ v \preceq_S f \circ t_X$. Given $x \in X$, we have that $u_X(x) = u \preceq_L v(x)$, that is, $v(x) \in \uparrow u \subseteq F$. Moreover $f(v(x)) \preceq_S (f \circ s_X)(x) = f(s)$ and $f(v(x)) \preceq_S (f \circ t_X)(x) = f(t)$. Consequently, $f(F)$ is filtered, therefore is a filter.

$(ii) \Rightarrow (iii)$ This is obvious since for each $a \in A$, $\uparrow a$ is a filter so by assumption $f(\uparrow a)$ is a filter, so filtered and $\uparrow f(a) \subseteq f(\uparrow a)$ since $f(\uparrow a)$ is an upper set.

$(iii) \Rightarrow (i)$ Let X be a nonempty set and $\{\mu_i : i \in I\}$ be a filter on $(L^X, \preceq_L)$. Let $\nu \in S^X$ such that $f \circ \mu_j \preceq_S \nu$ for some $j \in I$, that is, $\nu \in \Uparrow\{f \circ \mu_i : i \in I\}$. For any $x \in X$, we have by assumption that $\nu(x) \in\uparrow f(\mu_j(x)) \subseteq f(\uparrow \mu_j(x))$ so we can find $\eta(x) \in\uparrow \mu_j(x)$ such that $\nu(x) = f(\eta(x))$. Since x is arbitrary, we obtain an L-fuzzy set η on X such that $\eta \in\uparrow \mu_j \subseteq \{\mu_i : i \in I\}$ and $\nu = f \circ \eta$. Therefore, $\{f \circ \mu_i : i \in I\}$ is an upper set on $(S^X, \preceq_S)$.

On the other hand, fix $i, j \in I$. Since $\{\mu_i : i \in I\}$ is a filter on $(L^X, \preceq_L)$ we can find $k \in I$ such that $\mu_k \preceq_L \mu_i$ and $\mu_k \preceq_L \mu_j$. By hypothesis, for each $x \in X$ the set $f(\uparrow \mu_k(x))$ is filtered, so there exists $\eta(x) \in\uparrow \mu_k(x)$ such that $f(\eta(x)) \preceq_S f(\mu_i(x))$ and $f(\eta(x)) \preceq_S f(\mu_j(x))$. Since x is arbitrary, we obtain an L-fuzzy set η on X such that $\eta \in\uparrow \mu_k \subseteq \{\mu_i : i \in I\}$ and $f \circ \eta \preceq_S f \circ \mu_i$, $f \circ \eta \preceq_S f \circ \mu_j$. Hence $\{f \circ \mu_i : i \in I\}$ is filtered, so a filter. □

Corollary 1. *Let $(L, \preceq_L)$ and $(S, \preceq_S)$ be two partially ordered sets such that $\preceq_S$ is a total order. Let $f : (L, \preceq_L) \to (S, \preceq_S)$ be a function. The following statements are equivalent:*

(i) *f preserves filters of fuzzy sets;*
(ii) *f preserves filters;*
(iii) *f preserves upper families of fuzzy sets;*
(iv) *f preserves upper sets;*
(v) *$\uparrow f(a) \subseteq f(\uparrow a)$ for every $a \in L$.*

Proof. It is immediate from the above theorem and Proposition 3 using that if $(S, \preceq_S)$ is totally ordered then every subset is filtered. □

Remark 4. *In sight of Propositions 1 and 3, it is natural to wonder whether in Theorem 1 we can replace condition "$f(\uparrow a)$ is filtered" by isotonicity. It is obvious that if $f : (L, \preceq_L) \to (S, \preceq_S)$ is isotone then $f(\uparrow a)$ is filtered for all $a \in L$. Nevertheless, the converse is not true. For example, you can consider any non-isotone function $f : ([0,1], \leq) \to ([0,1], \leq)$. In this way, the function defined by:*

$$f(x) = \begin{cases} x & \text{if } 0 < x \leq 1 \\ 1 & \text{if } x = 0 \end{cases}$$

preserves, by the above corollary, filters fuzzy sets but it is not isotone.

Corollary 2. *Let $(L, \preceq_L)$ and $(S, \preceq_S)$ be two partially ordered sets. Let $f : (L, \preceq_L) \to (S, \preceq_S)$ be a function. Each of the following statements implies its successor:*

(i) f preserves filtered families and upper families of fuzzy sets,	*or equivalently,*	$\uparrow f(a) = f(\uparrow a)\ \ \forall a \in L$;
(ii) f preserves principal filters of fuzzy sets,	*or equivalently,*	$\forall a \in L, f(\uparrow a) =\uparrow f(b)$ *for some* $b \in\uparrow a$;
(iii) f preserves filters of fuzzy sets,	*or equivalently,*	$\uparrow f(a) \subseteq f(\uparrow a)$, $f(\uparrow a)$ *is filtered* $\ \forall a \in L$;
(iv) f preserves upper families of fuzzy sets,	*or equivalently,*	$\uparrow f(a) \subseteq f(\uparrow a)\ \ \forall a \in L$

Moreover, if f is isotone then all the statements are equivalent.

Proof. $(i) \Rightarrow (ii)$ By Remark 2, $\uparrow f(a) = f(\uparrow a)$ for all $a \in L$. This obviously implies that f preserves filters so the conclusion follows from Proposition 4.

$(ii) \Rightarrow (iii)$ Again by Proposition 4, f preserves principal filters so given $a \in L$, then $f(\uparrow a)$ is a principal filter and so is filtered. Moreover, $f(\uparrow a)$ is an upper set so $\uparrow f(a) \subseteq f(\uparrow a)$. Therefore, f preserves filters of fuzzy sets by Theorem 1.

The last implication follows from Theorem 1 and Proposition 3

If f is isotone, we have that (iv) implies (i) by Proposition 1. $\square$

Remark 5. *Observe that neither of the above implications can be reversed in general. Remark 3 provides a counterexample for $(ii) \Rightarrow (i)$ while Remark 4 gives a function f which preserves filters of fuzzy sets but not principal filters. Notice that $\uparrow 0$ is a principal filter but $f(\uparrow 0) =]0,1]$ is a filter which is not principal.*

Finally, we provide an easy example showing that, in general, (iv) does not imply (iii). Let us consider a set $X = \{a, b\}$ with two different points and consider the following two partial orders on X given by:

$$x \preceq_1 y \text{ if and only if } x = y \text{ or } x = a \text{ and } y = b;$$
$$x \preceq_2 y \text{ if and only if } x = y;$$

for every $x, y \in X$. Let $f : (X, \preceq_1) \to (X, \preceq_2)$ be the identity function. It is obvious that $\uparrow f(x) = f(x) \subseteq f(\uparrow x)$, so f preservers upper families of fuzzy sets. Nevertheless, $f(\uparrow a) = f(\{a, b\}) = \{a, b\}$ which is not filtered on $(X, \preceq_2)$. Hence, f does not satisfy (iii).

3. Functions Aggregating L-Probabilistic Quasi-Uniformities

This section is devoted to study functions that allow one to merge several (bases of) L-probabilistic quasi-uniformities into a single one. In this case, we will say that these functions aggregate (bases of) L-probabilistic quasi-uniformities. We are able to characterize completely those functions, which aggregate bases of L-probabilistic quasi-uniformities. Nevertheless, we only obtain different necessary and sufficient conditions for a function to aggregate L-probabilistic quasi-uniformities. The characterizations obtained in the previous sections about functions preserving upper families of fuzzy sets and filters of fuzzy sets give some keys for our characterization.

We first recollect some definitions related with L-probabilistic quasi-uniformities. In the following, if $(L, \preceq)$ is a lattice, we will denote by $x \wedge y$ and $x \vee y$ the infimum and supremum of $x, y \in L$.

Definition 4 ([36])**.** *Let $(L, \preceq)$ be a bounded lattice having $\top_L$ and $\bot_L$ as top and bottom elements respectively. A* triangular norm *or a* t-norm *$*$ on L is a binary operation $* \colon L \times L \to L$ such that $(L, *)$ becomes an Abelian monoid with unit $\top_L$ and such that $a * b \preceq c * d$ whenever $a \preceq c$ and $b \preceq d$, with $a, b, c, d \in L$.*

Example 2 ([36])**.** *Let $(L, \preceq)$ be a bounded lattice. Then the binary operation given by:*

$$x *_D y = \begin{cases} x \wedge y & \text{if } x = \top_L \text{ or } y = \top_L \\ \bot_L & \text{otherwise} \end{cases}$$

is a t-norm on L. Moreover it is the smallest t-norm that can be defined on L, that is, if $*$ is a t-norm on L then $x *_D y \preceq x * y$ for every $x, y \in L$. Furthermore, $\wedge$ is the greatest t-norm on L.

We also notice that if $L = \{0, 1\}$, then $*_D = \wedge$. So only one t-norm can be defined on $\{0, 1\}$.

Let $(L, \preceq)$ be a bounded lattice. Given a nonempty set X and $A \subseteq X$, then we will denote by χ_A^L the function $\chi_A^L : X \to L$ such that:

$$\chi_A^L(x) = \begin{cases} \top_L & \text{if } x \in A \\ \bot_L & \text{if } x \notin A \end{cases}.$$

If $L = \{0, 1\}$ we will omit the superscript L.

Let I be a set of indices. We will denote the elements of L^I by boldface letters $\boldsymbol{a}$ and we will write $\boldsymbol{a}_i$ instead of $\boldsymbol{a}(i)$ for all $i \in I$. Furthermore, L^I becomes a partially ordered set endowed with the cartesian partial order $\preceq^I$ given by:

$$\boldsymbol{a} \preceq^I \boldsymbol{b} \Leftrightarrow \boldsymbol{a}_i \preceq \boldsymbol{b}_i \text{ for all } i \in I.$$

Moreover L^I is bounded with top element $\top_{L^I}$ and bottom element $\perp_{L^I}$ where $(\top_{L^I})_i = \top_L$ and $(\perp_{L^I})_i = \perp_L$ for all $i \in I$. Furthermore, if $*$ is a t-norm on L we can define a t-norm $*^I$ on L^I (see [36] for a finite version) given by $(\boldsymbol{a} *^I \boldsymbol{b})_i = \boldsymbol{a}_i * \boldsymbol{b}_i$ for all $i \in I$.

Definition 5. *Let $\{(L_i, \preceq_i) : i \in I\}$ be a family of partially ordered sets possessing top element $\top_i$ for each $i \in I$. The* direct sum *of $\{(L_i, \preceq_i) : i \in I\}$ is the partially ordered set $(\bigoplus_{i\in I} L_i, \preceq^I)$ where:*

$$\bigoplus_{i\in I} L_i = \{\boldsymbol{l} \in \prod_{i\in I} L_i : \boldsymbol{l}_i \neq \top_i \text{ only for finitely many } i \in I\}$$

and $\boldsymbol{a} \preceq^I \boldsymbol{b}$ if and only if $\boldsymbol{a}_i \preceq_i \boldsymbol{b}_i$ for all $i \in I$.

Definition 6. *((cf. [27] [Definition 2.1]), [28]) Let $(L, \preceq)$ be a complete lattice with top element $\top_L$. An L-*probabilistic quasi-uniformity *on a nonempty set X is a pair $(\mathcal{U}, *)$, where $*$ is a t-norm on L and $\mathcal{U}$ is a filter on $(L^{X\times X}, \preceq)$ such that:*

(PQU1) $\mathsf{u}(x, x) = \top_L$ *for all* $\mathsf{u} \in \mathcal{U}$ *and* $x \in X$; *(reflexivity)*
(PQU2) *for each* $\mathsf{u} \in \mathcal{U}$ *there exists* $\mathsf{v} \in \mathcal{U}$ *such that:*

$$\mathsf{v} \circ_* \mathsf{v} \preceq \mathsf{u}$$

where $(\mathsf{v} \circ_* \mathsf{v})(x, y) = \bigvee_{z\in X} \mathsf{v}(x, z) * \mathsf{v}(z, y)$.

*In this case, the triple $(X, \mathcal{U}, *)$ is called an L-*probabilistic quasi-uniform space.
If $L = [0, 1]$ we simply say that $(\mathcal{U}, *)$ is a probabilistic quasi-uniformity.

In the following, we will always assume that $(L, \preceq_L)$ and $(S, \preceq_S)$ are complete lattices [34] unless stated otherwise.

Definition 7. *If $(\mathcal{U}, *)$ is an L-probabilistic quasi-uniformity on a nonempty set X, the pair $(\mathcal{B}, *)$ is said to be a* base *for $(\mathcal{U}, *)$ if $\mathcal{B}$ is a filter base for the filter $\mathcal{U}$, that is, for each $\mathsf{u} \in \mathcal{U}$ we can find $\mathsf{b} \in \mathcal{B}$ such that $\mathsf{b} \preceq \mathsf{u}$. If no confusion arises, we will omit the reference to the t-norm $*$.*

Given a nonempty filtered subset $\mathcal{B}$ of $(L^{X\times X}, \preceq)$ satisfying (PQU1) and (PQU2) for a certain t-norm $*$ on L, where instead of $\mathcal{U}$, the set $\mathcal{B}$ is used, then the pair $(\mathcal{B}, *)$ is a base for an L-probabilistic quasi-uniformity $(\mathcal{U}_{\mathcal{B}}, *)$ on X where $\mathcal{U}_{\mathcal{B}}$ is the filter $\uparrow \mathcal{B} \subseteq L^{X\times X}$.

Remark 6. *Let I be a set of indices and $(X_i, \mathcal{U}_i, *)$ be an L-probabilistic quasi-uniform space for every $i \in I$. Then for each $i \in I$, $\mathcal{U}_i$ becomes a partially ordered set as a subset of $(L^{X\times X}, \preceq)$. Furthermore, it has a top element $\top_{\mathcal{U}_i} = \chi^L_{X_i\times X_i}$. Therefore, the direct sum $\bigoplus_{i\in I} \mathcal{U}_i$ can be constructed.*

If for each $i \in I$, we consider a base $(\mathcal{B}_i, *)$ for an L-probabilistic quasi-uniformity on X_i, then $(\mathcal{B}_i, \preceq)$ is also a partially ordered set for every $i \in I$. However for some $i \in I$, $\mathcal{B}_i$ can fail to have a top element. However, we can add to $\mathcal{B}_i$ the top element $\chi^L_{X_i\times X_i}$ without changing the L-probabilistic quasi- uniformity generated by $(\mathcal{B}_i, *)$. Therefore, in the following, we will always consider that bases of L-probabilistic quasi-uniformities contain a top element, allowing the construction of the direct sum $\bigoplus_{i\in I} \mathcal{B}_i$.

Remark 7. *If $(\mathcal{B}, *)$ is a base for an L-probabilistic quasi-uniformity on a nonempty space X, we can define a preorder $\preceq_{\mathcal{B}}$ on X given by:*

$$x \preceq_{\mathcal{B}} y \text{ if and only if } \mathsf{b}(x,y) = \top_L \text{ for all } \mathsf{b} \in \mathcal{B}.$$

Definition 8. *An L-probabilistic quasi-uniformity $(\mathcal{U}, *)$ on a nonempty set X is said to be* transitive *if we can find a base $(\mathcal{B}, *)$ of $(\mathcal{U}, *)$ such that:*

$$\mathsf{b} \circ_* \mathsf{b} \preceq \mathsf{b}$$

for every $\mathsf{b} \in \mathcal{B}$. In this case, we say that $(\mathcal{B}, *)$ is a transitive base *for $(\mathcal{U}, *)$ and its elements are said to be $*$-transitive.*

We next define the functions of interest.

Definition 9. *A function $F : (L^I, \preceq_L^I) \to (S, \preceq_S)$ is said to:*

- Aggregate (bases of) L-probabilistic quasi-uniformities on products *if whenever $*$ is a t-norm on L and $\mathfrak{U} = \{(X_i, \mathcal{U}_i, *) : i \in I\}$ is a family such that for each $i \in I$, $(\mathcal{U}_i, *)$ is (a base of) an L-probabilistic quasi-uniformity on the nonempty set X_i, then given a t-norm $\star$ on S, the pair $(\{F \circ \Pi\mathbf{u} : \mathbf{u} \in \prod_{i \in I} \mathcal{U}_i\}, \star)$ is (a base of) an S-probabilistic quasi-uniformity $(\Pi\mathcal{U}_F^{\mathfrak{U}}, \star)$ on $\prod_{i\in I} X_i$ where $\Pi\mathbf{u} : (\prod_{i\in I} X_i)^2 \to L^I$ is given by:*

  $$(\Pi\mathbf{u}(\boldsymbol{x}, \boldsymbol{y}))_i = \mathbf{u}_i(\boldsymbol{x}_i, \boldsymbol{y}_i)$$

  for every $\boldsymbol{x}, \boldsymbol{y} \in \prod_{i\in I} X_i$.

  If F only satisfies the previous condition for fixed t-norms $*$ on L and $\star$ on S, then we say that F $\star$-aggregates (bases of) L-$*$-probabilistic quasi-uniformities on products;
- Aggregate (bases of) L-probabilistic quasi-uniformities on sets *if whenever $*$ is a t-norm on L and $\mathfrak{U} = \{(X, \mathcal{U}_i, *) : i \in I\}$ is a family such that for each $i \in I$, $(\mathcal{U}_i, *)$ is (a base of) an L-probabilistic quasi-uniformity on the nonempty set X, then given a t-norm $\star$ on S, the pair $(\{F \circ \triangle\mathbf{u} : \mathbf{u} \in \prod\limits_{i\in I} \mathcal{U}_i\}, \star)$ is (a base of) an S-probabilistic quasi-uniformity $(\triangle\mathcal{U}_F^{\mathfrak{U}}, \star)$ on X where $\triangle\mathbf{u} : X^2 \to L^I$ is given by:*

  $$(\triangle\mathbf{u}(x,y))_i = \mathbf{u}_i(x,y)$$

  for every $x, y \in X$.

  If F only satisfies the previous condition for fixed t-norms $*$ on L and $\star$ on S, then we say that F $\star$-aggregates L-$*$-probabilistic quasi-uniformities on sets.

If in the previous definitions the elements $\mathbf{u}$ belong to $\bigoplus_{i\in I} \mathcal{U}_i$ instead of $\prod_{i\in I} \mathcal{U}_i$, which implies that $\mathcal{U}_i$ has a top element for each $i \in I$ when it is a base, we say that:

- F directly aggregate (bases of) L-probabilistic quasi-uniformities on products *and the notation for the S-probabilistic quasi-uniformity is $(\overset{\oplus}{\Pi}\mathcal{U}_F^{\mathfrak{U}}, \star)$;*
- F directly aggregate (bases of) L-probabilistic quasi-uniformities on sets *and the notation for the S-probabilistic quasi-uniformity is $(\overset{\oplus}{\triangle}\mathcal{U}_F^{\mathfrak{U}}, \star)$.*

Notice that if in the above definition I is finite, there is no difference between saying that F aggregates (bases of) L-probabilistic quasi-uniformities or that F directly aggregates (bases of) L-probabilistic quasi-uniformities.

We also emphasize that the elements of an L-probabilistic quasi-uniformity $(\mathcal{U}, *)$ on a nonempty set X are L-fuzzy sets on $X \times X$, which are usually called L-fuzzy binary relations on X [18,37,38]. Observe that property (PQU2) of the elements of an L-probabilistic quasi-uniformity is, in some way, related with the property of $*$-transitivity. On the other hand, several authors have obtained characterizations of those functions to merge $[0,1]$-fuzzy binary relations satisfying certain properties into single $[0,1]$-fuzzy binary relation verifying the same properties (see [12,15,17,21,37]). These functions are said to preserve those properties of fuzzy binary relations. We emphasize that in [17] the concept of domination was introduced in order to characterize aggregation functions which preserve $*$-transitivity of fuzzy binary relations on products. When the dominated function is a t-norm, the notion of $*$-supmultiplicativity arises and it was used in [21] for characterizing functions aggregating fuzzy quasi-metrics into a single one. Hence it is natural to guess that the notion of supmultiplicativity will be important in our study due to its relation with the preservation of $*$-transitivity. Nevertheless, we will need a generalization of this concept as follows.

Definition 10 (cf. ([21], Remark 3.8), [17])**.** *Let $(L, \preceq_L)$, $(S, \preceq_S)$ be two bounded lattices and let $*, \star$ be t-norms on L and S respectively. A function $f : (L, \preceq_L) \to (S, \preceq_S)$ is said to be* $*, \star$-supmultiplicative *if:*

$$f(a) \star f(b) \preceq_S f(a * b)$$

for every $a, b \in L$.

Moreover, we will say that f is upper $*, \star$-supmultiplicative *if for each $a, b \in L$ there exist $c \in \uparrow a$ and $d \in \uparrow b$ such that:*

$$f(c) \star f(d) \preceq_S f(a * b).$$

If the previous conditions only satisfies for elements a, b belonging to a nonempty subset A of L, then we say that f is $*, \star$-supmultiplicative on A or f is upper $*, \star$-supmultiplicative on A respectively.

Remark 8. *Notice that if $f : (L, \preceq_L) \to (S, \preceq_S)$ is $*, \star$-supmultiplicative then $f^{-1}(\top_S)$ is filtered on $(L, \preceq_L)$. In fact, given $a, b \in f^{-1}(\top_S)$ then $f(a) \star f(b) = \top_S \star \top_S = \top_S \preceq_S f(a * b)$ so $a * b \in f^{-1}(\top_S)$. Since $a * b \preceq_L a$ and $a * b \preceq_L b$ we deduce that $f^{-1}(\top_S)$ is filtered.*

We give some examples of $*, \star$-supmultiplicative functions.

Example 3. *Let L be a complete lattice, $*$ be a t-norm on L, and I be a set of indices.*

- *For each $i \in I$, the ith projection function $P_i : L^I \to L$ given by $P_i(\boldsymbol{x}) = \boldsymbol{x}_i$ for every $\boldsymbol{x} \in L^I$ is $*^I, *$-supmultiplicative;*
- *The function $\mathsf{Inf} : L^I \to L$ given by $\mathsf{Inf}(\boldsymbol{x}) = \inf_{i \in I} \boldsymbol{x}_i$ for every $\boldsymbol{x} \in L^I$ is $*^I, *$-supmultiplicative;*
- *If $I = \{1, \ldots, n\}$ is finite, the function $T_* : L^n \to L$ given by $T_*(\boldsymbol{x}) = \boldsymbol{x}_1 * \ldots * \boldsymbol{x}_n$ is $*^I, *$-supmultiplicative.*

Finally, we introduce the last ingredient that we will need in our characterization: The (asymmetric) $*$-triangular triplets. The symmetric version of this concept was introduced in [15] meanwhile its asymmetric counterpart was first considered in [21]. This notion is important in order to characterize functions which merge fuzzy (quasi-)metrics into a single one [21]. However, as we will show, it is also key when we consider the aggregation of L-probabilistic quasi-uniformities. In [15,21], the notion was considered only on $[0,1]$ but we generalize it to an arbitrary bounded lattice.

Definition 11 (cf. [15,21])**.** *Let $(L, \preceq)$ be a bounded lattice and $*$ be a t-norm on L. A triplet $(a, b, c) \in L^3$ is said to be asymmetric $*$-triangular if:*

$$a * b \preceq c.$$

A triplet $(a, b, c) \in L^3$ is said to be $*$-triangular *if any permutation of the triplet is asymmetric $*$-triangular. If (a, b, c) is $*$-triangular (asymmetric $*$-triangular) for every t-norm $*$ on L then we say that (a, b, c) is a triangular triplet (an asymmetric triangular triplet).*

Example 4. *Given a bounded lattice $(L, \preceq)$ and a t-norm $*$ on L, the triplet $(a, b, a * b)$ is $*$-triangular for every $a, b \in L$.*

Furthermore, the triplet $(a, a, \top_L)$ is $*$-triangular for every $a \in L$ and every t-norm $*$ on L so it is triangular.

Example 5. *Let $(L, \preceq)$ be a complete lattice, X be a nonempty set, and $(\mathcal{U}, *)$ be an L-probabilistic quasi-uniformity on X. Given $\mathsf{u} \in \mathcal{U}$ there exists $\mathsf{v} \in \mathcal{U}$ such that $\mathsf{v} \circ_* \mathsf{v} \preceq \mathsf{u}$. Hence, for any $x, y, z \in X$, the triplet $(\mathsf{v}(x, z), \mathsf{v}(z, y), \mathsf{u}(x, y))$ is asymmetric $*$-triangular.*

A particular case of the following concept was key in [21] for characterizing those functions which aggregate fuzzy (quasi-)metrics.

Definition 12. *Let $(L, \preceq_L)$ and $(S, \preceq_S)$ be bounded lattices and let $*, \star$ be t-norms on L and S respectively. A function $f : (L, \preceq_L) \to (S, \preceq_S)$ $\star$-preserves $*$-triangular (asymmetric $*$-triangular) triplets if $(f(a), f(b), f(c))$ is an $\star$-triangular (an asymmetric $\star$-triangular) triplet whenever (a, b, c) is an $*$-triangular (an asymmetric $*$-triangular) triplet, where $a, b, c \in L$.*

Example 6. *Let $(L, \preceq_L)$ be a bounded lattice.*

- *The identity function $\mathrm{id}_L : (L, \preceq_L) \to (L, \preceq_L)$ $*$-preserves $*$-triangular triplets for every t-norm $*$ on L;*
- *Every constant function $f : (L, \preceq_L) \to ([0, 1], \leq)$ $\star$-preserves $*$-triangular (asymmetric $*$-triangular) triplets for any t-norm $*$ on L and any t-norm $\star$ on $[0, 1]$;*
- *If $(L, \preceq)$ is linearly ordered then $f : (L, \preceq_L) \to (L, \preceq_L)$ is isotone if and only if f $\wedge$-preserves $\wedge$-asymmetric triangular triplets.*

The following proposition, whose proof is straightforward, is an easy adaptation of ([21], Proposition 3.30, Corollary 3.35) to the more generalized concepts that are considered in this paper.

Proposition 5 (cf. [21])**.** *Let $f : (L, \preceq_L) \to (S, \preceq_S)$ be a function between two bounded lattices and let $*, \star$ be t-norms on L and S respectively. Each of the following statements implies its successor:*

(i) *f $\star$-preserves asymmetric $*$-triangular triplets;*
(ii) *f $\star$-preserves $*$-triangular triplets;*
(iii) *f is $*, \star$-supmultiplicative.*

If $f^{-1}(\top_S) \neq \varnothing$ then (i) implies isotonicity. Moreover, if f is isotone then all the above statements are equivalent.

The next result gives a complete characterization of those functions which aggregate bases of L-probabilistic quasi-uniformities.

Theorem 2. *Let $(L, \preceq_L)$ and $(S, \preceq_S)$ be two complete lattices and let I be a set of indices. Let $F : (L^I, \preceq_L^I) \to (S, \preceq_S)$ be a function and $*$,$\star$ be t-norms on L and S respectively. The following statements are equivalent:*

(i) *F (directly) $\star$-aggregates bases of L-$*$-probabilistic quasi-uniformities on products;*
(ii) *F (directly) $\star$-aggregates bases of L-$*$-probabilistic quasi-uniformities on sets;*
(iii) *F (directly) $\star$-aggregates transitive bases of L-$*$-probabilistic quasi-uniformities on products;*
(iv) *F (directly) $\star$-aggregates transitive bases of L-$*$-probabilistic quasi-uniformities on sets;*
(v) *$F(\top_{L^I}) = \top_S$, F is isotone and $*^I, \star$-supmultiplicative.*
(vi) *$F(\top_{L^I}) = \top_S$ and F $\star$-preserves $*^I$-asymmetric triangular triplets.*

Proof. $(i) \Rightarrow (ii)$ and $(iii) \Rightarrow (iv)$ are straightforward.

$(i) \Rightarrow (iii)$ Suppose that F $\star$-aggregates bases of L-$*$-probabilistic quasi-uniformities on products. For each $i \in I$, let $(\mathcal{B}_i, *)$ be a transitive base for an L-probabilistic quasi-uniformity on a nonempty set X_i. Let $\boldsymbol{b} \in \prod_{i \in I} \mathcal{B}_i$. Since $\boldsymbol{b}_i$ is $*$-transitive for every $i \in I$, then $(\{\boldsymbol{b}_i\}, *)$ is a transitive base for an L-probabilistic quasi-uniformity on X_i. By assumption, $(\{F \circ \Pi \boldsymbol{b}\}, \star)$ must be a base for an S-probabilistic quasi-uniformity so $F \circ \Pi \boldsymbol{b}$ must be $\star$-transitive. Since $\boldsymbol{b}$ was arbitrary, we deduce that $(\{F \circ \Pi \boldsymbol{b} : \boldsymbol{b} \in \prod_{i\in I} \mathcal{B}_i\}, \star)$ is a transitive base for an S-probabilistic quasi-uniformity on $\prod_{i\in I} X_i$.

If we suppose that F directly $\star$-aggregates bases of L-$*$-probabilistic quasi-uniformities on products, we can proceed as above but in this case $(\mathcal{B}_i, *)$ must have a top element for each $i \in I$. Observe that $\{\boldsymbol{b}\} = \bigoplus_{i\in I}\{\boldsymbol{b}_i\} = \prod_{i\in I}\{\boldsymbol{b}_i\}$.

$(ii) \Rightarrow (iv)$ is similar to $(i) \Rightarrow (iii)$.

$(iv) \Rightarrow (v)$ Suppose that F $\star$-aggregates transitive bases of L-$*$-probabilistic quasi-uniformities on sets. We start proving that $F(\top_{L^I}) = \top_S$. For each $i \in I$, suppose that $(\mathcal{B}_i, *)$ is a transitive base for an L-probabilistic quasi-uniformity on a nonempty set X. By assumption, $(\{F \circ \triangle\mathbf{u} : \mathbf{u} \in \prod_{i\in I} \mathcal{B}_i\}, \star)$ is a transitive base for an S-probabilistic quasi-uniformity on X. In particular, $F \circ \triangle\mathbf{u}$ must satisfy (PQU1) for every $\mathbf{u} \in \prod_{i\in I} \mathcal{B}_i$. Then, for any $x \in X$ we have that:

$$\top_S = (F \circ \triangle\mathbf{u})(x, x) = F((\mathbf{u}_i(x, x))_{i\in I}) = F(\top_{L^I}).$$

Let us show that F is isotone. Let $\boldsymbol{a}, \boldsymbol{b} \in L^I$ such that $\boldsymbol{a} \preceq_L^I \boldsymbol{b}$. Let $X = \{x_1, x_2, x_3\}$ be a set with three different elements. For each $i \in I$, define $\mathsf{u}_i : X \times X \to L$ as:

$$\mathsf{u}_i(x, y) = \begin{cases} \boldsymbol{b}_i & \text{if } x = x_1, y = x_3 \\ \top_L & \text{if } x = y \text{ or } x = x_2, y = x_3 \\ \boldsymbol{a}_i & \text{otherwise} \end{cases}.$$

It is straightforward to prove that $\mathsf{u}_i \circ_* \mathsf{u}_i \preceq_L \mathsf{u}_i$ and $\mathsf{u}_i(x, x) = \top_L$. Hence $\{(\{\mathsf{u}_i\}, *) : i \in I\}$ is a family of transitive bases of L-probabilistic quasi-uniformities on X. By assumption $(\{F \circ \triangle\mathbf{u}\}, \star)$ must be a transitive base of an S-probabilistic quasi-uniformity on X where $\mathbf{u}_i = \mathsf{u}_i$ for all $i \in I$. Hence $F \circ \triangle\mathbf{u}$ must be $\star$-transitive. Then the following inequality must be true:

$$\begin{gathered} (F \circ \triangle\mathbf{u}) \circ_\star (F \circ \triangle\mathbf{u})(x_1, x_3) \preceq_S F \circ \triangle\mathbf{u}(x_1, x_3) \\ \bigvee_{z\in X} F((\mathsf{u}_i(x_1, z))_{i\in I}) \star F((\mathsf{u}_i(z, x_3))_{i\in I}) \preceq_S F((\mathsf{u}_i(x_1, x_3))_{i\in I}) \\ (F(\boldsymbol{b}) \star F(\top_{\boldsymbol{L}^I})) \vee (F(\boldsymbol{a}) \star F(\top_{L^I})) = F(\boldsymbol{b}) \vee F(\boldsymbol{a}) \preceq_S F(\boldsymbol{b}). \end{gathered}$$

Therefore, $F(\boldsymbol{a}) \preceq_S F(\boldsymbol{b})$.

We check now that F is $*^I, \star$-supmultiplicative. Let $\boldsymbol{a}, \boldsymbol{b} \in L^I$. We can proceed in a similar way as above by considering the same set $X = \{x_1, x_2, x_3\}$ and for each $i \in I$, define $\mathsf{v}_i : X \times X \to L$ as:

$$\mathsf{v}_i(x, y) = \mathsf{v}_i(y, x) = \begin{cases} \boldsymbol{a}_i & \text{if } x = x_1, y = x_2, \\ \boldsymbol{b}_i & \text{if } x = x_2, y = x_3, \\ \boldsymbol{a}_i * \boldsymbol{b}_i & \text{if } x = x_1, y = x_3, \\ \top_L & \text{if } x = y, \end{cases}$$

for every $x, y \in X$. It is easy to check that $\mathsf{v}_i \circ_* \mathsf{v}_i \preceq_L \mathsf{v}_i$ and $\mathsf{v}_i(x, x) = \top_L$ for all $x \in X$. So $\{(\{\mathsf{v}_i\}, *) : i \in I\}$ is a family of transitive bases of L-probabilistic uniformities on X. By assumption $(\{F \circ \triangle\mathbf{v}\}, \star)$ must be a transitive base of an S-probabilistic quasi-uniformity on X where $\mathbf{v}$ is the only element of $\prod_{i\in I}\{\mathsf{v}_i\}$. Then:

$$\begin{gathered} ((F \circ \triangle\mathbf{v}) \circ_\star (F \circ \triangle\mathbf{v}))(x_1, x_3) \preceq_S F \circ \triangle\mathbf{v}(x_1, x_3) \\ \bigvee_{z\in X} F((\mathsf{v}_i(x_1, z))_{i\in I}) \star F((\mathsf{v}_i(z, x_3))_{i\in I}) \preceq_S F((\mathsf{v}_i(x_1, x_3))_{i\in I}) \\ F(\boldsymbol{a} *^I \boldsymbol{b}) \vee (F(\boldsymbol{a}) \star F(\boldsymbol{b})) \preceq_S F(\boldsymbol{a} *^I \boldsymbol{b}). \end{gathered}$$

Consequently, $F(\boldsymbol{a}) \star F(\boldsymbol{b}) \preceq_S F(\boldsymbol{a} *^I \boldsymbol{b})$ so F is $*^I, \star$-supmultiplicative.

If we suppose that F directly $\star$-aggregates transitive bases of L-$*$-probabilistic quasi-uniformities on sets, we can proceed as above but in this case $(\mathcal{B}_i, *)$ must have a top element for each $i \in I$. Observe that our constructions makes sense also on the direct sum since we have considered transitive bases of L-probabilistic quasi-uniformities with only one element.

$(v) \Rightarrow (vi)$ This follows from Proposition 5.

$(vi) \Rightarrow (i)$ For each $i \in I$, suppose that $(\mathcal{B}_i, *)$ is a base for an L-probabilistic quasi-uniformity on a nonempty set X_i. If $\mathbf{u} \in \prod_{i\in I} \mathcal{B}_i$ then $(F \circ \Pi\mathbf{u})(x, x) = F((\mathbf{u}_i(x_i, x_i)_{i\in I}) = F(\top_L) = \top_S$.

On the other hand, observe that the family $\{\Pi\mathbf{u} : \mathbf{u} \in \prod_{i\in I} \mathcal{B}_i\}$ is a filtered family of (L^I)-fuzzy sets on $(\prod_{i\in I} X_i)^2$. Since by Proposition 5 F is isotone then $\{F \circ \Pi\mathbf{u} : \mathbf{u} \in \prod_{i\in I} \mathcal{B}_i\}$ is filtered by Proposition 1.

Let $\mathbf{u} \in \prod_{i\in I} \mathcal{B}_i$. For each $i \in I$ we can find $\mathsf{v}_i \in \mathcal{B}_i$ such that $\mathsf{v}_i \circ_* \mathsf{v}_i \preceq_L \mathbf{u}_i$. Let $\mathbf{v} \in \prod_{i\in I} \mathcal{B}_i$ such that $\mathbf{v}_i = \mathsf{v}_i$ for all $i \in I$. We next prove that $(F \circ \Pi\mathbf{v}) \circ_\star (F \circ \Pi\mathbf{v}) \preceq_S F \circ \Pi\mathbf{u}$. In fact, let $\boldsymbol{x}, \boldsymbol{y} \in \prod_{i\in I} X_i$. Then $(\mathsf{v}_i \circ_* \mathsf{v}_i)(x_i, y_i) \preceq_L \mathbf{u}_i(x_i, y_i)$ for all $i \in I$. So for each $\boldsymbol{z} \in \prod_{i\in I} X_i$, $((\mathbf{v}_i(x_i, z_i))_{i\in I}, (\mathbf{v}_i(z_i, y_i))_{i\in I}, (\mathbf{u}_i(x_i, y_i))_{i\in I})$ is an asymmetric $*^I$-triangular triplet. Consequently, $(F((\mathbf{v}_i(x_i, z_i))_{i\in I}), F((\mathbf{v}_i(z_i, y_i))_{i\in I}), F((\mathbf{u}_i(x_i, y_i))_{i\in I}))$ is an asymmetric $\star$-triangular triplet. Therefore,

$$F((\mathbf{v}_i(x_i, z_i))_{i\in I}) \star F((\mathbf{v}_i(z_i, y_i))_{i\in I}) \preceq_S F((\mathbf{u}_i(x_i, y_i))_{i\in I})$$
$$F(\Pi\mathbf{v}(\boldsymbol{x}, \boldsymbol{z})) \star F(\Pi\mathbf{v}(\boldsymbol{z}, \boldsymbol{y})) \preceq_S F(\Pi\boldsymbol{u}(\boldsymbol{x}, \boldsymbol{y})).$$

Hence,

$$\bigvee_{\boldsymbol{z}\in\prod_{i\in I} X_i} F(\Pi\mathbf{v}(\boldsymbol{x}, \boldsymbol{z})) \star F(\Pi\mathbf{v}(\boldsymbol{z}, \boldsymbol{y})) = ((F \circ \Pi\mathbf{v}) \circ_\star (F \circ \Pi\mathbf{v}))\,(\boldsymbol{x}, \boldsymbol{y}) \preceq_S F(\Pi\boldsymbol{u}(\boldsymbol{x}, \boldsymbol{y})).$$

Therefore, $(\{F \circ \Pi\mathbf{u} : \mathbf{u} \in \prod_{i\in I} \mathcal{B}_i\}, \star)$ is a base for an S-probabilistic quasi-uniformity on $\prod_{i\in I} X_i$.

In case that for each $i \in I$, $(\mathcal{B}_i, *)$ is a base for an L-probabilistic uniformity on X_i such that $\mathcal{B}_i$ has a top element, a slight modification of the above process allows to show that $(\{F \circ \Pi\mathbf{u} : \mathbf{u} \in \bigoplus_{i\in I} \mathcal{B}_i\}, \star)$ is a base for an S-probabilistic uniformity on $\prod_{i\in I} X_i$. □

Example 7. *We provided in Example 3 three $*^I, *$-supmultiplicative functions. Since they are also isotone and maps $\top_{L^I}$ to $\top_S$ then they $*$-aggregate bases of L-$*$-probabilistic quasi-uniformities on products and on sets.*

Some authors have studied how to construct adjoint functors between the category of probabilistic quasi-uniformities and the category of classical quasi-uniformities [29,39]. Although their results have been obtained for L-probabilistic quasi-uniformities when $L = [0, 1]$, some of them, as the following one, are also valid when L is an arbitrary complete lattices.

Proposition 6 (cf. [39])**.** *Let $(L, \preceq)$ be a complete lattice and $*$ be a t-norm on L.*

(i) If $(X, \mathcal{U})$ is a quasi-uniform space then $(X, \mathfrak{U}^L_{\mathcal{U}}, *)$ is an L-probabilistic quasi-uniform space where $\mathfrak{U}^L_{\mathcal{U}}$ is the filter on $L^{X\times X}$ having as base $\{\chi^L_u : u \in \mathcal{U}\}$;

(ii) If $(X, \mathfrak{U}, *)$ is an L-probabilistic quasi-uniform space then $(X, \mathcal{U}_{\mathfrak{U}})$ is a quasi-uniform space where $\mathcal{U}_{\mathfrak{U}}$ is the filter $\{u \subseteq X \times X : \chi^L_u \in \mathfrak{U}\}$.

Moreover, $\mathcal{U} = \mathcal{U}_{\mathfrak{U}^L_{\mathcal{U}}}$.

Therefore, we can convert a classical quasi-uniform space $(X, \mathcal{U})$ into an L-probabilistic quasi-uniform space $(X, \mathfrak{U}^L_{\mathcal{U}}, *)$ by making use of the previous result. Notice that in case that $L = \{0, 1\}$ then $(\{\chi_u : u \in \mathcal{U}\}, *)$ is not only a base for $(\mathfrak{U}^L_{\mathcal{U}}, *)$ but it is the whole $\{0, 1\}$-probabilistic quasi-uniformity $(\mathfrak{U}^L_{\mathcal{U}}, *)$. Therefore, there is an equivalence between quasi-uniformities and $\{0, 1\}$-probabilistic quasi-uniformities. Consequently it is natural to study a particular case of Theorem 2 when we consider Boolean functions. To achieve this, we will make use of the following concept:

Definition 13 ([3,34]). *A function $f : L \to S$ between two semilattices is* minitive *or* preserves finite infs *if:*

$$f(x \wedge y) = f(x) \wedge f(y)$$

for every $x, y \in L$.

Corollary 3. *Let $F : \{0,1\}^I \to \{0,1\}$ be a Boolean function. The following statements are equivalent:*

(i) *F (directly) $\wedge$-aggregates bases of $\{0,1\}$-$\wedge$-probabilistic quasi-uniformities on products;*
(ii) *F (directly) $\wedge$-aggregates bases of $\{0,1\}$-$\wedge$-probabilistic quasi-uniformities on sets;*
(iii) *F (directly) $\wedge$-aggregates transitive bases of $\{0,1\}$-$\wedge$-probabilistic quasi-uniformities on products;*
(iv) *F (directly) $\wedge$-aggregates transitive bases of $\{0,1\}$-$\wedge$-probabilistic quasi-uniformities on sets;*
(v) *$F(\mathbf{1}) = 1$, F is isotone and $\wedge^I, \wedge$-supmultiplicative;*
(vi) *$F(\mathbf{1}) = 1$ and F is minitive.*

Proof. The first five statements are equivalent by Theorem 2 and (v) is obviously equivalent to (vi). $\square$

Remark 9. *Notice that if I is finite, then the conditions of the above result are also equivalent to "$F^{-1}(1)$ is a principal filter", that is, there exists $\boldsymbol{i} \in F^{-1}(1)$ such that $F^{-1}(1) = \uparrow \boldsymbol{i}$. In fact, set $\boldsymbol{i} = \bigwedge_{\boldsymbol{x} \in F^{-1}(1)} \boldsymbol{x}$ which exists since $F^{-1}(1)$ is finite and $\{0,1\}^I$ is a lattice. Since F is minitive we have that $F(\boldsymbol{i}) = F(\bigwedge_{\boldsymbol{x} \in F^{-1}(1)} \boldsymbol{x}) = \bigwedge_{\boldsymbol{x} \in F^{-1}(1)} F(\boldsymbol{x}) = 1$ so $\boldsymbol{i} \in F^{-1}(1)$ is the minimum.*

It is easy to check that if $F^{-1}(1)$ is a principal filter then $F(\mathbf{1}) = 1$ and F is minitive.

Remark 10. *We observe that we can easily obtain the minitive Boolean functions $F : \{0,1\}^I \to \{0,1\}$ with $F(\mathbf{1}) = 1$ when I is finite. In fact, suppose that $I = \{1, \ldots, n\}$. In this case, for each $\boldsymbol{x} \in \{0,1\}^n$ and for each $i \in \{1, \ldots, n\}$, consider $\mathbf{1x}_{\boldsymbol{i}} \in \{0,1\}^n$ given by $(\mathbf{1x}_{\boldsymbol{i}})_j = 1$ if $i \neq j$ and $(\mathbf{1x}_{\boldsymbol{i}})_i = \boldsymbol{x}_i$. Then we have that $F(\boldsymbol{x}) = F(\bigwedge_{i=1}^n \mathbf{1x}_{\boldsymbol{i}}) = \bigwedge_{i=1}^n F(\mathbf{1x}_{\boldsymbol{i}})$. Therefore, if we define $f_i : \{0,1\} \to \{0,1\}$ as $f_i(x) = F(1, \ldots, \overbrace{x}^{i}, \ldots, 1)$ we have that $f_i(\boldsymbol{x}_i) = F(\mathbf{1x}_{\boldsymbol{i}})$ for each $\boldsymbol{x} \in \{0,1\}^n$. Hence $F(\boldsymbol{x}) = \bigwedge_{i=1}^n f_i(\boldsymbol{x}_i)$. Since $f_i(1) = 1$ for all $i \in \{1, \ldots, n\}$ then $f_i = \mathrm{id}_{\{0,1\}}$ or $f_i = \chi_{\{0,1\}}$. Consequently, there exists $J_F \subseteq \{1, \ldots, n\}$, such that the function F must be of the form:*

$$F(\boldsymbol{x}) = \bigwedge_{j \in J_F} \boldsymbol{x}_j, \text{ for all } \boldsymbol{x} \in \{0,1\}^n. \tag{1}$$

Notice that if I is not finite, we can find minitive functions $F : \{0,1\}^I \to \{0,1\}$ satisfying $F(\mathbf{1}) = 1$ but F cannot be expressed as in equation (1). For example, let $I = \mathbb{N}$. Then it can be easily checked that the function $G(\boldsymbol{x}) = \liminf_{n \in \mathbb{N}} \boldsymbol{x} = \sup_{n \in \mathbb{N}} \inf_{k \geq n} \boldsymbol{x}_k$ is minitive and $G(\mathbf{1}) = 1$. Moreover, G cannot be expressed as given in formula (1). Indeed, let $J \subseteq \mathbb{N}$. Fix $j_0 \in J$ and consider $\boldsymbol{x} \in \{0,1\}^{\mathbb{N}}$ such that $\boldsymbol{x}_{j_0} = 0$ and $\boldsymbol{x}_n = 1$ for every $n \in \mathbb{N} \backslash \{j_0\}$. Then $G(\boldsymbol{x}) = \liminf_{n \in \mathbb{N}} \boldsymbol{x}_n = 1$ but $\bigwedge_{j \in J} \boldsymbol{x}_j = 0$.

Example 8. *Let $\{(X_i, \mathcal{U}_i) : i \in I\}$ be a family of quasi-uniform spaces. Then $\mathfrak{U} = \{(X_i, \mathfrak{U}_{\mathcal{U}_i}^{\{0,1\}}, \wedge) : i \in I\}$ is a family of $\{0,1\}$-probabilistic quasi-uniform spaces where $\mathfrak{U}_{\mathcal{U}_i}^{\{0,1\}} = \{\chi_u : u \in \mathcal{U}_i\}$ (see Proposition 6 and the comment after it). Consider $F : \{0,1\}^I \to \{0,1\}$ given by:*

$$F(\boldsymbol{x}) = \bigwedge_{i \in I} \boldsymbol{x}_i$$

for all $\boldsymbol{x} \in \{0,1\}^I$. Since $F(\mathbf{1}) = 1$ and F is minitive then, by Corollary 3 $\{F \circ \Pi\mathbf{u} : \mathbf{u} \in \bigoplus_{i \in I} \mathfrak{U}_{\mathcal{U}_i}^{\{0,1\}}\}$ must be a base for a $\{0,1\}$-probabilistic quasi-uniformity $(\overset{\oplus}{\Pi}\mathfrak{U}_F^{\mathfrak{U}}, \wedge)$ on $\prod_{i \in I} X_i$. Given $\mathbf{u} \in \bigoplus_{i \in I} \mathfrak{U}_{\mathcal{U}_i}^{\{0,1\}}$, for each $i \in I$, $\mathbf{u}_i = \chi_{u_i}$ for some $u_i \in \mathcal{U}_i$. Given $(\boldsymbol{x}, \boldsymbol{y}) \in \prod_{i \in I} X_i \times \prod_{i \in I} X_i$ then,

$$F \circ \Pi\mathbf{u}(\boldsymbol{x}, \boldsymbol{y}) = F((\chi_{u_i}(\boldsymbol{x}_i, \boldsymbol{y}_i))_{i \in I}) = \bigwedge_{i \in I} \chi_{u_i}(\boldsymbol{x}_i, \boldsymbol{y}_i) = 1 \Leftrightarrow (\boldsymbol{x}_i, \boldsymbol{y}_i) \in u_i \text{ for all } i \in I.$$

Since $u_i \neq X_i \times X_i$ only for finitely many $i \in I$, it is clear that $\mathcal{U}_{\underset{\Pi\mathfrak{U}_F^{\mathfrak{U}}}{\oplus}}$ is the product of quasi-uniformity on $\prod_{i\in I} X_i$ ([33], 1.16).

In case of considering a family of quasi-uniformities $\{\mathcal{U}_i : i \in I\}$ on the same set X, since F also $\wedge$-aggregates bases of $\{0,1\}$-$\wedge$-probabilistic quasi-uniformities on sets, we can proceed as above in order to obtain a quasi-uniformity $\mathcal{U}_{\underset{\Delta\mathfrak{U}_F^{\mathfrak{U}}}{\oplus}}$ on X. In this case $\mathcal{U}_{\underset{\Delta\mathfrak{U}_F^{\mathfrak{U}}}{\oplus}}$ is the supremum quasi-uniformity of the family $\{\mathcal{U}_i : i \in I\}$.

The next example shows that we can use this theory in order to construct preorders in the cartesian product of preordered sets.

Example 9. *Let $\{(X_i, \preceq_i) : i = 1, \ldots, n\}$ be a finite family of preordered sets. For each $i \in \{1, \ldots, n\}$, it is clear that $(\{\chi_{\preceq_i}\}, \wedge)$ is a base for a transitive $\{0,1\}$-probabilistic quasi-uniformity on X_i. Set $\mathfrak{B} = \{(X_i, \{\chi_{\preceq_i}\}, \wedge) : i = 1, \ldots, n\}$. Let $F : \{0,1\}^n \to \{0,1\}$ be a function which $\wedge$-aggregates bases of $\{0,1\}$-$\wedge$-probabilistic quasi-uniformities. By Remark 10, there exists $J \subseteq \{1, \ldots, n\}$ such that $F(\boldsymbol{x}) = \bigwedge_{j\in J} \boldsymbol{x}_j$ for all $\boldsymbol{x} \in \{0,1\}^n$. Denote by $\boldsymbol{\chi}_{\preceq}$ the only element of $\prod_{i=1}^n \{\chi_{\preceq_i}\}$. Then $(\Pi\mathcal{B}_F^{\mathfrak{B}}, \wedge) := (\{F \circ \Pi\boldsymbol{\chi}_{\preceq}\}, \wedge)$ is a base for a $\{0,1\}$-probabilistic quasi-uniformity on $\prod_{i=1}^n X_i$. By Remark 7, $(\Pi\mathcal{B}_F^{\mathfrak{B}}, \wedge)$ induces a preorder $\preceq_{\Pi\mathcal{B}_F^{\mathfrak{B}}}$ on $\prod_{i=1}^n X_i$ given by:*

$$\boldsymbol{x} \preceq_{\Pi\mathcal{B}_F^{\mathfrak{B}}} \boldsymbol{y} \iff (F \circ \Pi\boldsymbol{\chi}_{\preceq})(\boldsymbol{x}, \boldsymbol{y}) = 1 \iff \boldsymbol{x}_j \preceq_j \boldsymbol{y}_j \text{ for all } j \in J.$$

In particular, if $J = \{1, \ldots, n\}$ we obtain that $\preceq_{\Pi\mathcal{B}_F^{\mathfrak{B}}}$ is the cartesian preorder or the coordinatewise preorder.

Next we address the problem of characterizing those functions which aggregate L-probabilistic quasi-uniformities instead of bases.

Proposition 7. *Let $(L, \preceq_L)$ and $(S, \preceq_S)$ be two complete lattices and let I be a set of indices. Let $F : (L^I, \preceq_L^I) \to (S, \preceq_S)$ be a function and $*, \star$ be t-norms on L and S respectively. Each of the following statements implies its successor:*

- *(i) F $\star$-aggregates L-$*$-probabilistic quasi-uniformities on products;*
- *(ii) F $\star$-aggregates L-$*$-probabilistic quasi-uniformities on sets;*
- *(iii) $F(\top_{L^I}) = \top_S$; for each $\boldsymbol{a} \in L^I$, $\uparrow F(\boldsymbol{a}) \subseteq F(\uparrow \boldsymbol{a})$ and $F(\uparrow \boldsymbol{a})$ is filtered; F is upper $*^I, \star$-supmultiplicative;*
- *(iv) if $\mathfrak{U} = \{(X, \mathcal{U}_i, *) : i \in I\}$ is a family of L-probabilistic quasi-uniform spaces then $\triangle\mathcal{U}_F^{\mathfrak{U}}$ is a filter on $(L^{X\times X}, \preceq_L)$ satisfying property (PQU1).*

Moreover, if we replace in (iii) the condition "$F(\uparrow a)$ is filtered" by the stronger condition "F is isotone", that is, if we consider the statement:

$(\widehat{iii})$ $F(\top_{L^I}) = \top_S$; for each $\boldsymbol{a} \in L^I$, $\uparrow F(\boldsymbol{a}) \subseteq F(\uparrow \boldsymbol{a})$; F is isotone and upper $*^I, \star$-supmultiplicative;

then $(\widehat{iii})$ implies (ii).

Proof. $(i) \Rightarrow (ii)$ This is straightforward.

$(ii) \Rightarrow (iii)$ Let X be a nonempty set and $(\mathcal{U}, *)$ be an arbitrary L-probabilistic quasi-uniformity on X. Taking $\mathfrak{U} = \{(X, \mathcal{U}_i, *) : i \in I\}$ and $\mathcal{U}_i = \mathcal{U}$ for each $i \in I$ we have that $(\Delta\mathcal{U}_F^{\mathfrak{U}}, \star)$ is an S-probabilistic quasi-uniformity on X. Moreover, given $x \in X$ and $\mathsf{u} \in \mathcal{U}$ then $\mathsf{u}(x,x) = \top_L$. Taking $\mathbf{u} \in \prod_{i\in I} \mathcal{U}_i$ such that $\mathbf{u}_i = \mathsf{u}$ for every $i \in I$ then $F \circ \triangle\mathbf{u} \in \Delta\mathcal{U}_F^{\mathfrak{U}}$ and $\top_S = (F \circ \triangle\mathbf{u})(x,x) = F((\mathsf{u}(x,x))_{i\in I}) = F(\top_{L^I})$.

Let $\boldsymbol{a} \in L^I$ and let X be a nonempty set with at least two elements. For each $i \in I$, consider $\mathsf{u}_i \in L^{X\times X}$ given by:

$$\mathsf{u}_i(x,y) = \begin{cases} \boldsymbol{a}_i & \text{if } x \neq y \\ \top_L & \text{if } x = y \end{cases},$$

for all $x, y \in X$. It is obvious that for each $i \in I$, u_i is $*$-transitive and $\mathsf{u}_i(x,x) = \top_L$ for every $x \in X$. Then $(\{\mathsf{u}_i\}, *)$ is a base for a transitive L-probabilistic quasi-uniformity $(\mathcal{U}_i, *)$ on X for every $i \in I$. Set $\mathfrak{U} = \{(X, \mathcal{U}_i, *) : i \in I\}$. By assumption, $(\triangle\mathcal{U}_F^{\mathfrak{U}}, \star)$ is a probabilistic S-quasi-uniformity on X. Let $\mathbf{u} \in \prod_{i\in I} \mathcal{U}_i$ such that $\mathbf{u}_i = \mathsf{u}_i$ for all $i \in I$.

- Let $c \in \uparrow F(\boldsymbol{a})$. Then, given $x, y \in X$

$$F \circ \triangle\mathbf{u}(x,y) = F((\mathsf{u}_i(x,y))_{i\in I}) = \begin{cases} \top_S & \text{if } x = y \\ F(\boldsymbol{a}) & \text{if } x \neq y \end{cases} \preceq_S \mathsf{v}_c(x,y) := \begin{cases} \top_S & \text{if } x = y \\ c & \text{if } x \neq y \end{cases}.$$

  Therefore, $\mathsf{v}_c \in \triangle\mathcal{U}_F^{\mathfrak{U}}$ so we can find $\mathbf{v} \in \prod_{i\in I} \mathcal{U}_i$ such that $F \circ \triangle\mathbf{v} = \mathsf{v}_c$. Then $\mathbf{v}_i \in \mathcal{U}_i = \uparrow \mathsf{u}_i$. Hence, given two distinct points $x_0, y_0 \in X$ we have that $\mathsf{u}_i(x_0,y_0) = \boldsymbol{a}_i \preceq_L \mathbf{v}_i(x_0,y_0)$ for all $i \in I$, so $\triangle\mathbf{v}(x,y) \in \uparrow \boldsymbol{a}$ and $F(\triangle\mathbf{v}(x_0,y_0)) = \mathsf{v}_c(x_0,y_0) = c$. Hence $c \in F(\uparrow \boldsymbol{a})$ and $\uparrow F(\boldsymbol{a}) \subseteq F(\uparrow \boldsymbol{a})$.
- Next we prove that $F(\uparrow \boldsymbol{a})$ is filtered. Let $\boldsymbol{l}, \boldsymbol{m} \in \uparrow \boldsymbol{a}$. For each $i \in I$, define $\mathsf{v}_i, \mathsf{w}_i \in L^{X\times X}$ as:

$$\mathsf{v}_i(x,y) = \begin{cases} \boldsymbol{l}_i & \text{if } x \neq y \\ \top_L & \text{if } x = y \end{cases}, \qquad \mathsf{w}_i(x,y) = \begin{cases} \boldsymbol{m}_i & \text{if } x \neq y \\ \top_L & \text{if } x = y \end{cases}.$$

  Notice that $\mathsf{u}_i \preceq_L \mathsf{v}_i$ and $\mathsf{u}_i \preceq_L \mathsf{w}_i$ for all $i \in I$ so $\mathsf{v}_i, \mathsf{w}_i \in \mathcal{U}_i$. Consider $\mathbf{v}, \mathbf{w} \in \prod_{i\in I} \mathcal{U}_i$ such that $\mathbf{v}_i = \mathsf{v}_i, \mathbf{w}_i = \mathsf{w}_i$ for all $i \in I$. By assumption, $F \circ \triangle\mathbf{v}, F \circ \triangle\mathbf{w} \in \triangle\mathcal{U}_F^{\mathfrak{U}}$ so there exists $\mathbf{p} \in \prod_{i\in I} \mathcal{U}_i$ such that $F \circ \triangle\mathbf{p} \preceq_S F \circ \triangle\mathbf{v}$ and $F \circ \triangle\mathbf{p} \preceq_S F \circ \triangle\mathbf{w}$. Given two distinct points $x_0, y_0 \in X$ we have that $\mathbf{u}_i(x_0,y_0) = \boldsymbol{a}_i \preceq_L \mathbf{p}_i(x_0,y_0)$ for all $i \in I$ so $\boldsymbol{a} \preceq_L^I \triangle\mathbf{p}(x_0,y_0)$, that is, $\triangle\mathbf{p}(x_0,y_0) \in \uparrow \boldsymbol{a}$. Moreover, $F(\triangle\mathbf{p}(x_0,y_0)) \preceq_S F(\triangle\mathbf{v}(x_0,y_0)) = F(\boldsymbol{l})$ and $F(\triangle\mathbf{p}(x_0,y_0)) \preceq_S F(\triangle\mathbf{w}(x_0,y_0)) = F(\boldsymbol{m})$. Hence $F(\uparrow \boldsymbol{a})$ is filtered.

Finally, we prove that F is upper $*^I, \star$-supmultiplicative. Let $\boldsymbol{a}, \boldsymbol{b} \in L^I$. Let $X = \{x_1, x_2, x_3\}$ be a set with three different elements and for each $i \in I$, define $\mathsf{u}_i : X \times X \to L$ as:

$$\mathsf{u}_i(x,y) = \begin{cases} \boldsymbol{a}_i & \text{if } x = x_1, y = x_2 \\ \boldsymbol{b}_i & \text{if } x = x_2, y = x_3 \\ \top_L & \text{if } x = y \\ \boldsymbol{a}_i * \boldsymbol{b}_i & \text{otherwise} \end{cases}.$$

It is easy to check that, for each $i \in I$, u_i is $*$-transitive and $\mathsf{u}_i(x,x) = \top_L$ for all $x \in X$ so $(\{\mathsf{u}_i\}, *)$ is a base of an L-$*$-probabilistic quasi-uniformity $(\mathcal{U}_i, *)$ on X. Set $\mathfrak{U} = \{(X, \mathcal{U}_i, *) : i \in I\}$. By assumption $(\triangle\mathcal{U}_F^{\mathfrak{U}}, \star)$ must be an S-probabilistic quasi-uniformity on X. In particular, we can find $\mathbf{v} \in \prod_{i\in I} \mathcal{U}_i$ such that $(F \circ \triangle\mathbf{v}) \circ_\star (F \circ \triangle\mathbf{v}) \preceq_S F \circ \triangle\mathbf{u}$ where $\mathbf{u}_i = \mathsf{u}_i$ for all $i \in I$. Then:

$$\begin{aligned} ((F \circ \triangle\mathbf{v}) \circ_\star (F \circ \triangle\mathbf{v}))(x_1,x_3) &\preceq_S (F \circ \triangle\mathbf{u})(x_1,x_3) \\ \bigvee_{z\in X} F((\mathbf{v}_i(x_1,z))_{i\in I}) \star F((\mathbf{v}_i(z,x_3))_{i\in I}) &\preceq_S F((\mathsf{u}_i(x_1,x_3))_{i\in I}) \\ \bigvee_{z\in X} F((\mathbf{v}_i(x_1,z))_{i\in I}) \star F((\mathbf{v}_i(z,x_3))_{i\in I}) &\preceq_S F(\boldsymbol{a} *^I \boldsymbol{b}). \end{aligned}$$

In particular,

$$F(\triangle\mathbf{v}(x_1,x_2)) \star F(\triangle\mathbf{v}(x_2,x_3)) = F((\mathbf{v}_i(x_1,x_2))_{i\in I}) \star F((\mathbf{v}_i(x_2,x_3))_{i\in I}) \preceq_S F(\boldsymbol{a} *^I \boldsymbol{b}).$$

Moreover, since $\mathcal{U}_i$ has as base $\{\mathbf{u}_i\}$ for all $i \in I$ then $\triangle\mathbf{u}(x_1,x_2) = \boldsymbol{a} \preceq_L^I \triangle\mathbf{v}(x_1,x_2)$ and $\triangle\mathbf{u}(x_2,x_3) = \boldsymbol{b} \preceq_L^I \triangle\mathbf{v}(x_2,x_3)$. Hence F is upper $*^I, \star$-supmultiplicative.

$(iii) \Rightarrow (iv)$ It is easy to check that $\{\triangle \mathbf{u} : \mathbf{u} \in \prod_{i\in I} \mathfrak{U}_i\}$ is a filter of L^I-fuzzy sets on X^2. By Theorem 1, F preserves filters of fuzzy sets so $\triangle \mathcal{U}_F^{\mathfrak{U}} = \{F \circ \triangle \mathbf{u} : \mathbf{u} \in \prod_{i\in I} \mathfrak{U}_i\}$ is a filter of S-fuzzy sets on X^2.

Moreover, let $\mathbf{u} \in \prod_{i\in I} \mathfrak{U}_i$. Then $(F \circ \triangle \mathbf{u})(x,x) = F((\mathbf{u}_i(x,x))_{i\in I}) = F(\top_{L^I}) = \top_S$ for all $x \in X$ so $\triangle \mathcal{U}_F^{\mathfrak{U}}$ satisfies (PQU1).

$(\widehat{iii}) \Rightarrow (ii)$ Let $\mathfrak{U} = \{(X, \mathfrak{U}_i, *) : i \in I\}$ be a family of L-probabilistic quasi-uniform spaces. Since isotonicity implies that $F(\uparrow \boldsymbol{a})$ is filtered for all $\boldsymbol{a} \in L^I$, reasoning as in $(iii) \Rightarrow (iv)$, we obtain that $\triangle \mathcal{U}_F^{\mathfrak{U}} = \{F \circ \triangle \mathbf{u} : \mathbf{u} \in \prod_{i\in I} \mathfrak{U}_i\}$ is a filter of S-fuzzy sets on X^2 satisfying (PQU1).

Let $\mathbf{u} \in \prod_{i\in I} \mathfrak{U}_i$. For each $i \in I$, we can find $\mathrm{v}_i \in \mathfrak{U}_i$ such that $\mathrm{v}_i \circ_* \mathrm{v}_i \preceq_L \mathbf{u}_i$. Let $\mathbf{v} \in \prod_{i\in I} \mathfrak{U}_i$ such that $\mathbf{v}_i = \mathrm{v}_i$ for all $i \in I$. Since F is $*^I, \star$-upper supmultiplicative and isotone then it is $*^I, \star$-supmultiplicative. Then for each $x, y, z \in X$ we have:

$$\begin{aligned}(F \circ \triangle \mathbf{v})(x,z) \star (F \circ \triangle \mathbf{v})(z,y) &= F((\mathrm{v}_i(x,z))_{i\in I}) \star F((\mathrm{v}_i(z,y))_{i\in I}) \\ &\preceq_S F((\mathrm{v}_i(x,z) * \mathrm{v}_i(z,y))_{i\in I}) \\ &\preceq_S F((\boldsymbol{u}_i(x,y))_{i\in I}) = (F \circ \triangle \mathbf{u})(x,y).\end{aligned}$$

Consequently, $(F \circ \triangle \mathbf{v}) \circ_\star (F \circ \triangle \mathbf{v}) \preceq_S (F \circ \triangle \mathbf{u})$ and $(\triangle \mathcal{U}_F^{\mathfrak{U}}, \star)$ is an S-probabilistic quasi-uniformity on X. □

A simple adaptation of the above proof, allows to show the following.

Proposition 8. *Let $(L, \preceq_L)$ and $(S, \preceq_S)$ be two complete lattices and let I be a set of indices. Let $F : (L^I, \preceq_L^I) \to (S, \preceq_S)$ be a function and $*, \star$ be t-norms on L and S respectively. Each of the following statements implies its successor:*

(i) F directly $\star$-aggregates L-$*$-probabilistic quasi-uniformities on products;
(ii) F directly $\star$-aggregates L-$*$-probabilistic quasi-uniformities on sets;
(iii) $F(\top_{L^I}) = \top_S$; for each $\boldsymbol{a} \in \bigoplus_{i\in I} L$, $\uparrow F(\boldsymbol{a}) \subseteq F(\uparrow \boldsymbol{a})$ and $F(\uparrow \boldsymbol{a})$ is filtered; F is upper $*^I, \star$-supmultiplicative on $\bigoplus_{i\in I} L$;
(iv) if $\mathfrak{U} = \{(X, \mathfrak{U}_i, *) : i \in I\}$ is a family of L-probabilistic quasi-uniform spaces then $\overset{\oplus}{\triangle} \mathcal{U}_F^{\mathfrak{U}}$ is a filter on $(L^{X\times X}, \preceq_L)$ satisfying property (PQU1).

Moreover, if we replace in (iii) the condition "$F(\uparrow a)$ is filtered for every $\boldsymbol{a} \in \bigoplus_{i\in I} L$" by the stronger condition "F is isotone", that is, if we consider the statement:

$(\widehat{iii})$ $F(\top_{L^I}) = \top_S$; for each $\boldsymbol{a} \in \bigoplus_{i\in I} L$, $\uparrow F(\boldsymbol{a}) \subseteq F(\uparrow \boldsymbol{a})$; F is isotone and upper $*^I, \star$-supmultiplicative on $\bigoplus_{i\in I} L$;

then $(\widehat{iii})$ implies (ii).

In view of Theorem 2 and Proposition 7, we notice that in general, the function F which $\star$-aggregates bases of L-$*$-probabilistic quasi-uniformities could not $\star$-aggregates L-$*$-probabilistic quasi-uniformities. For example, we can consider the function $F : [0,1] \to [0,1]$ given by:

$$F(x) = \begin{cases} \frac{1}{2}x & \text{if } x \neq 1, \\ 1 & \text{if } x = 1. \end{cases}$$

It is straightforward to show that F satisfies all conditions of Theorem 2 (v) with respect to the product t-norm. However $\uparrow F(0) = [0,1] \not\subseteq F(\uparrow 0) = [0, \frac{1}{2}[\cup \{1\}$ so F does not verify the necessary conditions given in Proposition 7 (iii) for $\cdot$-aggregate $[0,1]$-$\cdot$-probabilistic quasi-uniformities.

Nevertheless, if F $\star$-aggregates bases of L-$*$-probabilistic quasi-uniformities and F preserves upper sets, that is, $\uparrow F(\boldsymbol{a}) \subseteq F(\uparrow \boldsymbol{a})$ for all $\boldsymbol{a} \in L^I$ (see Proposition 3), then F $\star$-aggregates L-$*$-probabilistic quasi-uniformities on sets by Theorem 2 and $(\widehat{iii}) \Rightarrow (ii)$ of Proposition 7. Notice that every Boolean function $F : \{0,1\}^I \to \{0,1\}$ verifying that $F(\mathbf{1}) = 1$ preserves upper sets so, in particular, we have the following.

Corollary 4. *Let $F : \{0,1\}^I \to \{0,1\}$ be a Boolean function. If F $\wedge$-aggregates bases of $\{0,1\}$-$\wedge$-probabilistic quasi-uniformities on sets then F $\wedge$-aggregates $\{0,1\}$-$\wedge$-probabilistic quasi-uniformities on sets.*

Proof. By Corollary 3 we know that $F(\mathbf{1}) = 1$, F is isotone and $\wedge, \wedge^I$-supmultiplicative. Moreover, given $\boldsymbol{a} \in \{0,1\}^I$ we have that:

$$\uparrow F(\boldsymbol{a}) = \begin{cases} \{1\} & \text{if } F(\boldsymbol{a}) = 1 \\ \{0,1\} & \text{if } F(\boldsymbol{a}) = 0 \end{cases}.$$

Since $F(\mathbf{1}) = 1$ and $\mathbf{1} \in\uparrow \boldsymbol{a}$, it is clear that $\uparrow F(\boldsymbol{a}) \subseteq F(\uparrow \boldsymbol{a})$. The conclusion follows from $(\widehat{iii}) \Rightarrow (ii)$ of Proposition 7. □

Example 10. *In [29] (see also [31]) Lowen introduced the so-called Lowen uniformities which are a particular case of probabilistic uniformities [30]. Moreover, he established adjoint functors between the categories of Lowen uniformities and classical uniformities. These functors can also be established in the more general categories of probabilistic quasi-uniformities and crisp quasi-uniformities [39]. In particular, if $(X, \mathcal{U}, *)$ is a probabilistic quasi-uniform space then:*

$$\iota(\mathcal{U}) = \{\mathsf{u}^{-1}((\alpha, 1]) : \mathsf{u} \in \mathcal{U},\ \alpha \in [0,1)\}$$

is a quasi-uniformity on X. We next show that we can apply our theory in order to obtain this result.

For every $\alpha \in [0,1[$, let $F_\alpha : [0,1] \to \{0,1\}$ given by:

$$F_\alpha(x) = \begin{cases} 1 & \text{if } \alpha < x \leq 1 \\ 0 & \text{if } 0 \leq x \leq \alpha \end{cases}$$

for all $x \in [0,1]$. Then it is obvious that F_α is isotone and: $\wedge, \wedge$-supmultiplicative (in fact it is minitive). Moreover, for each $\alpha \in [0,1[$, $F_\alpha(1) = 1$ and $F_\alpha(\uparrow a) =\uparrow F_\alpha(a)$ for all $a \in [0,1[$. Therefore, by Proposition 8, if $\mathfrak{U} = \{(X, \mathcal{U}, *)\}$ is a family with only one probabilistic quasi-uniform space then $(\overset{\oplus}{\triangle}\mathcal{U}^{\mathfrak{U}}_{F_\alpha}, \wedge) = (\triangle\mathcal{U}^{\mathfrak{U}}_{F_\alpha}, \wedge)$ is a $\{0,1\}$-probabilistic quasi-uniformity on X for all $\alpha \in [0,1[$. Notice that $\triangle\mathcal{U}^{\mathfrak{U}}_{F_\alpha}$ is formed by all the $\{0,1\}$-fuzzy sets $F_\alpha \circ \mathsf{u}$ on $X \times X$ where $\mathsf{u} \in \mathcal{U}$, and

$$(F_\alpha \circ \mathsf{u})(x,y) = \begin{cases} 1 & \text{if } \alpha < \mathsf{u}(x,y) \leq 1, \\ 0 & \text{if } 0 \leq \mathsf{u}(x,y) \leq \alpha, \end{cases}$$

for all $x, y \in X$. For simplicity, let us write $\mathcal{U}_{F_\alpha}$ instead of $\triangle\mathcal{U}^{\mathfrak{U}}_{F_\alpha}$ and consider the family $\mathfrak{G} = \{(X, \mathcal{U}_{F_\alpha}, \wedge) : \alpha \in [0,1[\}$ of probabilistic $\{0,1\}$-quasi-uniform spaces.

On the other hand, consider the function $G : \{0,1\}^{[0,1[} \to \{0,1\}$ given by $G(\boldsymbol{x}) = \inf_{\varepsilon \in [0,1[} \boldsymbol{x}_\varepsilon$ for all $\boldsymbol{x} \in \{0,1\}^{[0,1[}$. Then G is also isotone, minitive, $G(\mathbf{1}) = 1$ and $G(\uparrow \boldsymbol{a}) =\uparrow G(\boldsymbol{a})$ for all $\boldsymbol{a} \in \{0,1\}^{[0,1[}$. By Proposition 8, G directly $\wedge$-aggregates $\{0,1\}$-$\wedge$-probabilistic quasi-uniformities on sets. Then, $(\overset{\oplus}{\triangle}\mathcal{U}^{\mathfrak{G}}_G, \wedge)$ is a $\{0,1\}$-probabilistic quasi-uniformity on X having as base $\{G \circ \triangle\mathbf{u} : \mathbf{u} \in \bigoplus_{\alpha \in [0,1[} \mathcal{U}_{F_\alpha}\}$. Given $\mathbf{u} \in \bigoplus_{\alpha \in [0,1[} \mathcal{U}_{F_\alpha}$ we can find a finite subset $J \subset [0,1[$ such that $\mathbf{u}_\alpha = \chi_{X \times X}$ if and only if $\alpha \notin J$. Then

$$\begin{aligned} (G \circ \triangle\mathbf{u})(x,y) &= G((\mathbf{u}_\alpha(x,y))_{\alpha \in [0,1[}) = G(((F_\alpha \circ \mathbf{u}_\alpha)(x,y))_{\alpha \in [0,1[}) \\ &= \inf_{\alpha \in [0,1[} (F_\alpha \circ \mathbf{u}_\alpha)(x,y) = \inf_{\alpha \in J} (F_\alpha \circ \mathbf{u}_\alpha)(x,y) \\ &= \begin{cases} 1 & \text{if } \alpha < \mathbf{u}_\alpha(x,y) \leq 1 \text{ for all } \alpha \in J \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Consequently, $(G \circ \triangle\mathbf{u})(x,y) = 1$ *if and only if* $(x,y) \in \bigcap_{\alpha\in J} \mathbf{u}_\alpha^{-1}((\alpha,1])$. *Then it is obvious that the quasi-uniformity* $\mathcal{U}_{\overset{\oplus}{\triangle}\mathfrak{U}_G^{\mathfrak{G}}}$ *associated with the* $\{0,1\}$*-probabilistic quasi-uniformity* $\overset{\oplus}{\triangle}\mathfrak{U}_G^{\mathfrak{G}}$ *is the quasi-uniformity* $\iota(\mathfrak{U})$.

4. Conclusions

L-probabilistic quasi-uniformities are important structures in the fuzzy context as they play a similar role to quasi-uniformities in classical general topology. Therefore, it is important to know how we can define operations on L-probabilistic quasi-uniform spaces as sums or cartesian products, a theme which has not been previously considered in the literature. Even for crisp quasi-uniformities, the classical theory only deals with the usual product quasi-uniformity but other construction methods cannot be found. We investigated this topic and our results provided a general method to construct L-probabilistic quasi-uniformities in the cartesian product or in the direct sum of an arbitrary family of L-probabilistic quasi-uniform spaces. Moreover, as L is arbitrary, it could also deduce results for crisp quasi-uniformities when $L = \{0,1\}$, including the classical construction of the product quasi-uniformity. Furthermore we encompassed in our general theory a classical construction due to Lowen of a crisp quasi-uniformity starting from a probabilistic quasi-uniformity.

Nevertheless, more research can be developed in order to work with L-probabilistic quasi-uniformities rather than bases since, in this case, we have only obtained some necessary or sufficient conditions. However, notice that this is not an issue since every L-probabilistic quasi-uniformity admits a base (itself).

Author Contributions: Conceptualization, investigation, writing–original draft preparation, writing–review and editing, supervision, T.P. and J.R.-L.; funding acquisition, J.R.-L. All authors have read and agreed to the published version of the manuscript.

Funding: J. Rodríguez-López acknowledges financial support from FEDER/Ministerio de Ciencia, Innovación y Universidades-Agencia Estatal de Investigación Proyecto PGC2018-095709-B-C21.

Acknowledgments: We kindly acknowledge the comments of all the reviewers of this paper who contributed to improve it.

Conflicts of Interest: The authors declare no conflict of interest.

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