Article

Inequalities in Triangular Norm-Based ∗-Fuzzy $(L^+)^p$ Spaces

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Abstract: In this article, we introduce the ∗-fuzzy $(L^+)^p$ spaces for $1 \leq p < \infty$ on triangular norm-based ∗-fuzzy measure spaces and show that they are complete ∗-fuzzy normed space and investigate some properties in these space. Next, we prove Chebyshev’s inequality and Hölder’s inequality in ∗-fuzzy $(L^+)^p$ spaces.

Keywords: fuzzy measure space; fuzzy integration; t-norm; Chebyshev’s inequality; Hölder’s inequality

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Function spaces, especially $L^p$ spaces, play an important role in many parts in analysis. The impact of $L^p$ spaces follows from the fact that they offer a partial but useful generalization of the fundamental $L^1$ space of integrable functions. The standard analysis, based on sigma-additive measures and Lebesgue–Stieltjess integral, including also several integral inequalities, has been generalized in the past decades into set-valued analysis, including set-valued measures, integrals, and related inequalities. Some subsequent generalizations are based on fuzzy sets [1,2] and include fuzzy measures, fuzzy integrals and several fuzzy integral inequalities. Our aim is the further development of fuzzy set analysis, expanding our original proposal given in [3]. In fact, we use a new model of the fuzzy measure theory (∗-fuzzy measure) which is a dynamic generalization of the classical measure theory. Our model of the fuzzy measure theory created by replacing the non-negative real range and the additivity of classical measures with fuzzy sets and triangular norms. Moreover, the ∗-fuzzy measure theory has been motivated by defining new additivity property using triangular norms. Our approach is related to the idea of fuzzy metric spaces [4–7] and can be apply for decision making problems [8,9].

In this paper, we shall work on a fixed triangular norm-based ∗-fuzzy measure space $(X, C, \mu, ∗)$ introduced in [3] which was derived from the idea of fuzzy and probabilistic metric spaces [5–7,10,11]. Using the concept of fuzzy measurable functions and fuzzy integrable functions we define a special class of function spaces named by ∗-fuzzy $(L^+)^p$. After some overview given in Sections 2–4 and devoted to the basic information concerning ∗-fuzzy measures and related integration, in Section 5 we define a norm on ∗-fuzzy $(L^+)^p$ spaces and show these spaces are complete ∗-fuzzy normed space in the sense of Cheng-Mordeson and others [12–15]. This definition of ∗-fuzzy norm helps us to prove Chebyshev’s Inequality and Hölder’s Inequality.
1. *-Fuzzy Measure

First, we recall some basic concepts and notations that will be used throughout the paper. Let \( X \) be a non-empty set, \( C \) be a \( \sigma \)-algebra of subsets of \( X \). Unless stated otherwise, all subsets of \( X \) are supposed to belong to \( C \). Here, we let \( I = [0,1] \).

**Definition 1.** ([10,11]) A continuous triangular norm (shortly, a ct-norm) is a continuous binary operation \( * \) from \( I^2 = [0,1]^2 \) to \( I \) such that

(a) \( \varsigma * \tau = \tau * \varsigma \) and \( \varsigma * (\tau * \upsilon) = (\varsigma * \tau) * \upsilon \) for all \( \varsigma, \tau, \upsilon \in [0,1] \);

(b) \( \varsigma * 1 = \varsigma \) for all \( \varsigma \in I \);

(c) \( \varsigma * \tau \leq \upsilon * \iota \) whenever \( \varsigma \leq \upsilon \) and \( \tau \leq \iota \) for all \( \varsigma, \tau, \upsilon, \iota \in I \).

Some examples of the ct-norms are as follows.

1. \( \varsigma * \rho \tau = \varsigma \tau \) (: the product t-norm);
2. \( \varsigma * M \tau = \min\{\varsigma, \tau\} \) (: the minimum t-norm);
3. \( \varsigma * L \tau = \max\{\varsigma + \tau - 1, 0\} \) (: the Lukasiewicz t-norm);
4. \( \varsigma * H \tau = \begin{cases} 0, & \text{if } \varsigma = \tau = 0, \\ \frac{1}{\varsigma + \frac{1}{\tau} - 1}, & \text{otherwise,} \end{cases} \) (: the Hamacher product t-norm).

We define

\[ *^k_{i=1} \varsigma_i = \varsigma_1 * \varsigma_2 * \cdots * \varsigma_k, \]

for \( k \in \{2,3,\cdots\} \), which is well defined due to the associativity of the operation \( * \). Moreover,

\[ *^\infty_{i=1} \varsigma_i = \lim_{k \to \infty} *^k_{i=1} \varsigma_i, \]

which is well defined due to the monotonicity and boundedness of the operation \( * \).

Now, we introduce the concept of \( * \)-fuzzy measure.

**Definition 2 ([3]).** Let \( X \) be a set and \( C \) be a \( \sigma \)-algebra consisting of subsets of \( X \). A fuzzy measure on \( C \times (0,\infty) \) is a fuzzy set \( \mu : C \times (0,\infty) \to I \) such that

(i) \( \mu(\emptyset, \tau) = 1 \), \( \forall \tau \in (0,\infty) \);

(ii) if \( A_i \in C, i = 1,2,\cdots, \) are pairwise disjoint, then

\[ \mu(\cup_{i=1}^{\infty} A_i, \tau) = *_{i=1}^{\infty} \mu(A_i, \tau), \forall \tau \in (0,\infty). \]

Saying the \( A_i \) are pairwise disjoint means that \( A_i \cap A_j = \emptyset, \) if \( i \neq j \).

Definition 2 is known as countable \( * \)-additivity. We say a fuzzy measure \( \mu \) is finitely \( * \)-additive if, for any \( n \in \mathbb{N} \)

\[ \mu(\cup_{i=1}^{n} A_i, \tau) = *_{i=1}^{n} \mu(A_i, \tau), \forall \tau \in (0,\infty), \]

whenever \( A_1, \cdots, A_n \) are in \( C \) and are pairwise disjoint. The quadruple \( (X,C,\mu,*) \) is called a \( * \)-fuzzy measure space (in short, \( * \)-FMS).
Example 1. Let \((X, C, m)\) be a measurable space. Let \(* = *_H\) and define
\[
\mu_0(A, \tau) = \frac{\tau}{\tau + m(A)}, \quad \forall \tau \in (0, \infty),
\]
then \((X, C, \mu_0, *)\) is a \(*\)-FMS.

Example 2. Let \((X, C, m)\) be a measurable space. Let \(* = *_p\). Define
\[
\mu_0(A, \tau) = e^{-\frac{m(A)}{\tau}}, \quad \forall \tau \in (0, \infty).
\]

Then, \(\mu_0\) is a \(*\)-FMS on \(C \times (0, \infty)\).

2. \(*\)-Fuzzy Measurable Functions

Now, we review the concept of \(*\)-fuzzy normed spaces, for more details, we refer to the works in [12–15].

Definition 3. Let \(X\) be a vector space, \(*\) be a ct-norm and the fuzzy set \(N\) on \(X \times (0, \infty)\) satisfies the following conditions for all \(x, y \in X\) and \(\tau, \sigma \in (0, \infty)\),

(i) \(N(x, \tau) > 0\).
(ii) \(N(x, \tau) = 1 \iff x = 0\).
(iii) \(N(\alpha x, \tau) = N\left(x, \frac{\tau}{|\alpha|}\right)\) for every \(\alpha \neq 0\).
(iv) \(N(x, \tau) \cdot N(y, \sigma) \leq N(x + y, \tau + \sigma)\).
(v) \(N(x, \tau) : (0, \infty) \to (0, 1]\) is continuous.
(vi) \(\lim_{\tau \to 0} N(x, \tau) = 1\) and \(\lim_{\tau \to 0} N(x, \tau) = 0\).

Then, \(N\) is called a \(*\)-fuzzy norm on \(X\) and \((X, N, *)\) is called \(*\)-fuzzy normed space.

Assume that \((\mathbb{R}, |.|)\) is a standard normed space, we define: \(N(x, \tau) = \frac{\tau}{|x| + |\tau|}\) with \(* = *_p\), it is obvious \((\mathbb{R}, N, *_p)\) is a \(*\)-fuzzy normed space.

Let \((X, N, *)\) be a \(*\)-fuzzy normed space. We define the open ball \(B(x, r, \tau)\) and the closed ball \(B[x, r, \tau]\) with center \(x \in X\) and radius \(0 < r < 1\), \(\tau > 0\) as follows,

\[
B(x, r, \tau) = \{y \in X : N(x - y, \tau) > 1 - r\},
\]
\[
B[x, r, \tau] = \{y \in X : N(x - y, \tau) \geq 1 - r\}.
\]

Let \((X, N, *)\) be a \(*\)-fuzzy normed space. A set \(E \subset X\) is said to be open if for each \(x \in E\), there is \(0 < r_x < 1\) and \(\tau_x > 0\) such that \(B(x, r_x, \tau_x) \subset E\). A set \(F \subset X\) is said to be closed in \(X\) in case its complement \(\complement X = X - F\) is open in \(X\).

Let \((X, N, *)\) be a \(*\)-fuzzy normed space. A subset \(E \subset X\) is said to be fuzzy bounded if there exist \(\tau > 0\) and \(r \in (0, 1)\) such that \(N(x - y, \tau) > 1 - r\) for all \(x, y \in E\).

Let \((X, N, *)\) be a \(*\)-fuzzy normed space. A sequence \(\{x_n\} \subset X\) is fuzzy convergent to an \(x \in X\) in \(*\)-fuzzy normed space \((X, N, *)\) if for any \(\tau > 0\) and \(\epsilon > 0\) there exists a positive integer \(N_\epsilon > 0\) such that \(N(x_n - x, \tau) > 1 - \epsilon\) whenever \(n \geq N_\epsilon\).

Now, we define \(*\)-fuzzy measurable functions.

Definition 4. Let \((X, C)\) and \((Y, D)\) be \(*\)-fuzzy measurable spaces. A mapping \(f : X \to Y\) is called \(*\)-fuzzy \((C, D)\)-measurable if \(f^{-1}(E) \subset C\) for all \(E \in D\). If \(X\) is any \(*\)-fuzzy normed space, the \(\sigma\)-algebra generated by
the family of open sets in $X$ (or, equivalently, by the family of closed sets in $X$) is called the Borel $\sigma$-algebra on $X$ and is denoted by $B_X$.

3. $*$-Fuzzy Integration

In this section, we recall the concept of $*$-fuzzy integration by using fuzzy simple functions on the $*$-FMS $(X, C, *, \mu)$ and add some new results.

**Definition 5.** Let $(X, C, *, \mu)$ be $*$-FMS, we define

$$L_+ = \{ f : X \to [0, \infty) \mid f \text{ is fuzzy (C, } B_X) \text{-measurable function} \}.$$

If $\phi$ is a simple fuzzy ((C, $B_X$)-measurable) function in $L_+$ with standard representation $\phi = \sum_{i=1}^{n} a_i \chi_{E_i}$, where $a_i > 0$ and $E_i \in C$ for $i = 1, ..., n$, and $E_i \cap E_j = \emptyset$ for $i \neq j$, we define the fuzzy integral of $\phi$ as

$$\int_X \phi(x) d\mu(x, \tau) = \int_X \sum_{i=1}^{n} a_i \chi_{E_i} d\mu(x, \tau) = \sum_{i=1}^{n} a_i \mu(E_i, \frac{\tau}{a_i}).$$

In [3], the authors have shown that, with respect to $\mu(A, \tau)$, $\mu$ satisfies the following statement:

(i) $\mu : (A, \tau) : (0, \infty) \to [0, 1]$ is increasing and continuous.

(ii) $\mu \left( A, \frac{T}{a + b} \right) \geq \mu \left( \left( A, \frac{T}{a} \right) \ast \mu \left( \left( A, \frac{T}{b} \right) \right) \right)$ for every $a, b > 0, \tau \in (0, \infty)$.

(iii) $\lim_{\tau_n \to \tau_0} \mu(E, \tau_n) = \mu(E, \tau_0)$ for every $E \in C$ and $\tau \in (0, \infty)$.

(iv) $\lim_{\tau_n \to \tau_0} \mu(E, \tau_n) = \mu(E, \tau_0)$.

(v) $\lim_{\tau_n \to \tau_0} \lim_{m \to \infty} \mu \left( E, \frac{\tau_n}{a^m} \right) = \lim_{m \to \infty} \lim_{\tau_n \to \tau_0} \mu \left( E, \frac{\tau}{a^m} \right)$.

If $A \in C$, then $\phi \chi_A$ is also fuzzy simple function $\left( \phi \chi_A = \sum_{i=1}^{n} a_i \chi_{A \cap E_i} \right)$, and we define

$$\int \phi(x) d\mu(x, \tau) = \int \phi \chi_A d\mu(x, \tau).$$

**Theorem 1** ([3]). Let $\phi$ and $\psi$ be simple functions in $L_+$. Then, we have

(i) $\int_X \phi d\mu(x, \tau) = 1$.

(ii) If $c \in (0, 1)$ then $\int_X (c \phi)(x) d\mu(x, \tau) \geq c \int_X \phi(x) d\mu(x, \tau)$, and for $c \in [1, \infty)$ we have $\int_X (c \phi)(x) d\mu(x, \tau) \leq c \int_X \phi(x) d\mu(x, \tau)$, $\forall \tau \in (0, \infty)$.

(iii) If $\phi \leq \psi$, then $\int_X \phi(x) d\mu(x, \tau) \geq \int_X \psi(x) d\mu(x, \tau)$.

(iv) The map $A \to \int_A \phi(x) d\mu(x, \tau)$ is a fuzzy measure on $C$, $\forall \tau \in (0, \infty)$.

In the next theorem, we prove an important fuzzy integral inequality for fuzzy simple functions.

**Theorem 2.** Let $\phi$ and $\psi$ be fuzzy simple functions in $L_+$, then

$$\int \phi(x) d\mu(x, \tau) \geq \left( \int \phi(x) d\mu(x, \tau) \right) \ast \left( \int \psi(x) d\mu(x, \tau) \right).$$
Proof. Let $\phi$ and $\psi$ be fuzzy simple functions in $L_+$, then we have

\[
\int_X (\phi + \psi)(x)d\mu(x, \tau),
\]

\[
= \int_X \left( \left( \sum_{i=1}^n a_i \chi_{E_i}(x) \right) \chi_F(x) \right) d\mu(x, \tau),
\]

\[
= \int_X \left( \sum_{i,j} (a_i + b_j) \chi_{E_i \cap F_j}(x) \right) d\mu(x, \tau),
\]

\[
= \sum_{i=1}^n \sum_{j=1}^m \mu \left( (E_i \cap F_j), \frac{\tau}{a_i + b_j} \right).
\]

On the other hand,

\[
\left( \int_X \phi(x)d\mu(x, \tau) \right) * \left( \int_X \psi(x)d\mu(x, \tau) \right)
\]

\[
= \left( \sum_{i=1}^n \sum_{j=1}^m \mu \left( (E_i \cap F_j), \frac{\tau}{a_i} \right) \right) * \left( \sum_{j=1}^m \sum_{i=1}^n \mu \left( (E_i \cap F_j), \frac{\tau}{b_j} \right) \right),
\]

\[
= \sum_{i=1}^n \sum_{j=1}^m \mu \left( (E_i \cap F_j), \frac{\tau}{a_i} \right) * \mu \left( (E_i \cap F_j), \frac{\tau}{b_j} \right),
\]

\[
= \sum_{i=1}^n \sum_{j=1}^m \mu \left( (E_i \cap F_j), \frac{\tau}{(a_i + b_j)} \right).
\]

From (3) and (4), we get

\[
\int_X (\phi + \psi)(x)d\mu(x, \tau) \geq \left( \int_X \phi(x)d\mu(x, \tau) \right) * \left( \int_X \psi(x)d\mu(x, \tau) \right).
\]

\[
\square
\]

Now, we extend the concept of fuzzy integral to all functions in $L_+$.

Definition 6. Let $f$ be a fuzzy measurable function in $L_+$, we define fuzzy integral by

\[
\int_X f(x)d\mu(x, \tau)
\]

\[
= \inf \left\{ \int_X \phi(x)d\mu(x, \tau) \mid 0 \leq \phi \leq f, \ \phi \text{ is fuzzy simple function} \right\}.
\]

By Theorem 1 (iii), the two definitions of $\int f$ agree when $f$ is fuzzy simple function, as the family of fuzzy simple functions over which the infimum is taken includes $f$ itself. Moreover, it is obvious from the definition that $\int f \geq \int g$ whenever $f \leq g$, and $\int cf \geq c \int f$ for all $c \in (0, 1]$ and $\int cf \leq c \int f$ for all $c \in [1, \infty)$ and $\int (f + g) \geq (\int f) * (\int g)$.

Definition 7. If $f \in L_+$, we say that $f$ is fuzzy integrable if $\int f d\mu(x, \tau) > 0$ for each $\tau > 0$. Let $(X, C, \mu, *)$ be a $*-FMS$. We define

\[
L^+ := \left\{ f : X \to [0, \infty), f \text{ is measurable function and } \int f(x)d\mu(x, \tau) > 0 \right\}.
\]
Theorem 3 ([3]). (The fundamental convergence theorem) Let \((X, \mathcal{C}, \mu, \ast)\) be a \(\ast\)-FMS. Let \(f_n\) be a sequence in \(L^+\) such that \(f_n \to f\) almost everywhere, then \(f \in L^+\) and \(\int f = \lim_{n \to \infty} \int f_n\).

\(\ast\)-Fuzzy \(L^+\) Spaces

Here, we are ready to show that every \(L^+\) is a \(\ast\)-fuzzy normed space. It is clear if we define \(L := \{f : X \to \mathbb{R}, f \text{ is fuzzy measurable function}\}\), then \((L, +, \cdot, \mathbb{R})\) is a vector space. Moreover, in [3] the authors proved that if \(f, g \in L^+\), then \(|f - g| \in L^+\).

Using definition \(L\) and \(L^+\) we can show \(L^+ \subseteq L\). In \(L^+\) we define \(f \leq g\) if and only if \(f(x) \leq g(x)\) and so \((L^+, \leq)\) is a cone.

Note. Recall that, due to the continuity of t-norm \(\ast\), for any systems \(\{a_n\}_{n \in \mathbb{N}}\) and \(\{b_n\}_{n \in \mathbb{N}}\) of elements form \(I\) we have \(\inf\{a_n \ast b_n\} = \inf\{a_n\} \ast \inf\{b_n\}\).

In the next theorem we define a fuzzy norm on \(L^+\) and prove that \((L^+, N, \ast)\) is a \(\ast\)-fuzzy normed space.

Theorem 4. Let \(N : L^+ \times (0, \infty) \to (0, 1]\) be a fuzzy set, such that \(N(f, \tau) = \int f d\mu(x, \tau)\), then \((L^+, N, \ast)\) is a \(\ast\)-fuzzy normed space.

Proof.

(FN1) \(N(f, \tau) = \int f d\mu(x, \tau) > 0\).

(FN2) By theorem 4.5 of [3] we have

\[ N(f, \tau) = 1 \iff \int f d\mu(x, \tau) = 1 \iff f = 0 \]

almost everywhere.

(FN3) Let \(f = \phi = \sum_{i=1}^{n} a_i \chi_{E_i}\) and \(c > 0\) so,

\[ N(c\phi, \tau) = \int c\phi d\mu(x, \tau), \]

\[ = \int \sum_{i=1}^{n} a_i \chi_{E_i} d\mu(x, \tau), \]

\[ = \ast \sum_{i=1}^{n} \mu\left(E_i, \frac{\tau}{ca_i}\right). \]

On the other hand,

\[ N\left(\phi, \frac{\tau}{c}\right) = \int \phi d\mu\left(x, \frac{\tau}{c}\right), \]

\[ = \int \sum_{i=1}^{n} a_i \chi_{E_i} d\mu\left(x, \frac{\tau}{c}\right), \]

\[ = \ast \sum_{i=1}^{n} \mu\left(E_i, \frac{\tau}{ca_i}\right). \]

From (5) and (6) we conclude that

\[ N(c\phi, \tau) = N\left(\phi, \frac{\tau}{c}\right). \]
Now, if \( f \in L^+ \) we have \( \{ \phi_n \} \subseteq L^+ \) such that \( \phi_n \uparrow f \), then \( c\phi_n \uparrow cf \) so
\[
\int c\phi_n d\mu(x, \tau) \downarrow \int cf d\mu(x, \tau).
\]

By (7), we have \( \int c\phi_n d\mu(x, \tau) = \int \phi_n d\mu(x, \tau) \), and so
\[
\int \phi_n d\mu(x, \tau) \downarrow \int cf d\mu(x, \tau),
\]

(8)

On the other hand,
\[
\int \phi_n d\mu(x, \tau) \downarrow \int f d\mu(x, \tau),
\]

(9)

by (8) and (9) we have,
\[
\int \phi_n d\mu(x, \tau) \downarrow \int f d\mu(x, \tau),
\]

(10)

On the other hand
\[
N(cf, \tau) = N(f, \tau).
\]

(FN4) Let \( f = \sum_{i=1}^{m} a_i \chi_{E_i}, g = \sum_{j=1}^{n} b_j \chi_{F_j} \) then,
\[
N(\phi + \psi, s + \tau) = \int (\phi + \psi) d\mu(x, \tau + s),
\]
\[
= \int \sum_{i,j} (a_i + b_j) \chi_{E_i \cap F_j} d\mu(x, \tau + s),
\]
\[
= \ast_{i,j}(E_i \cap F_j, \frac{s + \tau}{a_i + b_j}).
\]

On the other hand
\[
N(\phi, s) \ast N(\psi, \tau) = \left( \int \phi d\mu(x, s) \right) \ast \left( \int \psi d\mu(x, \tau) \right),
\]
\[
= \left( \int \sum_{i,j} a_i \chi_{E_i \cap F_j} d\mu(x, s) \right) \ast \left( \int \sum_{i,j} b_j \chi_{E_i \cap F_j} d\mu(x, \tau) \right),
\]
\[
= \ast_{i,j}(E_i \cap F_j, \frac{s}{a_i}) \ast \left( \ast_{i,j}(E_i \cap F_j, \frac{\tau}{b_j}) \right),
\]
\[
= \ast_{i,j} \left( \mu(E_i \cap F_j, \frac{s}{a_i}) \ast \mu(E_i \cap F_j, \frac{\tau}{b_j}) \right),
\]
\[
\leq \ast_{i,j} \left( \min \left\{ \mu(E_i \cap F_j, \frac{s}{a_i}), \mu(E_i \cap F_j, \frac{\tau}{b_j}) \right\} \right).
\]

Now, we assume \( \frac{s}{a_i} < \frac{\tau}{b_j} \). From (10), we conclude
\[
N(\phi, s) \ast N(\psi, \tau) \leq \ast_{i,j}(E_i \cap F_j, \frac{s}{a_i}).
\]

(11)
Again, from \( \frac{s}{a_i} < \frac{\tau}{b_j} \), we get \( \frac{s}{a_i} < \frac{\tau+s}{a_i+b_j} \) because 
\[ bjs < a_i \tau, \]
then 
\[ a_is + bjs < a_is + a_i \tau, \]
and 
\[ (a_i + b_j)s < a_i(\tau + s), \]
and so 
\[ \frac{s}{a_i} < \frac{\tau+s}{a_i+b_j}. \]

Therefore, from (11) we have 
\[ N(\phi, s) \ast N(\psi, \tau) \leq \ast_{i,j} \mu \left( E_i \cap F_j, \frac{s}{a_i+b_j} \right), \] (12)
and 
\[ \ast_{i,j} \mu \left( E_i \cap F_j, \frac{s}{a_i} \right) \leq \ast_{i,j} \mu \left( E_i \cap F_j, \frac{\tau+s}{a_i+b_j} \right). \] (13)

From (12) and (13) we have 
\[ N(\phi, s) \ast N(\psi, \tau) \leq \ast_{i,j} \mu \left( E_i \cap F_j, \frac{\tau+s}{a_i+b_j} \right), \]
\[ = N(\phi + \psi, s + \tau + s). \]

Now let \( f, g \in L^+ \), then there exist \( \{\phi_n\} \subseteq L^+ \) such that \( \phi_n \uparrow f \). Similarly, there exist \( \{\psi_n\} \subseteq L^+ \) such that \( \psi_n \uparrow g \), and \( \phi_n + \psi_n \uparrow f + g \), then 
\[ \inf \left\{ \int (\phi_n + \psi_n) d\mu(x, \tau + s) \right\} = \int (f + g) d\mu(x, \tau + s). \]

Also according to (12), we get 
\[ \int (\phi_n + \psi_n) d\mu(x, \tau + s) \geq \int \phi_n d\mu(x, s) \ast \int \psi_n d\mu(x, \tau), \]
and 
\[ \int (f + g) d\mu(x, \tau + s) = \inf \left\{ \int (\phi_n + \psi_n) d\mu(x, \tau + s) \right\} \]
\[ \geq \inf \left\{ \int \phi_n d\mu(x, s) \ast \int \psi_n d\mu(x, \tau) \right\}, \]
\[ \geq \inf \left\{ \int \phi_n d\mu(x, s) \right\} \ast \inf \int \psi_n d\mu(x, \tau) \]
\[ = \int f d\mu(x, s) \ast \int g d\mu(x, \tau), \]
then
\[ \int \left( f + g \right) d\mu(x, \tau + s) \geq \int f d\mu(x, s) \ast \int g d\mu(x, \tau). \]

(FN5) Let \( f = \sum_{i=1}^{k} a_i \chi_{E_i} \), then
\[ N(f, \tau_n) = \int \sum_{i=1}^{k} a_i \chi_{E_i} d\mu(x, \tau_n), \]
\[ = \ast_{i=1}^{k} \mu \left( E_i, \frac{\tau_n}{a_i} \right), \]
and
\[ \lim_{\tau_n \to \tau_0} N(f, \tau_n) = \lim_{\tau_n \to \tau_0} \ast_{i=1}^{k} \mu \left( E_i, \frac{\tau_n}{a_i} \right). \]

According to Definition 5 (iii), we get
\[ \lim_{\tau_n \to \tau_0} N(f, \tau_n) = \lim_{\tau_n \to \tau_0} \ast_{i=1}^{k} \mu \left( E_i, \frac{\tau_n}{a_i} \right), \]
and by Definition 5 (i),
\[ \lim_{\tau_n \to \tau_0} N(f, \tau_n) = \ast_{i=1}^{k} \lim_{\tau_n \to \tau_0} \mu \left( E_i, \frac{\tau_n}{a_i} \right), \]
\[ = \int f d\mu(x, \tau_0), \]
\[ = N(f, \tau_0). \]

Now, let \( f \in L^+ \), then
\[ N(f, \tau_n) = \int f d\mu(x, \tau_n), \]
\[ = \inf \left\{ \int \phi_m d\mu(x, \tau_n) \mid \phi_m \uparrow f \right\}, \]
\[ = \lim_{m \to \infty} \int \phi_m d\mu(x, \tau_n). \]

and
\[ \lim_{\tau_n \to \tau_0} N(f, \tau_n) = \lim_{\tau_n \to \tau_0} \lim_{m \to \infty} \int \phi_m d\mu(x, \tau_n), \]
\[ = \lim_{\tau_n \to \tau_0} \lim_{m \to \infty} \int \sum_{i=1}^{k} \phi_i \chi_{E_i}^\mu d\mu(x, \tau_n), \]
\[ = \lim_{\tau_n \to \tau_0} \lim_{m \to \infty} \ast_{i=1}^{k} \mu \left( E_i, \frac{\tau_n}{a_i} \right). \]
According to Definition 5 (v), we get
\[
\lim_{\tau_n \to \tau_0} N(f, \tau_n) = \lim_{\tau_n \to \tau_0} \lim_{m \to \infty} \sum_{i=1}^{k} \mu \left( E_i, \frac{\tau_n}{a_i^m} \right),
\]
\[
= \lim_{m \to \infty} \lim_{\tau_n \to \tau_0} \sum_{i=1}^{k} \mu \left( E_i, \frac{\tau_n}{a_i^m} \right),
\]
and by Definition 5 (iii), we get
\[
\lim_{\tau_n \to \tau_0} N(f, \tau_n) = \lim_{m \to \infty} \lim_{\tau_n \to \tau_0} \sum_{i=1}^{k} \mu \left( E_i, \frac{\tau_n}{a_i^m} \right),
\]
\[
= \lim_{m \to \infty} \sum_{i=1}^{k} \mu \left( E_i, \frac{\tau_0}{a_i^m} \right),
\]
\[
= \int f d\mu(x, \tau_0),
\]
\[
= N(f, \tau_0).
\]

(FN6) Let \( f = \sum_{i=1}^{k} a_i \chi_{E_i} \), then
\[
N(f, \tau) = \int f d\mu(x, \tau),
\]
\[
= \int \sum_{i=1}^{n} a_i \chi_{E_i} d\mu(x, \tau),
\]
\[
= \sum_{i=1}^{k} \mu \left( E_i, \frac{\tau}{a_i} \right).
\]

and
\[
\lim_{\tau \to \tau_0} N(f, \tau) = \lim_{\tau \to \tau_0} \sum_{i=1}^{k} \mu \left( E_i, \frac{\tau}{a_i} \right).
\]

According to Definition 5 (iii), we have
\[
\lim_{\tau \to \tau_0} N(f, \tau) = \lim_{\tau \to \tau_0} \sum_{i=1}^{k} \mu \left( E_i, \frac{\tau}{a_i} \right),
\]
\[
= \sum_{i=1}^{k} \mu \left( E_i, \frac{\tau}{a_i} \right),
\]
\[
= N(f, \tau_0).
\]
and by Definition 5 (iv),
\[
\lim_{\tau \to 0} \, N(f, \tau) = \lim_{\tau \to 0} \, \lim_{m \to \infty} \mu \left( E_i, \frac{\tau}{a_i^m} \right),
\]
\[
= \lim_{\tau \to 0} \mu \left( E_i, \frac{\tau}{a_i^m} \right),
\]
\[
= 0.
\]

Now let \( f \in L^+ \), so
\[
N(f, \tau) = \int f d\mu(x, \tau) = \inf \left\{ \int \phi_m d\mu(x, \tau) \right\},
\]
\[
= \lim_{m \to \infty} \left\{ \int \phi_m d\mu(x, \tau) \right\},
\]
\[
= \lim_{m \to \infty} \{ N(\phi_m, \tau) \}.
\]

Then,
\[
\lim_{\tau \to 0} \, N(f, \tau) = \lim_{m \to \infty} \lim_{\tau \to 0} \{ N(\phi_m, \tau) \},
\]
\[
= \lim_{m \to \infty} \lim_{\tau \to 0} \mu \left( E_i, \frac{\tau}{a_i^m} \right),
\]
\[
= 0.
\]

According to Definition 5 (v), we get
\[
\lim_{\tau \to 0} \, N(f, \tau) = \lim_{m \to \infty} \lim_{\tau \to 0} \mu \left( E_i, \frac{\tau}{a_i^m} \right),
\]
\[
= \lim_{m \to \infty} \lim_{\tau \to 0} \mu \left( E_i, \frac{\tau}{a_i^m} \right),
\]
and from Definition 5 (iii), we get
\[
\lim_{\tau \to 0} \, N(f, \tau) = \lim_{m \to \infty} \lim_{\tau \to 0} \mu \left( E_i, \frac{\tau}{a_i^m} \right),
\]
\[
= \lim_{m \to \infty} \lim_{\tau \to 0} \mu \left( E_i, \frac{\tau}{a_i^m} \right).
\]

From Definition 5 (iv), we get
\[
\lim_{\tau \to 0} \, N(f, \tau) = \lim_{m \to \infty} \lim_{\tau \to 0} \mu \left( E_i, \frac{\tau}{a_i^m} \right),
\]
\[
= \lim_{m \to \infty} \lim_{\tau \to 0} \mu \left( E_i, \frac{\tau}{a_i^m} \right),
\]
\[
= 0.
\]

Similarly,
\[
\lim_{\tau \to \infty} \, N(f, \tau) = 1.
\]

\( \Box \)

We have proved \((L^+, N, *)\) is a \(*\)-fuzzy normed space. Define \( M : L^+ \times L^+ \times (0, \infty) \to (0, 1] \) by
Theorem 7. Define $N(F)$.

Let $N$.

By theorem 4.5 of [3] we have, $N$.

Theorem 6.

Theorem 5.

$max\ f$.

Proof. $f$.

Furthermore, if $f$.

Definition 8.

4. $*$-Fuzzy ($L^+$)$^p$ Spaces

In this section, by the concept of fuzzy measurable functions and fuzzy integrable functions we define a class of function spaces.

Definition 8. Let $(X, C, *)$ be a $*$-fuzzy measure space. We define

$$(L^+)^p = \left\{ f : X \rightarrow \mathbb{R}^+ \text{ in which } f \text{ is fuzzy measurable function and } \int f^p d\mu(x, \tau) > 0, \ p \geq 1 \right\}.$$  

There is an order on $((L^+)^p, \leq)$ such that $f, g \in (L^+)^p$ we have $f \leq g$ if and only if $f(x) \leq g(x)$. Furthermore, if $f, g \in (L^+)^p$ then $|f - g| \in (L^+)^p$, and $|f - g|^p \leq f^p$ or $g^p$ hence $\int |f - g|^p d\mu(x, \tau) \geq \max[\int f^p d\mu(x, \tau), \int g^p d\mu(x, \tau)]$.

In the next theorem we prove $*$-fuzzy $(L^+)^p$ is a *-fuzzy normed space.

Theorem 7. Define $N_p : (L^+)^p \times (0, \infty) \rightarrow (0, 1]$ by $N_p(f, \tau) = \int f^p d\mu(x, \tau)$ then $((L^+)^p, N_p, *)$ is a $*$-fuzzy normed space.

Proof.

(FN1) $N_p(f, \tau) = \int f^p d\mu(x, \tau) > 0$.

(FN2) By theorem 4.5 of [3] we have, $N_p(f, \tau) = 1 \iff \int f^p d\mu(x, \tau) = 1 \iff f^p = 0 \iff f = 0$, almost everywhere.

(FN3) Let $f = \phi = \sum_{i=1}^{n} a_i \chi_{E_i}$, then

$$N_p(c\phi, \tau) = \int (c\phi)^p d\mu,$$

$$= \int \left( \sum_{i=1}^{n} c a_i \chi_{E_i} \right)^p d\mu,$$

$$= \sum_{i=1}^{n} H \left( E_i, \frac{\tau}{c^p a_i^p} \right).$$
On the other hand,
\[
N_p(\phi, \frac{\tau}{c^p}) = \int \phi^p d\mu(x, \frac{\tau}{c^p}),
\]
\[
= \int \left( \sum_{i=1}^{n} a_i \chi_{E_i} \right)^p d\mu(x, \frac{\tau}{c^p}),
\]
\[
= \int \sum_{i=1}^{n} a_i^p \chi_{E_i} d\mu(x, \frac{\tau}{c^p}),
\]
\[
= \ast \sum_{i=1}^{n} \mu \left( E_i, \frac{\tau}{c^p} \right).
\]

From (14) and (15) we conclude that
\[
N_p(cf, \tau) = N_p(f, \frac{\tau}{c}).
\]

Now let \( f \in (L^+)^p \), then we have
\[
N_p(cf, \tau) = \int (cf)^p d\mu(x, \tau) = \inf \left\{ \int (c\phi_n)^p d\mu(x, \tau) : (c\phi_n)^p \uparrow (cf)^p \right\}.
\]

On the other hand,
\[
N_p(f, \frac{\tau}{c}) = \int f^p d\mu(x, \frac{\tau}{c})
\]
\[
= \inf \left\{ \int \phi_n^p d\mu(x, \frac{\tau}{c}) : \phi_n^p \uparrow f^p \right\}.
\]

From (14) and (15) we get
\[
\int (c\phi_n)^p d\mu(x, \tau) = N_p(c\phi_n, \tau) = N_p(\phi_n, \frac{\tau}{c}) = \int \phi_n^p d\mu(x, \frac{\tau}{c}).
\]

Using (16) and (17) we get
\[
N_p(cf, \tau) = N_p(f, \frac{\tau}{c}).
\]

(FN4) Let \( f = \phi \) and \( g = \psi \) be simple functions. Then,
\[
N_p \left( \phi + \psi, s + \tau \right) = N_p \left( \sum_{i=1}^{n} a_i \chi_{E_i} + \sum_{j=1}^{m} b_j \chi_{F_j}, s + \tau \right),
\]
\[
= N_p \left( \sum_{i,j} (a_i + b_j) \chi_{E_i \cap F_j}, s + \tau \right),
\]
\[
= \int \left( \sum_{i,j} (a_i + b_j) \chi_{E_i \cap F_j} \right)^p d\mu(x, s + \tau),
\]
\[
= \int \sum_{i,j} (a_i + b_j)^p \chi_{E_i \cap F_j} d\mu(x, s + \tau),
\]
\[
= \ast \sum_{i,j} \mu \left( E_i \cap F_j, \frac{s + \tau}{(a_i + b_j)^p} \right).
\]
On the other hand,

\[ N_p(\phi, s) * N_p(\psi, \tau) = \left( \int \phi^p d\mu(x, s) \right) * \left( \int \psi^p d\mu(x, \tau) \right), \]

\[ = \left( \int \left( \sum_{i=1}^{n} a_i \chi_{E_i \cap F} \right)^p d\mu(x, s) \right) * \left( \int \left( \sum_{j=1}^{m} b_j \chi_{E_j \cap F} \right)^p d\mu(x, \tau) \right), \]

\[ = \left( \int \sum_{i=1}^{n} a_i^p \chi_{E_i \cap F} d\mu(x, s) \right) * \left( \int \sum_{j=1}^{m} b_j^p \chi_{E_j \cap F} d\mu(x, \tau) \right), \]

\[ = \left( \ast_{i,j} \mu \left( E_i \cap F_j, \frac{s}{a_i^p} \right) \right) * \left( \ast_{i,j} \mu \left( E_i \cap F_j, \frac{\tau}{b_j^p} \right) \right), \]

\[ = \ast_{i,j} \mu \left( E_i \cap F_j, \frac{s + \tau}{(a_i + b_j)^p} \right), \]

\[ \leq \ast_{i,j} \mu \left( E_i \cap F_j, \min \left\{ \frac{s}{a_i^p}, \frac{\tau}{b_j^p} \right\} \right). \]

(FN5) Let \( f = \sum_{i=1}^{k} a_i \chi_{E_i} \), then

\[ N_p(f, \tau_n) = \int \left( \sum_{i=1}^{k} a_i \chi_{E_i} \right)^p d\mu(x, \tau_n), \]

\[ = \ast_{i=1}^{k} \mu \left( E_i, \frac{\tau_n}{(a_i)^p} \right), \]

and so

\[ \lim_{\tau_n \to \tau_0} N_p(f, \tau_n) = \lim_{\tau_n \to \tau_0} \ast_{i=1}^{k} \mu \left( E_i, \frac{\tau_n}{(a_i)^p} \right). \]

Using Definition 5 (iii), we get

\[ \lim_{\tau_n \to \tau_0} N_p(f, \tau_n) = \lim_{\tau_n \to \tau_0} \ast_{i=1}^{k} \mu \left( E_i, \frac{\tau_n}{(a_i)^p} \right), \]

\[ = \ast_{i=1}^{k} \lim_{\tau_n \to \tau_0} \mu \left( E_i, \frac{\tau_n}{(a_i)^p} \right), \]

and according to Definition 5 (i),

\[ \lim_{\tau_n \to \tau_0} N_p(f, \tau_n) = \ast_{i=1}^{k} \lim_{\tau_n \to \tau_0} \mu \left( E_i, \frac{\tau_n}{(a_i)^p} \right), \]

\[ = \ast_{i=1}^{k} \mu \left( E_i, \frac{\tau_0}{(a_i)^p} \right), \]

\[ = \int f^p d\mu(x, \tau_0), \]

\[ = N_p(f, \tau_0). \]
Now let $f \in (L^*)^p$, we have

\[
N_p(f, \tau_0) = \int f^p d\mu(x, \tau_0)
= \inf \left\{ \int (\phi_m)^p d\mu(x, \tau_0) | \phi_m \uparrow f \right\}
= \lim_{m \to \infty} \int (\phi_m)^p d\mu(x, \tau_0).
\]

Then,

\[
\lim_{\tau_0 \to t_0} N_p(f, \tau_0) = \lim_{\tau_0 \to t_0} \lim_{m \to \infty} \int (\phi_m)^p d\mu(x, \tau_0),
= \lim_{\tau_0 \to t_0} \lim_{m \to \infty} \int \left( \sum_{i=1}^k \left( a_{i,m}^m X_{E_{i,m}} \right)^p d\mu(x, \tau_0) \right)
= \lim_{\tau_0 \to t_0} \lim_{m \to \infty} \sum_{i=1}^k \mu \left( E_{i,m}^m \frac{\tau_0}{(a_{i,m}^m)^p} \right).
\]

Using Definition 5 (v), we get

\[
\lim_{\tau_0 \to t_0} N_p(f, \tau_0) = \lim_{\tau_0 \to t_0} \lim_{m \to \infty} \sum_{i=1}^k \mu \left( E_{i,m}^m \frac{\tau_0}{(a_{i,m}^m)^p} \right),
= \lim_{m \to \infty} \lim_{\tau_0 \to t_0} \sum_{i=1}^k \mu \left( E_{i,m}^m \frac{\tau_0}{(a_{i,m}^m)^p} \right).
\]

and according to Definition 5 (iii)

\[
\lim_{\tau_0 \to t_0} N_p(f, \tau_0) = \lim_{m \to \infty} \lim_{\tau_0 \to t_0} \sum_{i=1}^k \mu \left( E_{i,m}^m \frac{\tau_0}{(a_{i,m}^m)^p} \right),
= \lim_{m \to \infty} \sum_{i=1}^k \lim_{\tau_0 \to t_0} \mu \left( E_{i,m}^m \frac{\tau_0}{(a_{i,m}^m)^p} \right).
\]

By Definition 5 (i), we have

\[
\lim_{\tau_0 \to t_0} N_p(f, \tau_0) = \lim_{m \to \infty} \sum_{i=1}^k \lim_{\tau_0 \to t_0} \mu \left( E_{i,m}^m \frac{\tau_0}{(a_{i,m}^m)^p} \right),
= \lim_{m \to \infty} \sum_{i=1}^k \mu \left( E_{i,m}^m \frac{\tau_0}{(a_{i,m}^m)^p} \right),
= \lim_{m \to \infty} \int (\phi_m)^p d\mu(x, \tau_0),
= \inf \left\{ \int (\phi_m)^p d\mu(x, \tau_0) \right\},
= \int f^p d\mu(x, \tau_0),
= N_p(f, \tau_0).
\]
Let \( f = \sum_{i=1}^{k} a_i \chi_{E_i} \), then

\[
N_p(f, \tau) = \int f^p \, d\mu(x, \tau),
\]

\[
= \int \left( \sum_{i=1}^{k} a_i \chi_{E_i} \right)^p \, d\mu(x, \tau),
\]

\[
= \sum_{i=1}^{k} \mu \left( E_i, \frac{\tau}{(a_i)^p} \right),
\]

and so

\[
\lim_{\tau \to 0} N_p(f, \tau) = \lim_{\tau \to 0} \sum_{i=1}^{k} \mu \left( E_i, \frac{\tau}{(a_i)^p} \right).
\]

Using Definition 5 (iii),

\[
\lim_{\tau \to 0} N_p(f, \tau) = \lim_{\tau \to 0} \sum_{i=1}^{k} \mu \left( E_i, \frac{\tau}{(a_i)^p} \right),
\]

\[
= \sum_{i=1}^{k} \mu \left( E_i, \frac{\tau}{(a_i)^p} \right)
\]

and by Definition 5 (iv), we have

\[
\lim_{\tau \to 0} N_p(f, \tau) = \sum_{i=1}^{k} \mu \left( E_i, \frac{\tau}{(a_i)^p} \right),
\]

\[
= \sum_{i=1}^{k} 0,
\]

\[
= 0.
\]

Now, let \( f \in (L^+)^p \), then

\[
N_p(f, \tau) = \int f^p \, d\mu(x, \tau) = \inf \left\{ \int (\phi_m)^p \, d\mu(x, \tau) : \phi_m \uparrow f \right\},
\]

\[
= \lim_{m \to \infty} \left\{ \int (\phi_m)^p \, d\mu(x, \tau) \right\},
\]

and so

\[
\lim_{\tau \to 0} N_p(f, \tau) = \lim_{\tau \to 0} \lim_{m \to \infty} \left\{ N_p(\phi_m, \tau) \right\},
\]

\[
= \lim_{\tau \to 0} \lim_{m \to \infty} \sum_{i=1}^{k} \mu \left( E_i^m, \frac{\tau}{(a_i^m)^p} \right).
\]

Using Definition 5 (v), we get

\[
\lim_{\tau \to 0} N_p(f, \tau) = \lim_{\tau \to 0} \lim_{m \to \infty} \sum_{i=1}^{k} \mu \left( E_i^m, \frac{\tau}{(a_i^m)^p} \right),
\]

\[
= \lim_{m \to \infty} \lim_{\tau \to 0} \sum_{i=1}^{k} \mu \left( E_i^m, \frac{\tau}{(a_i^m)^p} \right).
\]
Theorem 8. For 

and by Definition 5 (iii), we have

\[ \lim_{\tau \to 0} N_p(f, \tau) = \lim_{m \to \infty} \lim_{\tau \to 0} \sum_{i=1}^{k} \mu \left( E_i^n, \frac{\tau}{(a_i^n)^p} \right), \]

\[ = \lim_{m \to \infty} \sum_{i=1}^{k} \lim_{\tau \to 0} \mu \left( E_i^n, \frac{\tau}{(a_i^n)^p} \right). \]

from Definition 5 (iv), we get

\[ \lim_{\tau \to 0} N_p(f, \tau) = \lim_{\tau \to 0} \sum_{i=1}^{k} 0, \]

\[ = 0. \]

\[ \square \]

We proved \(((L^+)^p, N_p, \ast)\) is a *-fuzzy normed space. Now, define the fuzzy set \( M: (L^+)^p \times (L^+)^p \times (0, \infty) \to (0, 1] \) by

\[ M(f, g, \tau) = N_p \left( |f - g|, \tau \right) = \int |f - g|^p d\mu(x, \tau). \]

Then, \( M \) is a fuzzy metric on *-fuzzy \((L^+)^p\) and \(((L^+)^p, M, \ast)\) is called the *-fuzzy metric space induced by the *-fuzzy normed space \(((L^+)^p, N_p, \ast)\). Now, we study further properties of *-fuzzy \((L^+)^p\).

**Theorem 8.** For \( 1 \leq p < \infty \), the set of simple functions \( g = \sum_{i=1}^{n} a_i \chi_{E_i} \) where \( \mu(E_i, \tau) > 0 \) for all \( i \in \{1, 2, \ldots, n\} \) and for all \( \tau > 0 \), is dense in *-fuzzy \((L^+)^p\).

**Proof.** Clearly simple functions \( g = \sum_{i=1}^{n} a_i \chi_{E_i} \) are in *-fuzzy \((L^+)^p\). Let \( f \in (L^+)^p \), by theorem 3.20 in [3] we can choose a sequence \( \{f_n\} \) of simple functions such that \( f_n \uparrow f \) almost everywhere, and so \( (f - f_n)^p \downarrow 0 \).

We assert \((f - f_n)^p \in L^+\) because

\[ (f - f_n)^p \leq f^p, \]

and so

\[ \int (f - f_n)^p d\mu(x, \tau) \geq \int f^p d\mu(x, \tau) > 0, \]

then \((f - f_n)^p \in L^+\) and \((f - f_n)^p \to 0\). Using the fundamental convergence Theorem 3, we get

\[ \lim_{n \to \infty} \int (f - f_n)^p d\mu(x, \tau) = \int 0 d\mu(x, \tau) = 1. \]

Then, \( \lim_{n \to \infty} N_p(f - f_n, \tau) = 1 \) i.e., \( f_n \xrightarrow{N_p} f. \) \[ \square \]

In the next theorem we prove that *-fuzzy \((L^+)^p\) spaces are complete.

**Theorem 9.** For \( 1 \leq p < \infty \), *-fuzzy \((L^+)^p\) is a *-fuzzy Banach space.

**Proof.** Let \( \{f_n\} \subseteq (L^+)^p \) be a Cauchy sequence, then for every \( x \in X \), \( \{f_n(x)\} \subseteq \mathbb{R} \) is a Cauchy sequence in \( \mathbb{R} \) and since \( \mathbb{R} \) is complete, there exist \( y \in \mathbb{R} \) such that \( f_n(x) \to y \), we define \( f: X \to \mathbb{R} \) by \( f(x) = y \). Since \( f_n \to f \) almost everywhere, so \( (f_n)^p \to (f)^p \) almost everywhere, and \((f_n)^p \in L^+\).
by the fundamental converge Theorem 3 we have \((f)^p \in L^+\) and \(\lim \int (f_n)^p d\mu(x, \tau) = \int (f)^p d\mu(x, \tau)\), hence \(f \in (L^+)^p\). □

5. Inequalities on \(*\)-Fuzzy \((L^+)^p\)

In this section, we are ready to prove some important inequalities on \(*\)-fuzzy \((L^+)^p\).

Lemma 1 ([16]). If \(a \geq 0, b \geq 0, \) and \(0 < \lambda < 1\), then

\[a^\lambda b^{1-\lambda} \leq \lambda a + (1 - \lambda)b,\]

we have equality if and only if \(a = b\).

Theorem 10 (Hölder’s Inequality). Suppose \(1 < p < \infty\) and \(\frac{1}{p} + \frac{1}{q} = 1\). If \(f\) and \(g\) are fuzzy measurable functions on \(X\) then,

\[N(fg, \tau) \geq N_p\left(f, (p)^\frac{1}{q} \tau\right) * N_q\left(g, (q)^\frac{1}{q} \tau\right).\]

Proof. We apply Lemma 1 with \((f(x))^p = a, b = (g(x))^q,\) and \(\lambda = \frac{1}{p}\) to obtain

\[\left((f(x))^p\right)^\frac{1}{p} \cdot \left((g(x))^q\right)^{1-p} \leq \frac{1}{p}(f(x))^p + (1 - \frac{1}{p})(g(x))^q,\]

then

\[f(x)g(x) \leq \left((\frac{1}{p})^\frac{1}{p} f(x)\right)^p + \left((\frac{1}{q})^\frac{1}{q} g(x)\right)^q.\]

Takeing integral of both sides, we get

\[\int f(x)g(x)d\mu(x, \tau) \geq \int \left[\left((\frac{1}{p})^\frac{1}{p} f(x)\right)^p + \left((\frac{1}{q})^\frac{1}{q} g(x)\right)^q \right]d\mu(x, \tau),\]

\[\geq \left(\int \left((\frac{1}{p})^\frac{1}{p} f(x)\right)^p d\mu(x, \tau)\right) * \left(\int \left((\frac{1}{q})^\frac{1}{q} g(x)\right)^q d\mu(x, \tau)\right),\]

\[= N_p\left((\frac{1}{p})^\frac{1}{p} f, \tau\right) * N_q\left((\frac{1}{q})^\frac{1}{q} g, \tau\right),\]

\[= N_p\left(f, (p)^\frac{1}{q} \tau\right) * N_q\left(g, (q)^\frac{1}{q} \tau\right).\]

Then,

\[N_1(fg, \tau) \geq N_p\left(f, (p)^\frac{1}{q} \tau\right) * N_q\left(g, (q)^\frac{1}{q} \tau\right).\]

□

In the next theorem we compare two \(*\)-fuzzy \((L^+)^p\) spaces.

Theorem 11. If \(0 < p < q < r < \infty\), then \((L^+)^q \subseteq (L^+)^p + (L^+)^r\), that is, each \(f \in (L^+)^q\) is the sum of a function in \(*\)-fuzzy \((L^+)^p\) and a function in \(*\)-fuzzy \((L^+)^r\).
Proof. If $f \in (L^+)^q$, let $E = \{x : f(x) > 1\}$ and set $g = f^E$ and $h = f^{E'}$, then

$$f = f . 1,$$
$$= f^E + f^{E'},$$
$$= g + h.$$

However,

$$g^p = (f^E)^p = f^p \leq f^q,$$
then,

$$\int g^p d\mu \geq \int f^q d\mu > 0,$$
then,

$$g \in (L^+)^p.$$

On the other hand,

$$h^r = (f^{E'})^r = f^r \leq f^q,$$
then,

$$\int h^r d\mu \geq \int f^q d\mu > 0,$$
and so

$$h \in (L^+)^r.$$

Now, we apply Hölder’s inequality Theorem 10 to prove next theorem.

**Theorem 12.** If $0 < p < q < r < \infty$, then $L^p \cap L^r \subseteq L^q$ and

$$N_q(f, \tau) \geq N_p\left(f, \left(\frac{p}{\lambda q}\right)^\frac{1}{r}\right) \ast N_r\left(f, \left(\frac{r}{(1-\lambda)q}\right)^\frac{1}{r}\right),$$

where $\lambda \in (0, 1)$ is defined by $\lambda = \frac{1}{p} - \frac{1}{r}.$
Proof. From \( \int f^q d\mu(x, \tau) = \int f^{\lambda q} f^{(1-\lambda)q} d\mu(x, \tau) \) and Hölder’s inequality Theorem 10, we have

\[
\int f^q d\mu(x, \tau) = \int f^{\lambda q} f^{(1-\lambda)q} d\mu(x, \tau),
\]

\[
\geq \left( \int \left( \frac{\lambda q}{p} \right)^\frac{1}{q} f^{\lambda q} \right)^\frac{1}{\frac{1}{q}} d\mu(x, \tau) \cdot \left( \int \left( \frac{1-\lambda}{r} \right)^\frac{1}{r} f^{(1-\lambda)q} \right)^\frac{1}{\frac{1}{r}} d\mu(x, \tau),
\]

\[
\geq \left( \int \frac{\lambda q}{p} f^p d\mu(x, \tau) \right) \cdot \left( \int \frac{1-\lambda}{r} f^r d\mu(x, \tau) \right),
\]

\[
= \left( \int \left( \frac{\lambda q}{p} \right)^\frac{1}{p} f^p \right) d\mu(x, \tau) \cdot \left( \int \left( \frac{1-\lambda}{r} \right)^\frac{1}{r} f^r \right) d\mu(x, \tau),
\]

\[
= N_p \left( \left( \frac{\lambda q}{p} \right)^\frac{1}{p} f, \tau \right) \cdot N_r \left( \left( \frac{1-\lambda}{r} \right)^\frac{1}{r} f, \tau \right),
\]

then,

\[
N_q(f, \tau) \geq N_p \left( f, \left( \frac{\lambda q}{p} \right)^\frac{1}{p} \right) \cdot N_r \left( f, \left( \frac{1-\lambda}{r} \right)^\frac{1}{r} \right).
\]

Another application of Hölder’s inequality Theorem 10 helps us to prove next theorem.

**Theorem 13.** If \( \mu(X, \tau) > 0 \) and \( 0 < p < q < \infty \), then \( L^p(\mu) \supset L^q(\mu) \) and,

\[
N_p(f, \tau) \geq N_q \left( f, \left( \frac{q}{p} \right)^\frac{1}{p} \right) \cdot \mu \left( X, \left( \frac{q}{q-p} \right)^\frac{1}{q} \right).
\]

**Proof.** By Theorem 7 and Hölder’s inequality Theorem 10, we get

\[
N_p(f, \tau) = \int f^p \cdot 1 d\mu(x, \tau),
\]

\[
\geq N_q \left( f^p, \left( \frac{q}{p} \right)^\frac{1}{p} \right) \cdot N_q \left( 1, \left( \frac{q}{q-p} \right)^\frac{1}{q} \right),
\]

\[
= \left( \int f^p \right)^\frac{q}{p} d\mu \left( x, \left( \frac{q}{p} \right)^\frac{1}{p} \right) \cdot \left( \int 1 d\mu \left( x, \left( \frac{q}{q-p} \right)^\frac{1}{q} \right) \right),
\]

\[
= \int f^q d\mu \left( x, \left( \frac{q}{p} \right)^\frac{1}{p} \right) \cdot \mu \left( X, \left( \frac{q}{q-p} \right)^\frac{1}{q} \right),
\]

\[
= N_q \left( f, \left( \frac{q}{p} \right)^\frac{1}{p} \right) \cdot \mu \left( X, \left( \frac{q}{q-p} \right)^\frac{1}{q} \right).
\]

Finally, we prove the Chebyshev’s Inequality in \( * \)-fuzzy \( (L^+)^p \) spaces.

**Theorem 14 (Chebyshev’s Inequality).** If \( f \in (L^+)^p(0 < p < \infty) \) then for any \( a > 0 \), \( N_p(f, \tau) \leq N_p(\chi_{E_\alpha}, \frac{a}{p}) \) with respect to \( E_\alpha = \{ x : f(x) > a \} \).
Proof. We have,
\[ f^p > (f\chi_{E_a})^p = f^p\chi_{E_a}, \]
then
\[ \int f^p d\mu(x, \tau) \leq \int f^p d\mu(x, \tau)\chi_{E_a} = \int_{E_a} f^p d\mu(x, \tau), \tag{20} \]
and on \( E_a \) we have
\[ \int_{E_a} f^p d\mu(x, \tau) \leq \int_{E_a} a^p d\mu(x, \tau) = \int a^p\chi_{E_a} d\mu(x, \tau). \tag{21} \]
By (20) and (21) we get
\[ \int f^p d\mu(x, \tau) \leq \int a^p\chi_{E_a} d\mu(x, \tau), \]
\[ \quad = \int \left(a\chi_{E_a}\right)^p d\mu(x, \tau). \]
Then,
\[ N_p(f, \tau) \leq N_p(a\chi_{E_a}, \tau), \]
\[ \quad = N_p(\chi_{E_a}, \frac{\tau}{a}). \]

\[ \square \]

6. Conclusions

We have considered an uncertainty measure \( \mu \) based on the concept of fuzzy sets and continuous triangular norms named by \(*\)-fuzzy measure. In fact, we worked on a new model of the fuzzy measure theory (\(*\)-fuzzy measure) which is a dynamic generalization of the classical measure theory. \(*\)-fuzzy measure theory has gotten by replacing the non-negative real range and the additivity of classical measures with fuzzy sets and triangular norms. Moreover, the \(*\)-fuzzy measure theory has been motivated by defining new additivity property using triangular norms. Our approach can be apply for decision making problems [8,9].

We have restricted fuzzy measurable functions and fuzzy integrable functions and defined important classes of function spaces named by \(*\)-fuzzy \((L^+)^p\). Moreover, we have got a norm on \(*\)-fuzzy \((L^+)^p\) spaces and proved that \(*\)-fuzzy \((L^+)^p\) spaces are \(*\)-fuzzy Banach spaces. Finally, we have proved Chebyshev’s Inequality and Hölder’s Inequality.

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