On the \( \sigma \)-Length of Maximal Subgroups of Finite \( \sigma \)-Soluble Groups

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Abstract: Let \( \sigma = \{ \sigma_i : i \in I \} \) be a partition of the set \( \mathbb{P} \) of all prime numbers and let \( G \) be a finite group. We say that \( G \) is \( \sigma \)-primary if all the prime factors of \( |G| \) belong to the same member of \( \sigma \). \( G \) is said to be \( \sigma \)-soluble if every chief factor of \( G \) is \( \sigma \)-primary, and \( G \) is \( \sigma \)-nilpotent if it is a direct product of \( \sigma \)-primary groups. It is known that \( G \) has a largest normal \( \sigma \)-nilpotent subgroup which is denoted by \( F_\sigma(G) \). Let \( n \) be a non-negative integer. The \( n \)-term of the \( \sigma \)-Fitting series of \( G \) is defined inductively by \( F_0(G) = 1 \), and \( F_{n+1}(G)/F_n(G) = F_\sigma(G/F_n(G)) \). If \( G \) is \( \sigma \)-soluble, there exists a smallest \( n \) such that \( F_n(G) = G \). This number \( n \) is called the \( \sigma \)-nilpotent length of \( G \) and it is denoted by \( l_\sigma(G) \). If \( \mathcal{F} \) is a subgroup-closed saturated formation, we define the \( \sigma \)-\( \mathcal{F} \)-length \( n_{\mathcal{F}}(G, \mathcal{F}) \) of \( G \) as the \( \sigma \)-nilpotent length of the \( \mathcal{F} \)-residual \( G^{\mathcal{F}} \) of \( G \). The main result of the paper shows that if \( A \) is a maximal subgroup of \( G \) and \( G \) is \( \sigma \)-soluble, then \( n_{\mathcal{F}}(A, \mathcal{F}) = n_{\mathcal{F}}(G, \mathcal{F}) − i \) for some \( i \in \{0, 1, 2\} \).

Keywords: finite group; \( \sigma \)-solubility; \( \sigma \)-nilpotency; \( \sigma \)-nilpotent length

1. Introduction

All groups considered in this paper are finite.

Skiba [1] (see also [2]) generalised the concepts of solubility and nilpotency by introducing \( \sigma \)-solubility and \( \sigma \)-nilpotency, in which \( \sigma \) is a partition of \( \mathbb{P} \), the set of all primes. Hence \( \mathbb{P} = \bigcup_{i \in I} \mathbb{P}_{\sigma_i} \) with \( \mathbb{P}_{\sigma_i} \cap \mathbb{P}_{\sigma_j} = \emptyset \) for all \( i \neq j \).

In the sequel, \( \sigma \) will be a partition of the set of all primes \( \mathbb{P} \).

A group \( G \) is called \( \sigma \)-primary if all the prime factors of \( |G| \) belong to the same member of \( \sigma \).

Definition 1. A group \( G \) is said to be \( \sigma \)-soluble if every chief factor of \( G \) is \( \sigma \)-primary. \( G \) is said to be \( \sigma \)-nilpotent if it is a direct product of \( \sigma \)-primary groups.

We note in the special case that \( \sigma \) is the partition of \( \mathbb{P} \) containing exactly one prime each, the class of \( \sigma \)-soluble groups is just the class of all soluble groups and the class of \( \sigma \)-nilpotent groups is just the class of all nilpotent groups.

Many normal and arithmetical properties of soluble groups and nilpotent groups still hold for \( \sigma \)-soluble and \( \sigma \)-nilpotent groups (see [2]) and, in fact, the class \( \mathcal{N}_\sigma \) of all \( \sigma \)-nilpotent groups behaves in \( \sigma \)-soluble groups as nilpotent groups in soluble groups. In addition, every \( \sigma \)-soluble group has a conjugacy class of Hall \( \sigma_i \)-subgroups and a conjugacy class of Hall \( \sigma'_i \)-subgroups, for every \( \sigma_i \in \sigma \).

Recall that a class of groups \( \mathcal{F} \) is said to be a formation if \( \mathcal{F} \) is closed under taking epimorphic images and every group \( G \) has a smallest normal subgroup with quotient in \( \mathcal{F} \). This subgroup is called
the \( \mathfrak{F} \)-residual of \( G \) and it is denoted by \( G^\mathfrak{F} \). A formation \( \mathfrak{F} \) is called subgroup-closed if \( X^\mathfrak{F} \) is contained in \( G^\mathfrak{F} \) for all subgroups \( X \) of every group \( G \); \( \mathfrak{F} \) is saturated if it is closed under taking Frattini extensions.

A class of groups \( \mathfrak{F} \) is said to be a Fitting class if \( \mathfrak{F} \) is closed under taking normal subgroups and every group \( G \) has a largest normal subgroup in \( \mathfrak{F} \). This subgroup is called the \( \mathfrak{F} \)-radical of \( G \).

The following theorem which was proved in [1] (Corollary 2.4 and Lemma 2.5) turns out to be crucial in our study.

**Theorem 1.** \( \mathcal{N}_{\sigma} \) is a subgroup-closed saturated Fitting formation.

The \( \mathcal{N}_{\sigma} \)-radical of a group \( G \) is called the \( \sigma \)-Fitting subgroup of \( G \) and it is denoted by \( F_\sigma(G) \). Clearly, \( F_\sigma(G) \) is the product of all normal \( \sigma \)-nilpotent subgroups of \( G \). If \( \sigma \) is the partition of \( \mathbb{P} \) containing exactly one prime each, then \( F_\sigma(G) \) is just the Fitting subgroup of \( G \).

If \( G \) is \( \sigma \)-soluble, then every minimal normal subgroup \( N \) of \( G \) is \( \sigma \)-primary so that \( N \) is \( \sigma \)-nilpotent and it is contained in \( F_\sigma(G) \). In particular, \( F_\sigma(G) \neq 1 \) if \( G \neq 1 \).

Let \( n \) be a non-negative integer. The \( n \)-term of the \( \sigma \)-Fitting series of \( G \) is defined inductively by \( F_0(G) = 1 \), and \( F_{n+1}(G) = F_n(G) / F_n(G / F_n(G)) \). If \( G \) is \( \sigma \)-soluble, there exists a smallest \( n \) such that \( F_n(G) = G \). This number \( n \) is called the \( \sigma \)-nilpotent length of \( G \) and it is denoted by \( l_\sigma(G) \) (see [3,4]). The nilpotent length \( l(G) \) of a group \( G \) is just the \( \sigma \)-nilpotent length of \( G \) for \( \sigma \) the partition of \( \mathbb{P} \) containing exactly one prime each.

The \( \sigma \)-nilpotent length is quite useful in the structural study of \( \sigma \)-soluble groups (see [3,4]), and allows us to extend some known results.

The central concept of this paper is the following:

**Definition 2.** Let \( \mathfrak{F} \) be a saturated formation. The \( \sigma \)-\( \mathfrak{F} \)-length \( n_\sigma(G, \mathfrak{F}) \) of a group \( G \) is defined as the \( \sigma \)-nilpotent length of the \( \mathfrak{F} \)-residual \( G^\mathfrak{F} \) of \( G \).

Applying [5] (Chapter IV, Theorem (3.13) and Proposition (3.14)) (see also [3] (Lemma 4.1)), we have the following useful result.

**Proposition 1.** The class of all \( \sigma \)-soluble groups of \( \sigma \)-length at most \( l \) is a subgroup-closed saturated formation.

It is clear that the \( \mathfrak{F} \)-length \( n_\mathfrak{F}(G) \) of a group \( G \) studied in [6] is just the \( \sigma \)-\( \mathfrak{F} \)-length of \( G \) for \( \sigma \) the partition of \( \mathbb{P} \) containing exactly one prime each, and the \( \sigma \)-nilpotent length of \( G \) is just the \( \sigma \)-\( \mathfrak{F} \)-length of \( G \) for \( \mathfrak{F} = \{1\} \).

Ballester-Bolinches and Pérez-Ramos [6] (Theorem 1), extending a result by Doerk [7] (Satz 1), proved the following theorem:

**Theorem 2.** Let \( \mathfrak{F} \) be a subgroup-closed saturated formation and \( M \) be a maximal subgroup of a soluble group \( G \). Then \( n_\mathfrak{F}(M) = n_\mathfrak{F}(G) - i \) for some \( i \in \{0, 1, 2\} \).

Our main result shows that Ballester-Bolinches and Pérez-Ramos’ theorem still holds for the \( \sigma \)-\( \mathfrak{F} \)-length of maximal subgroups of \( \sigma \)-soluble groups.

**Theorem A.** Let \( \mathfrak{F} \) be a saturated formation. If \( A \) is a maximal subgroup of a \( \sigma \)-soluble group \( G \), then \( n_\sigma(A, \mathfrak{F}) = n_\sigma(G, \mathfrak{F}) - i \) for some \( i \in \{0, 1, 2\} \).

2. Proof of Theorem A

**Proof.** Suppose that the result is false. Let \( G \) be a counterexample of the smallest possible order. Then \( G \) has a maximal subgroup \( A \) such that \( n_\sigma(A, \mathfrak{F}) \neq n_\sigma(G, \mathfrak{F}) - i \) for every \( i \in \{0, 1, 2\} \).

Since \( A^\mathfrak{F} \) is contained in \( G^\mathfrak{F} \) because \( \mathfrak{F} \) is subgroup-closed, we have that \( G^\mathfrak{F} \neq 1 \). Moreover,
$n_\sigma(A, \bar{\mathfrak{g}}) \leq n_\sigma(G, \bar{\mathfrak{g}}) = n$ and $n \geq 1$. We proceed in several steps, the first of which depends heavily on the fact that the $\mathfrak{g}$-residual is epimorphism-invariant.

**Step 1.** If $N$ is a normal $\sigma$-nilpotent subgroup of $G$, then $N$ is contained in $A$, $n_\sigma(A, \bar{\mathfrak{g}}) = n_\sigma(A/N, \bar{\mathfrak{g}})$ and $n_\sigma(G/N, \bar{\mathfrak{g}}) = n - 1$.

Let $N$ be a normal $\sigma$-nilpotent subgroup of $G$. Applying [7] (Chapter II, Lemma (2.4)), we have that $G^\sigma N/N = (G/N)^\sigma$. Consequently, either $n_\sigma(G/N, \bar{\mathfrak{g}}) = n$ or $n_\sigma(G/N, \bar{\mathfrak{g}}) = n - 1$.

Assume that $N$ is not contained in $A$. Then $G = AN$ and so $G/N \cong A/A \cap N$. Observe that either $n_\sigma(A/A \cap N, \bar{\mathfrak{g}}) = n_\sigma(G/N, \bar{\mathfrak{g}}) = n$ or $n_\sigma(A/A \cap N, \bar{\mathfrak{g}}) = n_\sigma(G/N, \bar{\mathfrak{g}}) = n - 1$. Therefore $n - 1 \leq n_\sigma(A, \bar{\mathfrak{g}}) \leq n$. Consequently, either $n_\sigma(A, \bar{\mathfrak{g}}) = n$ or $n_\sigma(A, \bar{\mathfrak{g}}) = n - 1$, contrary to assumption.

Therefore, $N$ is contained in $A$. The minimal choice of $G$ implies that $n_\sigma(A/N, \bar{\mathfrak{g}}) = n_\sigma(G/N, \bar{\mathfrak{g}}) - i$ for some $i \in \{0, 1, 2\}$, and so either $n_\sigma(A/N, \bar{\mathfrak{g}}) = n - i$ or $n_\sigma(A/N, \bar{\mathfrak{g}}) = n - i - 1$. Suppose that $n_\sigma(A, \bar{\mathfrak{g}}) \neq n_\sigma(A/N, \bar{\mathfrak{g}})$. Then $n_\sigma(A, \bar{\mathfrak{g}}) = n_\sigma(A/N, \bar{\mathfrak{g}}) + 1$. Hence either $n_\sigma(A, \bar{\mathfrak{g}}) = n_i + 1$ or $n_\sigma(A, \bar{\mathfrak{g}}) = n_i$. In the first case, $i > 0$ because $n \geq n_\sigma(A, \bar{\mathfrak{g}})$. Hence $n_\sigma(A, \bar{\mathfrak{g}}) = n - j$ for some $j \in \{0, 1, 2\}$, which contradicts our supposition. Consequently, $n_\sigma(A, \bar{\mathfrak{g}}) = n_\sigma(A/N, \bar{\mathfrak{g}})$.

Suppose that $n_\sigma(G/N, \bar{\mathfrak{g}}) = n$. The minimality of $G$ yields $n_\sigma(A/N, \bar{\mathfrak{g}}) = n - i$ for some $i \in \{0, 1, 2\}$. Therefore $n_\sigma(A, \bar{\mathfrak{g}}) = n_\sigma(G, \bar{\mathfrak{g}}) - i$ for some $i \in \{0, 1, 2\}$. This is a contradiction since we are assuming that $G$ is a counterexample. Consequently, $n_\sigma(G/N, \bar{\mathfrak{g}}) = n - 1$.

**Step 2.** $\text{soc}(G)$ is a minimal normal subgroup of $G$ which is not contained in $\Phi(G)$, the Frattini subgroup of $G$.

Assume that $N$ and $L$ are two distinct minimal normal subgroups of $G$. Then, by Step 1, $n_\sigma(G/L, \bar{\mathfrak{g}}) = n - 1$. Since the class of all $\sigma$-soluble groups of $\sigma$-$\bar{\mathfrak{g}}$-length at most $n - 1$ is a saturated formation by Proposition 1 and $N \cap L = 1$, it follows that $n_\sigma(G, \bar{\mathfrak{g}}) = n - 1$. This contradiction proves that $N = \text{soc}(G)$ is the unique minimal normal subgroup of $G$.

Assume that $N$ is contained in $\Phi(G)$. Since $n_\sigma(G/N, \bar{\mathfrak{g}}) = n - 1$ and the class of all $\sigma$-soluble groups of $\sigma$-$\bar{\mathfrak{g}}$-length at most $n - 1$ is a saturated formation by Proposition 1, we have that $n_\sigma(G, \bar{\mathfrak{g}}) = n - 1$, a contradiction. Therefore $N$ is not contained in $\Phi(G)$ as desired.

According to Step 2, we have that $N = \text{soc}(G)$ is a minimal normal subgroup of $G$ which is not contained in $\Phi(G)$. Hence $G$ has a core-free maximal subgroup, $M$. Say. Then $G = NM$ and, by [5] (Chapter A, (15.2)), either $N$ is abelian and $C_G(N) = N$ or $N$ is non-abelian and $C_G(N) = 1$. Since $G$ is $\sigma$-soluble, it follows that $N$ is $\sigma$-primary. Thus, $N$ is a $\sigma_i$-group for some $\sigma_i \in \sigma$.

**Step 3.** Let $H$ be a subgroup of $G$ such that $N \subseteq H$. Then $F_\sigma(H) = \text{oc}_\sigma(H)$.

Since $N$ is contained in $F_\sigma(H)$, it follows that every Hall $\sigma_i$-subgroup of $F_\sigma(H)$ centralises $N$. Since $C_H(N) = N$ or $C_H(N) = 1$, we conclude that $F_\sigma(H)$ is a $\sigma_i$-group, i.e., $F_\sigma(H) = \text{oc}_\sigma(H)$.

**Step 4.** We have a contradiction.

Let $X = F_\sigma(G)$, and $T/X = F_\sigma(G/X)$. Suppose that $T$ is not contained in $A$. Then $G = AT$, $G/T \cong A/A \cap T$, and $n_\sigma(G/T, \bar{\mathfrak{g}}) = n_\sigma(A/\infty T, \bar{\mathfrak{g}})$. By Step 1, $n_\sigma(G/X, \bar{\mathfrak{g}}) = n - 1$. Hence $n_\sigma(G/T, \bar{\mathfrak{g}}) \in \{n - 2, n - 1\}$. Now, $X \subseteq A$ and $n_\sigma(A, \bar{\mathfrak{g}}) = n_\sigma(A/X, \bar{\mathfrak{g}})$ by Step 1. Consequently, $n_\sigma(A/\infty T, \bar{\mathfrak{g}}) \in \{n - 2, n - 1, n_\sigma(A, \bar{\mathfrak{g}}) - 1\}$. This means that $n_\sigma(A, \bar{\mathfrak{g}}) = n - j$ for some $j \in \{0, 1, 2\}$. This contradiction yields $T \subseteq A$.

By Step 3, we have that $X = \text{oc}_\sigma(G)$. Assume that $E/X$ and $F/X$ are the Hall $\sigma_i$-subgroup and the Hall $\sigma_i$-subgroup of $T/X$ respectively. Then $T/X = E/X \times F/X$ and $E$ and $F$ are normal subgroups of $G$. Since $X$ and $E/X$ are $\sigma_i$-groups, it follows that $E$ is a $\sigma_i$-group and hence $E \subseteq X$. In particular, $T/X$ is a $\sigma_i$-group.

On the other hand, $F_\sigma(A) = \text{oc}_\sigma(A)$ by Step 3. Consequently $F_\sigma(A)/X \subseteq \text{oc}_\sigma(T/X)$. Applying [1] (Corollary 11), we conclude that $C_A(T/X) \subseteq T/X$. Therefore $X = F_\sigma(A)$.

By Step 1, $n_\sigma(A, \bar{\mathfrak{g}}) = n_\sigma(A/X, \bar{\mathfrak{g}})$. Now $n_\sigma(A/X, \bar{\mathfrak{g}}) = l_\sigma(A^\bar{\mathfrak{g}} X/X)$. Since $A^\bar{\mathfrak{g}} A^\bar{\mathfrak{g}} \cap X = A^\bar{\mathfrak{g}} / F_\sigma(A^\bar{\mathfrak{g}})$, it follows that $n_\sigma(A/X, \bar{\mathfrak{g}}) = n_\sigma(A, \bar{\mathfrak{g}}) - 1$ which yields the desired contradiction. 

**3. Applications**

As it was said in the introduction, the $\bar{\mathfrak{g}}$-length $n_\bar{\mathfrak{g}}(G)$ of a group $G$ which is defined in [6] is just the $\sigma$-$\bar{\mathfrak{g}}$-length of $G$ for $\sigma$ the partition of $\mathbb{P}$ containing exactly one prime each, and the $\sigma$-nilpotent length of $G$ is just the $\sigma$-$\bar{\mathfrak{g}}$-length of $G$ for $\bar{\mathfrak{g}} = \{1\}$. 


Therefore the following results are direct consequences of our Theorem A.

**Corollary 1.** If $A$ is a maximal subgroup of a $\sigma$-soluble group $G$, then $l_\sigma(A) = l_\sigma(G) - i$ for some $i \in \{0, 1, 2\}$.

**Corollary 2** ([6] (Theorem 1)). If $A$ is a maximal subgroup of a soluble group $G$ and $\mathcal{F}$ is a saturated formation, then $n_\mathcal{F}(A) = n_\mathcal{F}(G) - i$ for some $i \in \{0, 1, 2\}$.

**Corollary 3** ([7] (Satz 1)). If $A$ is a maximal subgroup of a soluble group $G$, then $l(A) = l(G) - i$ for some $i \in \{0, 2\}$.

### 4. An Example

In [6], some examples showing that each case of Corollary 2 is possible for the partition $\sigma$ of $\mathbb{P}$ containing exactly one prime each. We give an example of slight different nature.

**Example 1.** Assume that $\sigma = \{\{2, 3, 5, 7\}, \{211\}, \{2, 3, 5, 7, 211\}\}$. Let $X$ be a cyclic group of order 7 and let $Y$ be an irreducible and faithful $X$-module over the finite field of 211 elements. Applying [5] (Chapter B, Theorem (9.8)), $Y$ is a cyclic group of order 211. Let $L = [Y]X$ be the corresponding semidirect product. Consider now $G = A_5 \wr L$ the regular wreath product of $A_5$, the alternating group of degree 5, with $L$. Then $F_\sigma(G) = A_5 \wr L$, the base group of $G$. Then $l_\sigma(G) = 3$. Let $A_1 = A_5 \wr X$. Then $A_1$ is a maximal subgroup of $G$ and $l_\sigma(A_1) = 1$. Let $A_2 = A_5 \wr Y$. Then $A_2$ is a maximal subgroup of $G$ and $l_\sigma(A_2) = 2$.


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