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Multiple Comparisons for Exponential Median Lifetimes with the Control Based on Doubly Censored Samples

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Abstract: Under doubly censoring, the one-stage multiple comparison procedures with the control in terms of exponential median lifetimes are presented. The uniformly minimum variance unbiased estimator for median lifetime is found. The upper bounds, lower bounds and two-sided confidence intervals for the difference between each median lifetimes and the median lifetime of the control population are developed. Statistical tables of critical values are constructed for the practical use of our proposed procedures. Users can use these simultaneous confidence intervals to determine whether the performance of treatment populations is better than or worse than the control population in agriculture and pharmaceutical industries. At last, one practical example is provided to illustrate the proposed procedures.

Keywords: one-stage procedure; doubly censored samples; multiple comparison with a control



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1. Introduction

In reliability studies, the lifetimes of some products may not have normal distribution. The exponential distribution we focus on in this study is one type of frequently used lifetime distribution and some examples can be seen in Johnson et al. [1]. For this type of lifetime distribution, Ng et al. [2] proposed multiple comparisons with a control for location parameters when the scale parameters are equal. For the case of unequal scale parameters (heteroscedasticity), Lam and Ng [3] proposed design-oriented two-stage multiple comparison procedures for location parameters with the control. However, the additional sample size for the second stage may be large when scale parameters are large. Wu [4] proposed one-stage multiple comparison procedures with the control for location parameter based on the doubly censored sample. Wu [5] presented a modified procedure to improve the coverage probabilities and confidence length in Wu [4]. Instead of making multiple comparison with the control, Wu and Wu [6] investigated the multiple comparison procedures with the average for exponential location parameters using the two-stage sampling procedures. Wu et al. [7] considered one-stage procedures comparing with the average instead of two-stage procedures. Based on the doubly censored sample, Wu [8] proposed multiple comparisons with the average for exponential location parameters under heteroscedasticity. For multiply type II censored sample, Wu [9] proposed a prediction interval for the future observation using the estimator of general weighted moment estimator (GWME). If the experimenters are interested in comparing the mean lifetimes instead of location parameters, Wu [10] proposed the one-stage multiple comparisons for exponential mean lifetimes with the control. Wu [11] proposed the one-stage multiple comparison procedures for exponential mean lifetimes with the control based on the doubly type II censored sample. Since the shape of the distribution of exponential lifetime is a right-skewed, median lifetimes should be used to provide more robust measurements for the central tendency of exponential distribution. Therefore, Wu [12] proposed procedures to compare the median lifetimes of exponential lifetime distributions with the control. In some experiments, the experimenters are not able to collect a complete sample due to the financial budget limit or experimental difficulties. Thus the censoring occurred. In this

research, we are focusing on type II censoring since it is a frequently used type of censoring. This type of censoring is briefly introduced as follows: Suppose that there are n items that are put in a life test. If the first r lifetimes and the last s lifetimes are missing, the middle observed lifetimes are called the doubly Type II censored sample, where r is the size of sample for left censoring and s is the size of sample for right censoring. In this research, $k (\geq 2)$ treatment populations denoted by π_1, \dots, π_k are considered and the lifetime for the i th population has exponential distribution denoted by $E(\theta_i, \sigma_i), i = 1, \dots, k$, where the location parameters $\theta_1, \dots, \theta_k$ are unknown and scale parameters $\sigma_1, \dots, \sigma_k$ are unknown and unequal. The cdf (cumulative distribution function) of the i th exponentially distributed population is $F(x) = 1 - \exp\left(-\frac{x-\theta_i}{\sigma_i}\right)$. Letting $F(x) = 0.5$. Then the median δ_i is found to be $\delta_i = \theta_i + \ln 2 \sigma_i, i = 1, \dots, k$. Let π_k be the control population. The goal of this research is to find the uniformly minimum variance unbiased estimator (UMVUE) for the median lifetime and the results are presented in Section 2. The other goal is to provide the multiple comparisons of $k-1$ treatment median lifetimes with the control population and the research methods are presented in Section 3. In Section 4, an example of doing the multiple comparisons with the control by comparing the median duration time of remission under the treatment of four drugs is given to demonstrate the main results of this paper. In the end, we summarized the main conclusions in Section 5.

2. The Uniformly Minimum Variance Unbiased Estimator (UMVUE) for the Median Lifetime

Firstly, we give a brief definition of the doubly censored samples as follows: Let X_{i1}, \dots, X_{in} be the random sample of size $n (\geq 2)$ from π_i and $X_{i(1)} < \dots < X_{i(n)}$ are the ordered sample. Suppose that the first r lifetimes $X_{i(1)}, \dots, X_{i(r)}$ and the last s lifetimes $X_{i(n-s+1)}, \dots, X_{i(n)}$ are censored. Then the middle $n-s-r$ observations $X_{i(r+1)}, \dots, X_{i(n-s)}$ are so-called the doubly Type II censored sample collected for population $\pi_i, i = 1, \dots, k$. From Wu [8], after taking the standardizing transformation to each $X_{i(j)}$ as $X_{i(j)}^* = \frac{X_{i(j)} - \theta_i}{\sigma_i}, j = r + 1, \dots, n - s$, we can obtain the doubly censored sample $X_{i(r+1)}^*, \dots, X_{i(n-s)}^*$ from the standard exponential distribution. The generalized spacings $Z_{i(r+2)} = (n - r - 1)(X_{i(r+2)}^* - X_{i(r+1)}^*), \dots, Z_{i(n-s)} = (s + 1)(X_{i(n-s)}^* - X_{i(n-s-1)}^*)$ are independent and identically distributed from a standard exponential distribution and they are independent with $X_{i(r+1)}^*$.

Let $\tilde{S}_i = \sum_{j=r+2}^{n-s} Z_{i(j)} / \nu = \sum_{j=r+2}^{n-s} (X_{i(j)} - X_{i(r+1)}) / \nu$. It is well known that $U_i = 2\nu\tilde{S}_i / \sigma_i$ has a chi-squared distribution with 2ν degrees of freedom, where $\nu = n - r - s - 1$. Therefore \tilde{S}_i is the unbiased estimator of σ_i . From Kambo [13], it has shown that the complete sufficient statistics for (θ_i, σ_i) is $(X_{i(r+1)}, \tilde{S}_i)$. Thus, the UMVUE for θ_i is $X_{i(r+1)} + \nu\tilde{S}_i \ln((n - r)/n) / (n - r - s)$ and the UMVUE for σ_i is \tilde{S}_i . Furthermore, the UMVUE for the i th median lifetime $\delta_i = \theta_i + \ln 2 \sigma_i$ is $\hat{\delta}_i = X_{i(r+1)} + \nu\tilde{S}_i \ln((n - r)/n) / (n - r - s) + \ln 2\tilde{S}_i = X_{i(r+1)} + ((\nu + 1) \ln 2 + \nu \ln((n - r)/n))\tilde{S}_i / (\nu + 1) = X_{i(r+1)} + \nu^*\tilde{S}_i$, where $\nu^* = ((\nu + 1) \ln 2 + \nu \ln((n - r)/n)) / (\nu + 1)$. If we are interested in the difference between the i th treatment median lifetime with the control population (the k th population), i.e., $\delta_i - \delta_k$, the UMVUE for this parameter is $X_{i(r+1)} - X_{k(r+1)} + \nu^*(\tilde{S}_i - \tilde{S}_k), i = 1, \dots, k - 1$.

3. Multiple Comparisons with the Control for Exponential Median Lifetimes Based on Doubly Censored Samples Using One-Stage Procedures

Based on this UMVUE for the i th median lifetime $\delta_i = \theta_i + \ln 2 \sigma_i$ in Section 2, we could construct a pivotal quantity

$$\begin{aligned} \tilde{G}_i &= \frac{\delta_i - \hat{\delta}_i}{2\nu\tilde{S}_i} = \frac{\delta_i - X_{i(r+1)} - \nu^*\tilde{S}_i}{2\nu\tilde{S}_i} = \frac{(\theta_i + \ln 2 \sigma_i - X_{i(r+1)} - \nu^*\tilde{S}_i) / \sigma_i}{2\nu\tilde{S}_i / \sigma_i} = \frac{-T_i + \ln 2 - \nu^*\tilde{S}_i / \sigma_i}{2\nu\tilde{S}_i / \sigma_i} \\ &= \frac{-T_i + \ln 2}{U_i} - \frac{\nu^*}{2\nu}, \text{ where } T_i = \frac{X_{i(r+1)} - \theta_i}{\sigma_i}. \end{aligned}$$

It is very hard to find the exact distribution of \tilde{G}_i . The distribution of \tilde{G}_i is only depending on random variables of T_i and U_i , where $\exp(-T_i)$ has a beta distribution with parameters $n-r$ and $r+1$ and U_i has a chi-squared distribution with 2ν degrees of freedom, where $\nu = n - r - s - 1$ and they are independent. Therefore, we can generate independent random variables T_i and U_i to generate random variable \tilde{G}_i . By the Monte-Carlo simulation method, we can find the empirical distribution of \tilde{G}_i .

Making use of the pivotal quantities \tilde{G}_i 's, we can start to develop the multiple comparison procedures for each median lifetime with the control (the k th population is regarded as the control population), i.e., $\delta_i - \delta_k, i = 1, \dots, k - 1$ based on the doubly censored samples in the following theorem:

Theorem 1. Let $\tilde{c} = \max_{i=1, \dots, k} 2\nu\tilde{S}_i$ and $\nu^* = ((\nu + 1) \ln 2 + \nu \ln((n - r)/n))/(\nu + 1)$. For a given confidence coefficient $0 < P^* < 1$, we have the upper confidence bounds, lower confidence bounds and simultaneous confidence intervals for $\delta_i - \delta_k, i = 1, \dots, k - 1$ as follows:

(a) $X_{i(r+1)} - X_{k(r+1)} + \nu^*(\tilde{S}_i - \tilde{S}_k) + \tilde{c}\ddot{h}_U$ are the upper confidence bounds for $\delta_i - \delta_k, i = 1, \dots, k - 1$ with confidence level of P^* , where \ddot{h}_U is the $100P^{th}$ percentile of the distribution of $G_1 = \max(\tilde{G}_i, -\tilde{G}_k, \tilde{G}_i - \tilde{G}_k, i = 1, \dots, k - 1)$.

(b) $X_{i(r+1)} - X_{k(r+1)} + \nu^*(\tilde{S}_i - \tilde{S}_k) - \tilde{c}\ddot{h}_L$ are the lower confidence bounds for $\delta_i - \delta_k, i = 1, \dots, k - 1$ with confidence level of P^* where \ddot{h}_L is the $100P^{th}$ percentile of the distribution of $G_2 = \max(-\tilde{G}_i, \tilde{G}_k, -\tilde{G}_i + \tilde{G}_k, i = 1, \dots, k - 1)$.

(c) $(Y_i - Y_k + \nu^*(\tilde{S}_i - \tilde{S}_k) \pm \tilde{c}\ddot{h}_t)$ are simultaneous two-sided confidence intervals for $\delta_i - \delta_k, i = 1, \dots, k - 1$ with confidence coefficient P^* where \ddot{h}_t is the $100P^{th}$ percentile of the distribution of $G_3 = \max(|\tilde{G}_i|, |\tilde{G}_k|, |\tilde{G}_k - \tilde{G}_i|, i = 1, \dots, k - 1)$.

Proof of Theorem 1.

For (a), we have

$$\begin{aligned} &P(\delta_i - \delta_k \leq X_{i(r+1)} - X_{k(r+1)} + \nu^*(\tilde{S}_i - \tilde{S}_k) + \tilde{c}\ddot{h}_U, i = 1, \dots, k - 1) \\ &= P(\theta_i + \ln 2\sigma_i - \theta_k - \ln 2\sigma_k \leq X_{i(r+1)} - X_{k(r+1)} + \nu^*(\tilde{S}_i - \tilde{S}_k) + \tilde{c}\ddot{h}_U, i = 1, \dots, k - 1) \\ &= P\left(2\nu\tilde{S}_i \frac{\sigma_i}{\tilde{S}_i} \frac{-\nu^*\tilde{S}_i + \ln 2\sigma_i + \theta_i - Y_i}{2\nu\sigma_i} \right. \\ &\leq 2\nu\tilde{S}_k \frac{\sigma_k}{\tilde{S}_k} \frac{-\nu^*\tilde{S}_k + \ln 2\sigma_k + \theta_k - Y_k}{2\nu\sigma_k} + \tilde{c}\ddot{h}_U, i = 1, \dots, k - 1) \\ &= P\left(2\nu\tilde{S}_k\tilde{G}_k \geq 2\nu\tilde{S}_i\tilde{G}_i - \tilde{c}\ddot{h}_U, i = 1, \dots, k - 1\right) \\ &\geq E_{S_1, \dots, S_k} P\left(2\nu\tilde{S}_k\tilde{G}_k \geq 2\nu\tilde{S}_i\tilde{G}_i - \max(2\nu\tilde{S}_i, 2\nu\tilde{S}_k) \ddot{h}_U, i = 1, \dots, k - 1\right) \\ &\geq P\left(\tilde{G}_i \leq \ddot{h}_U, \tilde{G}_k \geq -\ddot{h}_U, \tilde{G}_k \geq \tilde{G}_i - \ddot{h}_U, i = 1, \dots, k - 1\right) \end{aligned}$$

(the above inequality holds by using the Lemma in Lam [14,15] by setting $a = 2\nu\tilde{S}_k$ and $b = 2\nu\tilde{S}_i$.)

$$\begin{aligned} &= P\left(\tilde{G}_i \leq \ddot{h}_U, -\tilde{G}_k \leq \ddot{h}_U, \tilde{G}_i - \tilde{G}_k \leq \ddot{h}_U, i = 1, \dots, k - 1\right) \\ &= P\left(\max(\tilde{G}_i, -\tilde{G}_k, \tilde{G}_i - \tilde{G}_k, i = 1, \dots, k - 1) \leq \ddot{h}_U\right) = P(G_1 \leq h_U) = P^*, \end{aligned}$$

where $G_1 = \max(\tilde{G}_i, -\tilde{G}_k, \tilde{G}_i - \tilde{G}_k, i = 1, \dots, k - 1)$.

Solving the above equation, we obtain \ddot{h}_U as the $100P^{th}$ percentile of the distribution of G_1 and the proof is thus completed.

For (b), we have

$$\begin{aligned}
 &P(\delta_i - \delta_k \geq X_{i(r+1)} - X_{k(r+1)} + v^*(\tilde{S}_i - \tilde{S}_k) - \tilde{c}\ddot{h}_L, i = 1, \dots, k-1) \\
 &= P(\theta_i + \ln 2\sigma_i - \theta_k - \ln 2\sigma_k \geq X_{i(r+1)} - X_{k(r+1)} + v^*(\tilde{S}_i - \tilde{S}_k) - \tilde{c}\ddot{h}_L, i = 1, \dots, k-1) \\
 &= P\left(2\nu\tilde{S}_i \frac{\sigma_i}{\tilde{S}_i} \frac{-v^*\tilde{S}_i + \ln 2\sigma_i + \theta_i - Y_i}{2\nu\sigma_i} \right. \\
 &\quad \left. \geq 2\nu\tilde{S}_k \frac{\sigma_k}{\tilde{S}_k} \frac{-v^*\tilde{S}_k + \ln 2\sigma_k + \theta_k - Y_k}{2\nu\sigma_k} - \tilde{c}\ddot{h}_L, i = 1, \dots, k-1\right) \\
 &= P\left(2\nu\tilde{S}_i\tilde{G}_i \geq 2\nu\tilde{S}_k\tilde{G}_k - \max\left(\frac{\tilde{S}_i}{\tilde{m}}, \frac{\tilde{S}_k}{\tilde{m}}\right)\ddot{h}_L, i = 1, \dots, k-1\right) \\
 &\geq E_{S_1, \dots, S_k} P\left(2\nu\tilde{S}_i\tilde{G}_i \geq 2\nu\tilde{S}_k\tilde{G}_k - \max\left(\frac{\tilde{S}_i}{\tilde{m}}, \frac{\tilde{S}_k}{\tilde{m}}\right)\ddot{h}_L, i = 1, \dots, k-1\right) \\
 &\geq P\left(\tilde{G}_i \geq -\ddot{h}_L, \tilde{G}_k \leq \ddot{h}_L, \tilde{G}_i \geq \tilde{G}_k - \ddot{h}_L, i = 1, \dots, k-1\right)
 \end{aligned}$$

(the above inequality holds by using the Lemma in Lam [14,15] by setting $a = 2\nu\tilde{S}_i$ and $b = 2\nu\tilde{S}_k$.)

$$\begin{aligned}
 &\geq P\left(-\tilde{G}_i \leq \ddot{h}_L, \tilde{G}_k \leq \ddot{h}_L, \tilde{G}_k - \tilde{G}_i \leq \ddot{h}_L, i = 1, \dots, k-1\right) \\
 &= P(G_2 \leq \ddot{h}_L) = P^*, \text{ where } G_2 = \max(-\tilde{G}_i, \tilde{G}_k, -\tilde{G}_i + \tilde{G}_k, i = 1, \dots, k-1).
 \end{aligned}$$

Solving the above equation, we obtain \ddot{h}_L as the $100P^{th}$ percentile of the distribution of G_2 and the proof is thus completed.

For (c), combining (a) and (b), we have

$$\begin{aligned}
 &P\left(X_{i(r+1)} - X_{k(r+1)} + v^*(\tilde{S}_i - \tilde{S}_k) - \tilde{c}\ddot{h}_t \leq \delta_i - \delta_k \leq \right. \\
 &\quad \left. X_{i(r+1)} - X_{k(r+1)} + v^*(\tilde{S}_i - \tilde{S}_k) + \tilde{c}\ddot{h}_t, i = 1, \dots, k-1\right) \\
 &= E_{S_1, \dots, S_k} P\left(-\tilde{G}_i \leq \ddot{h}_t, \tilde{G}_k \leq \ddot{h}_t, \tilde{G}_k - \tilde{G}_i \leq \ddot{h}_t \cap \right. \\
 &\quad \left. -\tilde{G}_k \leq \ddot{h}_t, \tilde{G}_i \leq \ddot{h}_t, \tilde{G}_i - \tilde{G}_k \leq \ddot{h}_t, i = 1, \dots, k-1\right) \\
 &\geq P\left(\max(|\tilde{G}_i|, |\tilde{G}_k|, |\tilde{G}_k - \tilde{G}_i|, i = 1, \dots, k-1) \leq \ddot{h}_t, i = 1, \dots, k-1\right) = P(G_3 \leq \ddot{h}_t) = P^*,
 \end{aligned}$$

where $G_3 = \max(|\tilde{G}_i|, |\tilde{G}_k|, |\tilde{G}_k - \tilde{G}_i|, i = 1, \dots, k-1)$.

Solving the above equation, we obtain \ddot{h}_t as the $100P^{th}$ percentile of the distribution of G_3 and the proof is thus obtained. \square

From Theorem 1, the critical values of \ddot{h}_U, \ddot{h}_L and \ddot{h}_t are the $100P^{th}$ percentiles of the distributions of G_1, G_2 and G_3 . Using Monte-Carlo simulation methods, the critical values are the percentiles of the empirical distribution of G_1, G_2 and G_3 . They are listed in the following table for $k = 3(1)10, n = 20, 30, 60, r = 1, 2, 3, s = 0, 1, 2$, and $P^* = 0.90, 0.95$ and 0.975 . The critical values for any given k, n, r, s and P^* are available at the author’s site. Refer to part (c) of Theorem 1, the confidence length is $L_1 = 2\tilde{c}\ddot{h}_t$. From Table 1, since \ddot{h}_t is an increasing function of k , the length of simultaneous confidence interval (SCI) is getting larger when we compare more treatments for fixed n, r, s and P^* .

Table 1. Critical values \ddot{h}_U, \ddot{h}_L and \ddot{h}_t .

k	n	r	s	P* = 0.90			P* = 0.95			P* = 0.975			
				\ddot{h}_U	\ddot{h}_L	\ddot{h}_t	\ddot{h}_U	\ddot{h}_L	\ddot{h}_t	\ddot{h}_U	\ddot{h}_L	\ddot{h}_t	
3	20	1	0	0.0113	0.0104	0.0135	0.0140	0.0130	0.0160	0.0165	0.0155	0.0185	
				0.0121	0.0113	0.0145	0.0150	0.0140	0.0172	0.0177	0.0166	0.0199	
				0.0131	0.0124	0.0157	0.0161	0.0153	0.0186	0.0190	0.0180	0.0214	
		2	1	0.0124	0.0113	0.0147	0.0153	0.0141	0.0175	0.0181	0.0168	0.0203	
				0.0133	0.0123	0.0158	0.0164	0.0152	0.0188	0.0194	0.0182	0.0218	
				0.0144	0.0135	0.0172	0.0177	0.0167	0.0204	0.0210	0.0198	0.0237	
		3	2	2	0.0136	0.0123	0.0161	0.0168	0.0154	0.0193	0.0200	0.0185	0.0224
					0.0146	0.0134	0.0174	0.0181	0.0167	0.0208	0.0215	0.0200	0.0241
					0.0159	0.0148	0.0190	0.0196	0.0184	0.0227	0.0233	0.0219	0.0263

Table 1. Cont.

<i>k</i>	<i>n</i>	<i>r</i>	<i>s</i>	<i>P</i> [*] = 0.90			<i>P</i> [*] = 0.95			<i>P</i> [*] = 0.975		
				\ddot{h}_U	\ddot{h}_L	\ddot{h}_t	\ddot{h}_U	\ddot{h}_L	\ddot{h}_t	\ddot{h}_U	\ddot{h}_L	\ddot{h}_t
3	30	0	1	0.0057	0.0053	0.0067	0.0070	0.0065	0.0079	0.0082	0.0077	0.0091
			2	0.0060	0.0056	0.0070	0.0073	0.0068	0.0083	0.0085	0.0081	0.0095
			3	0.0062	0.0059	0.0074	0.0076	0.0072	0.0087	0.0089	0.0085	0.0099
		1	1	0.0060	0.0056	0.0071	0.0074	0.0069	0.0084	0.0086	0.0082	0.0096
			2	0.0063	0.0059	0.0075	0.0077	0.0072	0.0088	0.0090	0.0085	0.0100
			3	0.0066	0.0062	0.0078	0.0081	0.0076	0.0092	0.0094	0.0089	0.0105
		2	1	0.0064	0.0059	0.0076	0.0078	0.0073	0.0089	0.0092	0.0086	0.0102
			2	0.0067	0.0062	0.0079	0.0082	0.0076	0.0093	0.0096	0.0090	0.0107
			3	0.0070	0.0066	0.0083	0.0086	0.0080	0.0098	0.0100	0.0095	0.0112
	60	0	1	0.0019	0.0018	0.0022	0.0023	0.0022	0.0026	0.0026	0.0025	0.0029
			2	0.0019	0.0018	0.0023	0.0023	0.0022	0.0026	0.0027	0.0026	0.0030
			3	0.0020	0.0019	0.0023	0.0024	0.0022	0.0027	0.0027	0.0026	0.0030
		1	1	0.0019	0.0018	0.0023	0.0023	0.0022	0.0026	0.0027	0.0026	0.0030
			2	0.0020	0.0019	0.0023	0.0024	0.0023	0.0027	0.0027	0.0026	0.0030
			3	0.0020	0.0019	0.0024	0.0024	0.0023	0.0027	0.0028	0.0027	0.0031
		2	1	0.0020	0.0019	0.0023	0.0024	0.0023	0.0027	0.0028	0.0027	0.0031
			2	0.0020	0.0019	0.0024	0.0024	0.0023	0.0028	0.0028	0.0027	0.0031
			3	0.0021	0.0020	0.0024	0.0025	0.0024	0.0028	0.0029	0.0028	0.0032
4	20	0	1	0.0128	0.0112	0.0146	0.0155	0.0137	0.0172	0.0180	0.0162	0.0197
			2	0.0136	0.0122	0.0157	0.0165	0.0149	0.0184	0.0192	0.0175	0.0211
			3	0.0146	0.0135	0.0170	0.0177	0.0163	0.0200	0.0207	0.0192	0.0228
		1	1	0.0140	0.0121	0.0160	0.0169	0.0149	0.0188	0.0197	0.0177	0.0216
			2	0.0149	0.0133	0.0172	0.0181	0.0162	0.0203	0.0212	0.0192	0.0233
			3	0.0161	0.0146	0.0186	0.0195	0.0178	0.0219	0.0228	0.0209	0.0252
		2	1	0.0154	0.0132	0.0176	0.0186	0.0163	0.0208	0.0218	0.0194	0.0240
			2	0.0165	0.0144	0.0190	0.0200	0.0178	0.0224	0.0235	0.0211	0.0258
			3	0.0178	0.0160	0.0206	0.0217	0.0195	0.0243	0.0254	0.0230	0.0280
	30	0	1	0.0064	0.0057	0.0073	0.0076	0.0069	0.0085	0.0088	0.0081	0.0096
			2	0.0067	0.0060	0.0076	0.0079	0.0073	0.0089	0.0092	0.0085	0.0100
			3	0.0069	0.0063	0.0080	0.0083	0.0076	0.0093	0.0096	0.0089	0.0105
		1	1	0.0068	0.0060	0.0077	0.0081	0.0073	0.0090	0.0094	0.0086	0.0102
			2	0.0070	0.0063	0.0081	0.0084	0.0077	0.0094	0.0097	0.0090	0.0106
			3	0.0074	0.0067	0.0085	0.0088	0.0081	0.0098	0.0102	0.0094	0.0112
		2	1	0.0072	0.0063	0.0082	0.0086	0.0077	0.0095	0.0099	0.0091	0.0108
			2	0.0075	0.0067	0.0086	0.0090	0.0081	0.0100	0.0104	0.0095	0.0113
			3	0.0078	0.0071	0.0090	0.0093	0.0086	0.0104	0.0108	0.0100	0.0119
60	0	1	0.0021	0.0019	0.0024	0.0025	0.0023	0.0027	0.0028	0.0026	0.0031	
		2	0.0021	0.0019	0.0024	0.0025	0.0023	0.0028	0.0029	0.0027	0.0031	
		3	0.0022	0.0020	0.0025	0.0025	0.0024	0.0028	0.0029	0.0028	0.0032	
	1	1	0.0021	0.0020	0.0024	0.0025	0.0023	0.0028	0.0029	0.0027	0.0031	
		2	0.0022	0.0020	0.0025	0.0026	0.0024	0.0029	0.0029	0.0028	0.0032	
		3	0.0022	0.0020	0.0025	0.0026	0.0024	0.0029	0.0030	0.0028	0.0033	
	2	1	0.0022	0.0020	0.0025	0.0026	0.0024	0.0029	0.0030	0.0028	0.0032	
		2	0.0022	0.0021	0.0026	0.0026	0.0025	0.0029	0.0030	0.0029	0.0033	
		3	0.0023	0.0021	0.0026	0.0027	0.0025	0.0030	0.0031	0.0029	0.0034	

Table 1. Cont.

k	n	r	s	P* = 0.90			P* = 0.95			P* = 0.975		
				\ddot{h}_U	\ddot{h}_L	\ddot{h}_t	\ddot{h}_U	\ddot{h}_L	\ddot{h}_t	\ddot{h}_U	\ddot{h}_L	\ddot{h}_t
6	20	0	1	0.0146	0.0121	0.0161	0.0172	0.0147	0.0187	0.0198	0.0172	0.0212
			2	0.0155	0.0133	0.0173	0.0184	0.0159	0.0200	0.0212	0.0186	0.0228
			3	0.0167	0.0147	0.0187	0.0198	0.0175	0.0217	0.0228	0.0204	0.0246
		1	1	0.0159	0.0131	0.0176	0.0189	0.0159	0.0205	0.0218	0.0187	0.0233
			2	0.0170	0.0144	0.0189	0.0202	0.0173	0.0220	0.0233	0.0203	0.0251
			3	0.0184	0.0159	0.0206	0.0218	0.0191	0.0239	0.0252	0.0222	0.0272
		2	1	0.0176	0.0143	0.0194	0.0208	0.0174	0.0226	0.0241	0.0205	0.0257
			2	0.0189	0.0157	0.0209	0.0224	0.0190	0.0243	0.0259	0.0223	0.0278
			3	0.0204	0.0174	0.0227	0.0243	0.0209	0.0265	0.0281	0.0245	0.0302
	30	0	1	0.0072	0.0062	0.0080	0.0085	0.0074	0.0092	0.0096	0.0086	0.0103
			2	0.0075	0.0065	0.0083	0.0088	0.0078	0.0096	0.0100	0.0090	0.0107
			3	0.0079	0.0069	0.0087	0.0092	0.0082	0.0100	0.0105	0.0094	0.0112
		1	1	0.0077	0.0065	0.0085	0.0090	0.0078	0.0097	0.0102	0.0091	0.0109
			2	0.0080	0.0068	0.0088	0.0093	0.0082	0.0101	0.0106	0.0095	0.0114
			3	0.0083	0.0072	0.0093	0.0098	0.0086	0.0106	0.0111	0.0100	0.0119
		2	1	0.0081	0.0068	0.0090	0.0095	0.0082	0.0103	0.0108	0.0096	0.0116
			2	0.0085	0.0072	0.0094	0.0099	0.0086	0.0108	0.0113	0.0100	0.0121
			3	0.0088	0.0076	0.0098	0.0104	0.0091	0.0113	0.0118	0.0106	0.0127
60	0	1	0.0023	0.0021	0.0026	0.0027	0.0025	0.0029	0.0030	0.0028	0.0033	
		2	0.0024	0.0021	0.0026	0.0027	0.0025	0.0030	0.0031	0.0029	0.0033	
		3	0.0024	0.0022	0.0027	0.0028	0.0026	0.0031	0.0032	0.0029	0.0034	
	1	1	0.0024	0.0021	0.0027	0.0028	0.0025	0.0030	0.0031	0.0029	0.0033	
		2	0.0024	0.0022	0.0027	0.0028	0.0026	0.0031	0.0032	0.0029	0.0034	
		3	0.0025	0.0022	0.0028	0.0029	0.0026	0.0031	0.0032	0.0030	0.0035	
	2	1	0.0025	0.0022	0.0027	0.0028	0.0026	0.0031	0.0032	0.0030	0.0034	
		2	0.0025	0.0022	0.0028	0.0029	0.0026	0.0032	0.0033	0.0030	0.0035	
		3	0.0026	0.0023	0.0028	0.0030	0.0027	0.0032	0.0033	0.0031	0.0036	

4. Example

One example of comparing the duration of remission under the use of four drugs in the treatment of leukemia was used in this section. For each drug, n = 20 patients were treated. The data of duration times by four drugs can be found in Table 1 of Wu and Wu [6] and they claimed that the data is exponentially distributed on the treatment of each drug and the scale parameters are unequal.

We applied Theorem 1 to this example to compare the median duration of remission times by using Drugs 1–3 with Drug 4 (the control population) denoted by $\delta_i - \delta_4$, $i = 1, 2, 3$. Three cases of censoring schemes of $(r,s) = (1,1), (2,1), (1,2)$ are considered for demonstration. The design variable $\tilde{c} = 150.7832$ under $(r,s) = (1,1)$; $\tilde{c} = 150.0608$ under $(r,s) = (2,1)$; $\tilde{c} = 136.9536$ under $(r,s) = (1,2)$. The critical values \ddot{h}_U, \ddot{h}_L and \ddot{h}_t for $P^* = 0.90, 0.95$ and 0.975 can be found in Table 1. For example when $P^* = 0.90$, the critical values $\ddot{h}_U = 0.0140, \ddot{h}_L = 0.0121$ and $\ddot{h}_t = 0.016$ under $(r,s) = (1,1)$; $\ddot{h}_U = 0.0149, \ddot{h}_L = 0.0133$ and $\ddot{h}_t = 0.0172$ under $(r,s) = (2,1)$; $\ddot{h}_U = 0.0154, \ddot{h}_L = 0.0132$ and $\ddot{h}_t = 0.0176$ under $(r,s) = (1,2)$. The UMVUE for $\delta_i - \delta_4, i = 1, 2, 3$ are $X_{i(r+1)} - X_{4(r+1)} + v^* (\tilde{S}_i - \tilde{S}_4) = -5.5, -4.0817$ and -1.9723 under $(r,s) = (1,1)$; $X_{i(r+1)} - X_{4(r+1)} + v^* (\tilde{S}_i - \tilde{S}_4) = -5.3410, -3.9415$ and -1.9252 under $(r,s) = (2,1)$; $X_{i(r+1)} - X_{4(r+1)} + v^* (\tilde{S}_i - \tilde{S}_4) = -5.4049, -4.1248$ and -1.8676 under $(r,s) = (1,2)$.

Applying parts (a) and (b) of Theorem 1 to this example, we could obtain the upper and lower confidence bounds for $\delta_i - \delta_4, i = 1, 2, 3$ with confidence levels of 0.90, 0.95 and 0.975 in Table 2. Since the upper bounds for Drugs 1 and 2 were negative, these two drugs were selected in a subset of worse than the control populations to reach the probabilities of

correct selection $P^* = 0.9, 0.95$ and 0.975 . Since all lower bounds were negative, no drugs were selected in a subset of better than the Drug 4 by comparing their median lifetimes

Table 2. The upper and lower confidence bounds for three drugs compared with the control drug (drug 4).

$(r,s) = (1,1)$	$X_{i(r+1)} - X_{4(r+1)} + v^* (\tilde{S}_i - \tilde{S}_4) + \tilde{c}\ddot{h}_U$ $X_{i(r+1)} - X_{4(r+1)} + v^* (\tilde{S}_i - \tilde{S}_4) - \tilde{c}\ddot{h}_L, i = 1, 2, 3$		
	90%	95%	97.5%
$\delta_1 - \delta_4$	-3.389, -7.324	-2.982, -7.747	-2.530, -8.169
$\delta_2 - \delta_4$	-1.971, -5.906	-1.564, -6.328	-1.111, -6.751
$\delta_3 - \delta_4$	0.139, -3.797	0.546, -4.219	0.998, -4.641
$(r,s) = (2,1)$	90%	95%	97.5%
$\delta_1 - \delta_4$	-3.105, -7.337	-2.625, -7.772	-2.160, -8.222
$\delta_2 - \delta_4$	-1.706, -5.937	-1.225, -6.372	-0.760, -6.823
$\delta_3 - \delta_4$	0.311, -3.921	0.791, -4.356	1.256, -4.806
$(r,s) = (1,2)$	90%	95%	97.5%
$\delta_1 - \delta_4$	-3.296, -7.213	-2.858, -7.637	-2.419, -8.062
$\delta_2 - \delta_4$	-2.016, -5.933	-1.577, -6.357	-1.139, -6.782
$\delta_3 - \delta_4$	0.241, -3.675	0.680, -4.100	1.118, -4.524

Applying part (c) of Theorem 1 to this example, we could obtain the two-sided confidence intervals for $\delta_i - \delta_4, i = 1, 2, 3$ under confidence levels of 0.90, 0.95 and 0.975 in Table 3. Since the upper limits for Drugs 1 and 2 were negative, we could conclude that the median lifetimes of these two drugs were worse than Drug 4. Looking at the UMVUE for the difference between these two drugs with Drug 4 under three censoring cases, the performance of Drug 1 was worse than Drug 2 comparing with Drug 4. Since the confidence interval for Drug 3 contained zero, we could claim that the median lifetime of Drug 3 was not much different from drug 4.

Table 3. The two-sided confidence intervals for three drugs compared with the control drug (drug 4).

$(r,s) = (1,1)$	$(X_{i(r+1)} - X_{4(r+1)} + v^* (\tilde{S}_i - \tilde{S}_4) \pm \tilde{c}\ddot{h}_t), i = 1, 2, 3$		
	90%	95%	97.5%
$\delta_1 - \delta_4$	(-7.913, -3.087)	(-8.335, -2.665)	(-8.757, -2.243)
$\delta_2 - \delta_4$	(-6.494, -1.669)	(-6.916, -1.247)	(-7.339, -0.825)
$\delta_3 - \delta_4$	(-4.385, 0.440)	(-4.807, 0.862)	(-5.229, 1.285)
$(r,s) = (2,1)$	90%	95%	97.5%
$\delta_1 - \delta_4$	(-7.922, -2.760)	(-8.387, -2.295)	(-8.837, -1.845)
$\delta_2 - \delta_4$	(-6.523, -1.360)	(-6.988, -0.895)	(-7.438, -0.445)
$\delta_3 - \delta_4$	(-4.506, 0.656)	(-4.971, 1.121)	(-5.422, 1.571)
$(r,s) = (1,2)$	90%	97.5%	95%
$\delta_1 - \delta_4$	(-7.815, -2.995)	(-8.254, -2.556)	(-8.692, -2.118)
$\delta_2 - \delta_4$	(-6.535, -1.714)	(-6.973, -1.276)	(-7.412, -0.838)
$\delta_3 - \delta_4$	(-4.278, 0.543)	(-4.716, 0.981)	(-5.154, 1.419)

5. Conclusions

When a sampling procedure in an experiment was unexpectedly terminated earlier so that the additional sample for the two-stage sample was not available, the one-stage procedure could be employed for the multiple comparison with the control. Applying Lam’s [14,15] technique, we developed multiple comparison procedures with a control for exponential median lifetimes under heteroscedasticity based on doubly censored sam-

ple. At last, we used one example in the treatment of leukemia to find the upper and lower confidence bounds and for each median lifetime of treatment populations compared to the control population. The two-sided confidence intervals were also obtained and analyzed.

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